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On a Relationship between Pascal Matrix and Vandermonde Matrix

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Abstract: EI-Mikkawy M obtained that the symmetric Pascal matrix Q_n and the Vandermonde matrix V_n are connected by the equation $Q_n = T_n V_n$, where T_n is a stochastic matrix in [1]. In this paper, a decomposition of the matrix T_n is given via the Stirling matrix of the first kind, and a recurrence relation of the elements of the matrix T_n is obtained, so an open problem proposed by EI-Mikkawy[2] is solved. Some combinatorial identities are also given.

Key words: Stirling number; Stirling matrix; Vandermonde matrix; Pascal matrix.
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1. Introduction

The lower triangular Pascal matrix P_n and the symmetric Pascal matrix Q_n which derived naturally from the Pascal triangle were studied by many authors in recent years. In [3–5], the authors studied the generalized Pascal matrices and gave some combinatorial identities. The Stirling matrix of the first kind s_n and the Stirling matrix of the second kind S_n obtained from the Stirling numbers of the first kind $s(i,j)$ and of the second kind $S(i,j)$ respectively have been discussed recently[6]. In [1], the author investigated a connection between the Pascal, Vandermonde and Stirling matrices, and showed by using MAPLE that a stochastic matrix T_n links together these matrices. In [2], the author raised that to generate the elements of the matrix T_n for any arbitray n using only one or two recurrence relations is an open question. In this paper, some relations between the Stirling matrix S_n and the Pascal matrix P_n are obtained, and a factorization of the matrix T_n^{-1} is given by the using the Stirling matrix of the second kind. Furthermore, a decomposition of the matrix T_n is given via the Stirling matrix of the first kind, and a recurrence relation of the elements of the matrix T_n is obtained, so an open problem proposed by EI-Mikkawy[2] is solved. As a consequence we obtain some combinatorial identities related to the Stirling numbers.

2. Preliminary results

Let n,k be nonnegative integers and $n \geq k$, the Stirling numbers of the first kind $s(n,k)$ and

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of the second kind $S(n,k)$ can be defined as the coefficients in the following expansion of a variable x : $(x)_n=\sum_{k=0}^n s(n,k)x^k$ and $x^n=\sum_{k=0}^n S(n,k)(x)_k$, where $(x)_k=x(x-1)(x-2)\cdots(x-k+1)$ for any integer $k>0$, and $(x)_0=1$. $s(k,k)=S(k,k)=1$ for $k\geq 0$, and $s(k,0)=S(k,0)=0$ for $k>0$.

It is known that the Stirling numbers have the following recurrence relations[7]

$$s(n,k)=s(n-1,k-1)-(n-1)s(n-1,k), \tag{1}$$

$$S(n,k)=S(n-1,k-1)+kS(n-1,k). \tag{2}$$

The $n\times n$ Pascal matrix P_n is defined by[8,9] $P_n=[\binom{i-1}{j-1}]_{1\leq i,j\leq n}$, where $\binom{i}{j}=0$, if $i<j$. It is known that $P_n^{-1}=J_nP_nJ_n$, where $J_n=\mathrm{diag}(1,-1,\cdots,(-1)^{n-1})$.

The Stirling matrix of the first kind s_n and the Stirling matrix of the second kind S_n are defined respectively by $s_n=[s(i,j)]_{1\leq i,j\leq n}$, $S_n=[S(i,j)]_{1\leq i,j\leq n}$, where $s(i,j)=0, S(i,j)=0$ if $i<j$. It is easy to see that $S_ns_n=I_n, S_n^{-1}=s_n$.

