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Construction of Minimal Surfaces with Special Type Ends

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Abstract We proved that there exists a family of complete oriented minimal surfaces in $\mathbb{R}^3$ with finite total curvature $-4n\pi$, each of which has 0 genus and two ends, and both of the ends have winding order n, where $n \in \mathbb{N}$, and discussed the symmetric property for special parameters.

Keywords minimal surface; total curvature; end; winding order.

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1. Introduction

It is generally admitted that the investigations about minimal surfaces started with Lagrange in 1760. Since then, great progress has been made in the classical theory of minimal surfaces in $\mathbb{R}^3$. Before Costa's surface occurs, the plane, the catenoid and the helicoid were the only known embedded minimal surfaces, the others were immersed.

Chern-Osserman [1] proved a fundamental classical result to complete minimal surfaces of finite total curvature, which states that such a surface M is conformally equivalent to compact Riemann surface $\bar{M}$ punctured in a finite number of points. Furthermore, the punctured points coincide with the ends of M. Embedded minimal surfaces have many remarkable properties, such as, each end is itself intersection-free, and the limiting tangent planes (specifically the plane orthogonal to the limiting value of the Gauss map at the end) are parallel with each other, so that the the Gauss map at the ends have at most two directions. Since 1980s, great progress has been made in minimal surfaces with embedded ends. Using the elliptic functions, Costa [5] exhibited an example of a complete minimal immersion of a torus minus three points in $\mathbb{R}^3$, and proved these three ends are embedded. Later, Hoffman [9] and Meeks [4] proved that the whole Costa's surface is in fact embedded, and constructed a large family of embedded minimal surfaces which similar to Costa's surface, and the limiting surface of the family is planar and catenoid types. Xiao [10] proved that there exist preduo-embedded minimal surfaces with k ends in $\mathbb{R}^3$ for any $k \neq 3, 4, 5$.

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In 1982, Chen-Gackstatter [8] constructed surfaces which have one Enneper type end and are of genus one and genus two, whose end's winding order are both 3. In 1994, Sato [2] constructed a family of complete minimal surfaces with genus jk and a Enneper type end with winding order $2k+1$, for all $j, k \in \mathbb{N}$. Lopez [6] gave the classification of complete oriented minimal surfaces with total curvature greater than -12π, which included a family of genus zero examples with two ends whose winding order are both two.

The aim of this paper is to construct a family of complete immersion minimal surfaces with higher winding order. Moreover, we present computer-generated images. Actually we have the following result.

Therorem 5 *There exists a family of complete oriented minimal surfaces in* $\mathbb{R}^3$ *with finite total curvature* $-4n\pi$, *each of which has* 0 *genus and two ends, and both of the ends have winding order* n, *where* $n \in \mathbb{N}$.

We denote $\mathcal{M}_n$ as the surfaces described in the Theorem 5.

In Section 2, we summarize some basic results about the minimal surface theory. In Section 3, we construct one of Lopez's examples, and discuss its symmetric property. In Section 4, we present the Weierstrass data for the case $1 \leq n \leq 5$, then proved Theorem 5. Finally we discuss the symmetric property of $\mathcal{M}_n$ for special parameters.

At the end of this section, we present some computer-generated images of the examples described in Theorem 5, viewing from different directions. These images were produced by Zhanchang Zhang at Dalian University of Technology, using Java3D.

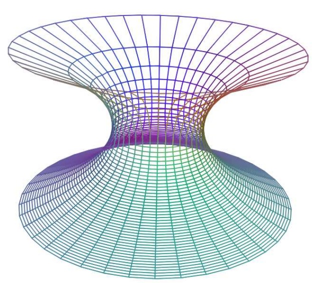


Figure 1 Surface $\mathcal{M}_1$ (Catenoid)

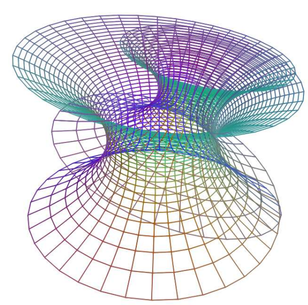


Figure 2 Another form of Surface $\mathcal{M}_2$

2. Basic facts

We begin by stating Weierstrass Representation Theorem (say WR-Theorem briefly).