Lemma 1[6] *$S_n=P_n([1]\oplus S_{n-1})$; $s_n=([1]\oplus s_{n-1})P_n^{-1}$.*

Lemma 2[6] *Define V_n be the $n\times n$ Vandermonde matrix by $V_n(i,j)=j^{i-1}, 1\leq i,j\leq n$. Then $V_n=S_nD_nP_n^T$, where $D_n=\mathrm{diag}(0!,1!,2!,\cdots,(n-1)!)$.*

Lemma 3[1] *Let $Q_n=[\binom{i+j-2}{j-1}]_{1\leq i,j\leq n}$ be the $n\times n$ symmetric Pascal matrix, then we have following factorization*

$$Q_n=P_nP_n^T=P_nD_n^{-1}s_nV_n;$$

Denote $T_n=P_nD_n^{-1}s_n=P_nD_n^{-1}([1]\oplus s_{n-1})P_n^{-1}$, then

$$T_n^{-1}=S_nD_nP_n^{-1}=P_n([1]\oplus S_{n-1})D_nP_n^{-1},$$

and T_n links the symmetric Pascal matrix Q_n and the Vandermonde matrix V_n by $Q_n=T_nV_n$.

3. Main results

Lemma 4[10] *$V_n=([1]\oplus S_{n-1})D_n\Delta_nP_n^T$, where Δ_n is the $n\times n$ lower triangular matrix whose $(i,j)-$ entry is $\binom{1}{i-j}$ if $i\geq j$ and otherwise is 0.*

Lemma 5 *$\Delta_nP_n^{-1}=([1]\oplus P_{n-1}^{-1})=([1]\oplus P_{n-1})^{-1}$.*

Proof

$$\begin{aligned}(\Delta_nP_n^{-1})(i,j)&=\Delta_n(i,i-1)P_n^{-1}(i-1,j)+\Delta_n(i,i)P_n^{-1}(i,j)\\&=1\cdot(-1)^{i-j-1}\binom{i-2}{j-1}+1\cdot(-1)^{i-j}\binom{i-1}{j-1}\\&=(-1)^{i-j}\left(\binom{i-1}{j-1}-\binom{i-2}{j-1}\right)\\&=(-1)^{i-j}\binom{i-2}{j-2}=([1]\oplus P_{n-1}^{-1})(i,j).\end{aligned}$$

Lemma 6 $T_n^{-1}=([1]\oplus S_{n-1})D_n([1]\oplus P_{n-1}^{-1})$.

Proof By Lemmas 3, 4 and 5, we have

$$\begin{aligned}T_n^{-1}=V_nQ_n^{-1}&=(([1]\oplus S_{n-1})D_n\Delta_nP_n^T)((P_n^T)^{-1}P_n^{-1})\\&=([1]\oplus S_{n-1})D_n\Delta_nP_n^{-1}=([1]\oplus S_{n-1})D_n([1]\oplus P_{n-1}^{-1}).\end{aligned}$$

Lemma 7 *For each $i,j=1,2,3,\cdots,n, i\geq j$, we have*

$$S(i,j)j!=\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!\binom{k-1}{j-1}. \tag{3}$$

Proof For a fixed positive integer j, we prove the statement by induction on i. For $i=j$, the statement holds since the right hand of (3) equals $(-1)^{j-j}S(j,j)j!\binom{j-1}{j-1}=S(j,j)j!$, it is exact the left hand of (3). Suppose it holds for $\leq i$, and we want to prove it for $i+1$. Using the recurrence relation (2) and the induction hypothesis, we obtain

$$\begin{aligned}&S(i+1,j)j!=j!(S(i,j-1)+jS(i,j))\\&=j!(\frac{1}{(j-1)!}\sum_{k=j-1}^{i}(-1)^{i-k}S(i,k)k!\binom{k-1}{j-2}+\frac{j}{j!}\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!\binom{k-1}{j-1})\\&=j\sum_{k=j-1}^{i}(-1)^{i-k}S(i,k)k!\binom{k-1}{j-2}+j\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!\binom{k-1}{j-1},\end{aligned}$$

that is $S(i+1,j)j!=j\sum_{k=j-1}^{i}(-1)^{i-k}S(i,k)k!\binom{k-1}{j-2}+j\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!\binom{k-1}{j-1}$.

On the other hand,

$$\begin{aligned}&\sum_{k=j}^{i+1}(-1)^{i+1-k}S(i+1,k)k!\binom{k-1}{j-1}=\sum_{k=j}^{i+1}(-1)^{i+1-k}(kS(i,k)+S(i,k-1))k!\binom{k-1}{j-1}\\&=\sum_{k=j}^{i+1}(-1)^{i+1-k}S(i,k)k!k\binom{k-1}{j-1}+\sum_{k=j}^{i+1}(-1)^{i+1-k}S(i,k-1)k!\binom{k-1}{j-1}\\&=\sum_{k=j}^{i}(-1)^{i+1-k}S(i,k)k!k\binom{k-1}{j-1}+\sum_{k=j}^{i+1}(-1)^{i+1-k}S(i,k-1)k!\binom{k-1}{j-1}\\&=-\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!k\binom{k-1}{j-1}+\sum_{t=j-1}^{i}(-1)^{i-t}S(i,t)t!(t+1)\binom{t}{j-1}\\&=j\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!\binom{k-1}{j-1}-\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!(k+j)\binom{k-1}{j-1}+\\&\qquad(-1)^{i-j+1}S(i,j-1)j!+\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!(k+1)\binom{k}{j-1}\end{aligned}$$

$$= j\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!\binom{k-1}{j-1}+$$
$$(-1)^{i-j+1}S(i,j-1)j!+\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!\left((k+1)\binom{k}{j-1}-(k+j)\binom{k-1}{j-1}\right)$$
$$= j\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!\binom{k-1}{j-1}+(-1)^{i-j+1}S(i,j-1)j!+\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!j\binom{k-1}{j-2}$$
$$= j\sum_{k=j}^{i}(-1)^{i-k}S(i,k)k!\binom{k-1}{j-1}+j\sum_{k=j-1}^{i}(-1)^{i-k}S(i,k)k!\binom{k-1}{j-2}.$$

Therefore, $S(i+1,j)j! = \sum_{k=j}^{i+1}(-1)^{i+1-k}S(i+1,k)k!\binom{k-1}{j-1}$. This completes the proof.

Using Lemma 7 and considering the matrix equality, we obtain the following results.

Theorem 1 $\widetilde{S_n}= J_n\,\widetilde{S_n}\,J_nP_n$ *, and* $\widetilde{S_n}\,P_n^{-1} = J_n\,\widetilde{S_n}\,J_n$*, where* $\widetilde{S_n}= S_n\mathrm{diag}(1,2!,\cdots,n!)$.

Theorem 2 *The matrix* T_n^{-1} *has the following decomposition and properties*

(a) $T_n^{-1} = J_n([1]\oplus \widetilde{S}_{n-1})J_n$;

(b) $T_n^{-1}(i,j) = (-1)^{i-j}S(i-1,j-1)(j-1)!$;

(c) $T_n^{-1}(i,j) = [T_n^{-1}(i-1,j-1) - T_n^{-1}(i-1,j)](j-1)$.

Proof (a) Using Lemma 6 and Theorem 1, we have

$$\begin{aligned}T_n^{-1} &= ([1]\oplus S_{n-1})D_n([1]\oplus P_{n-1}^{-1}) = ([1]\oplus \widetilde{S}_{n-1})([1]\oplus P_{n-1}^{-1})\\ &= [1]\oplus(\widetilde{S}_{n-1}\,P_{n-1}^{-1}) = [1]\oplus(J_{n-1}\,\widetilde{S}_{n-1}\,J_{n-1}) = J_n([1]\oplus \widetilde{S}_{n-1})J_n.\end{aligned}$$

(b) From (a), we have

$$T_n^{-1}(i,j) = (J_n([1]\oplus \widetilde{S}_{n-1})J_n)(i,j) = (-1)^{i-j}S(i-1,j-1)(j-1)!;$$

(c) From (b) and the recurrence relation (2), we obtain

$$\begin{aligned}&[T_n^{-1}(i-1,j-1) - T_n^{-1}(i-1,j)](j-1)\\ &= [(-1)^{i-j}S(i-2,j-2)(j-2)! - (-1)^{i-j-1}S(i-2,j-1)(j-1)!](j-1)\\ &= (-1)^{i-j}(j-2)![S(i-2,j-2)+S(i-2,j-1)(j-1)](j-1)\\ &= (-1)^{i-j}(j-1)!S(i-1,j-1) = T_n^{-1}(i,j).\end{aligned}$$

EI-Mikkawy[2] pointed out that to generate the elements of the matrix T_n for any arbitray n using only one or two recurrence relations is an open question. We are now in a position to give a answer to this problem.

Theorem 3 *The matrix* T_n *has the following decomposition and properties*

(a) $T_n = J_nD_n^{-1}([1]\oplus s_{n-1})J_n$;

(b) $T_n(i,j) = (-1)^{i-j}\frac{s(i-1,j-1)}{(i-1)!}$;

(c) T_n *is a stochastic matrix;*

(d) $T_n(i,j) = \frac{1}{i-1}T_n(i-1,j-1) + \frac{i-2}{i-1}T_n(i-1,j)$.

Proof (a) From Theorem 2 (a), $T_n^{-1} = J_n([1]\oplus \widetilde{S}_{n-1})J_n$, so

$$T_n = (J_n([1]\oplus \widetilde{S}_{n-1})J_n)^{-1} = (J_n([1]\oplus S_{n-1})D_nJ_n)^{-1} = J_nD_n^{-1}([1]\oplus s_{n-1})J_n;$$

(b) By (a), we have $T_n(i,j) = (J_nD_n^{-1}([1]\oplus s_{n-1})J_n)(i,j) = (-1)^{i-j}\frac{s(i-1,j-1)}{(i-1)!}$;

(c) Since $\sum_{k=1}^{i}(-1)^k s(i,k) = (-1)^i i!$, we have

$$\sum_{j=1}^{i} T_n(i,j) = \sum_{j=1}^{i}(-1)^{i-j}s(i-1,j-1)\frac{1}{(i-1)!} = \frac{(-1)^i}{(i-1)!}\sum_{j=1}^{i}(-1)^j s(i-1,j-1)$$
$$= \frac{(-1)^i}{(i-1)!}\sum_{k=1}^{i-1}(-1)^{k+1}s(i-1,k) = \frac{(-1)^i}{(i-1)!}(-1)^i(i-1)! = 1.$$

Therefore, T_n is a stochastic matrix.

(d) From (b) and recurrence relation (1), we have

$$\begin{aligned}
&\frac{1}{i-1}T_n(i-1,j-1) + \frac{i-2}{i-1}T_n(i-1,j))\\
&= \frac{1}{i-1}[(-1)^{i-j}s(i-2,j-2)\frac{1}{(i-2)!} + (i-2)(-1)^{i-j-1}s(i-2,j-1)\frac{1}{(i-2)!}]\\
&= \frac{1}{i-1}(-1)^{i-j}\frac{1}{(i-2)!}[s(i-2,j-2)-(i-2)s(i-2,j-1)]\\
&= (-1)^{i-j}s(i-1,j-1)\frac{1}{(i-1)!}\\
&= T_n(i,j).
\end{aligned}$$

4. Some combinatorial identities

By applying the two different representations: $T_n^{-1} = S_nD_nP_n^{-1}$, and $T_n^{-1} = J_n([1]\oplus \widetilde{S}_{n-1}$ $)J_n$, the following results hold

$$J_n([1]\oplus \widetilde{S}_{n-1})J_n = S_nD_nP_n^{-1},\ S_nD_n = J_n([1]\oplus \widetilde{S}_{n-1})J_nP_n; \tag{4}$$

$$J_nD_n^{-1}([1]\oplus s_{n-1})J_n = P_nD_n^{-1}s_n,\ D_n^{-1}s_n = P_n^{-1}J_nD_n^{-1}([1]\oplus s_{n-1})J_n; \tag{5}$$

$$Q_n = J_nD_n^{-1}([1]\oplus s_{n-1})J_nV_n,\ P_n = J_nD_n^{-1}([1]\oplus s_{n-1})J_nS_nD_n. \tag{6}$$

Considering the matrix equality (4), we have the following identities for the Stirling numbers of the second kind

$$S(i-1,j-1)(j-1)! = \sum_{k=j}^{i}(-1)^{i-k}S(i,k)(k-1)!\binom{k-1}{j-1}, \tag{7}$$

$$S(i,j)(j-1)! = \sum_{k=j}^{i}(-1)^{i-k}S(i-1,k-1)(k-1)!\binom{k-1}{j-1}. \tag{8}$$

Using (5) yields the following identities for the Stirling numbers of the first kind

$$(-1)^{i-j}\frac{s(i-1,j-1)}{(i-1)!}=\sum_{k=j}^{i}\binom{i-1}{k-1}\frac{s(k,j)}{(k-1)!}, \tag{9}$$

$$\frac{s(i,j)}{(i-1)!}=\sum_{k=j}^{i}(-1)^{i-j}\binom{i-1}{k-1}\frac{s(k-1,j-1)}{(k-1)!}. \tag{10}$$

From (6) we obtain the following identities

$$\binom{i+j-2}{j-1}=\frac{1}{(i-1)!}\sum_{k=1}^{i}(-1)^{i-k}s(i-1,k-1)j^{k-1}, \tag{11}$$

$$\binom{i-1}{j-1}=\frac{j!}{(i-1)!}\sum_{k=j}^{i}(-1)^{i-k}s(i-1,k-1)S(k,j). \tag{12}$$

In particular for $j=1,2$, the identity (3) gives

$$\sum_{k=1}^{i}(-1)^{i-k}S(i,k)k!=1, \tag{13}$$

$$\sum_{k=2}^{i}(-1)^{i-k}S(i,k)k!(k-1)=2^{i}-2. \tag{14}$$

In particular for $j=1,2,3$, the identity (7) gives

$$\sum_{k=1}^{i}(-1)^{i-k}S(i,k)(k-1)!=0, \tag{15}$$

$$\sum_{k=2}^{i}(-1)^{i-k}S(i,k)(k-1)!(k-1)=1, \tag{16}$$

$$\sum_{k=3}^{i}(-1)^{i-k}S(i,k)(k-1)!\frac{(k-1)(k-2)}{2}=2^{i-1}-2. \tag{17}$$

In particular for $j=1,2$, the identity (8) gives

$$\sum_{k=1}^{i}(-1)^{i-k}S(i-1,k-1)(k-1)!=1, \tag{18}$$

$$\sum_{k=2}^{i}(-1)^{i-k}S(i-1,k-1)(k-1)!(k-1)=2^{i-1}-1. \tag{19}$$

In particular for $j=2$, the identity (9) gives

$$\sum_{k=2}^{i}\binom{i-1}{k-1}\frac{s(k,2)}{(k-1)!}=\frac{1}{i-1}. \tag{20}$$

In particular for $j = 2$, the identity (10) gives

$$s(i,2) = (i-1)!\sum_{k=2}^{i}(-1)^{i-k}\binom{i-1}{k-1}\frac{1}{(k-1)}. \tag{21}$$

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Pascal 矩阵与 Vandermonde 矩阵的关系

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摘要: EI-Mikkawy M 证明了对称 Pascal 矩阵 Q_n 和 Vandermonde 矩阵 V_n 之间满足矩阵方程 $Q_n = T_nV_n$, 这里 T_n 是一个随机矩阵. 本文证明了随机矩阵 T_n 能够分解成第一类 Stirling 矩阵和对角矩阵的乘积，得到了矩阵 T_n 的元素之间的递推关系，从而回答了 EI-Mikkawy M 的一个公开问题. 同时得到了一些与 Stirling 数相关的组合恒等式.

关键词: Stirling 数; Stirling 矩阵; vandermonde 矩阵; Pascal 矩阵.