Therorem 1 ([3]) *Let* M *be a Riemann surface,* η *a holomorphic one-form on* M *and* $g : M \to \mathbb{C} \cup \{\infty\}$ *a meromorphic function. Consider the vector valued one-form*

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) = \Big(\frac{1}{2}(1-g^2)\eta, \frac{i}{2}(1+g^2)\eta, g\eta\Big). \tag{1}$$

Then

$$X(p) = \mathrm{Re}\int_{p_0}^{p} \alpha \tag{2}$$

is a conformal minimal immersion which is well-defined on M and regular, provided that

(a) No component of α in (1) has a real period on M;

(b) The poles of g coincide with zeros of η and the order of a pole of g if precisely half of the order of a zero of η.

Conversely, every regular conformal minimal immersion $X : M \to \mathbb{R}^3$ can be expressed (up to translation) in the form (2) for some meromorphic function g and holomorphic one-form η. Moreover, g is the stereographic projection to $\mathbb{C} \cup \{\infty\}$ of the Gauss map $N : M \to S^2$ of X.

Remark 1 $\{g, \eta\}$ is called the Weierstrass data.

(a) (b)

Figure 3 (a) Half of the surface $\mathcal{M}_2$; (b) Surface $\mathcal{M}_4$ (not symmetric)

(c) (d)

Figure 4 Surface $\mathcal{M}_2$

With WR-Theorem, the first fundamental form and Gaussian curvature of minimal immersion can be easily presented by

$$\mathrm{d}s^2 = \frac{1}{4}|f|^2(1+|g|^2)^2|\mathrm{d}z|^2 \tag{3}$$

$$K = -\Big[\frac{4|g'|}{|f|(1+|g|^2)^2}\Big]^2. \tag{4}$$

Therorem 2 ([7]) *Let $X : M \to \mathbb{R}^3$ be a complete conformal minimal immersion with finite total curvature. Then:*

(a) M is conformal diffeomorphic to $\overline{M}_k \backslash \{p_1, \ldots, p_r\}$ where $\overline{M}_k$ is a closed Riemann surface of genus k and $p_1, \ldots, p_r$ are points in $\overline{M}_k$, $r \geqslant 1$;

(b) X is proper;

(c) The Gauss map $N : M \to S^2$, which is meromorphic on M, extends to a meromorphic function on $\overline{M}_k$; the holomorphic one-form η extends to a meromorphic one-form on $\overline{M}_k$.

(d) If the Weierstrass data g covers $S^2(1)$ m times, then the total curvature is

$$\int_M K \mathrm{d}A = -4m\pi \tag{5}$$

and satisfies

$$\int_M K \mathrm{d}A \leqslant -4\pi(k + r - 1) \tag{6}$$

where k and r are integers defined in Statement (a).

Suppose that $X : M \to \mathbb{R}^3$ is a complete regular conformal minimal immersion with finite total curvature where $M = \overline{M}_k \backslash \{p_1, \ldots, p_r\}$. Assume that X does not factor through another minimal immersion as a covering space.

Let D_j be a punctured neighborhood of $p_j \in \overline{M}_k$, for each $j = 1, \ldots, r$. We will refer to $X(D_j) = E_j$ as an end of M, and denote by $Y_{R,j}$, the intersection of E_j with the sphere of radius R with center at the origin. Let $X_{R,j}$ be the radial projection of $Y_{R,j}$ onto $S^2(1)$, i.e.,

$$X_{R,j} = \frac{1}{R} Y_{R,j}. \tag{7}$$

The winding order I of end and the relations between I and the total curvature are stated in the following theorem:

Therorem 3 ([3]) *Let $X_{R,j}$ be defined as in (7). Then*

(a) $X_{R,j}$ converges smoothly as $R \to \infty$ to a great circle, covered an integral number of times;

(b) Let I_j be the multiplicity of the great circle $\lim_{R\to\infty} X_{R,j}$, which was called winding order of the jth end. Then

$$\int_M K \mathrm{d}A = 2\pi\Big(2(1-k) - r - \sum_{j=1}^{r} I_j\Big) = 2\pi\Big(\chi(M) - \sum_{j=1}^{r} I_j\Big) \tag{8}$$

where $\chi(M)$ represents the Euler characteristic of M.

3. Construction of $\mathcal{M}_2$

In [6], Lopez present the following results.

Therorem 4 ([6]) *Let g, η be the Weierstrass data of $\mathcal{M}_2$, and $b_g(p)$ is the branch number of g at p. Suppose that $I_1 = I_2 = 2$, $b_g(\infty) = b_g(0) = 1$. Then, up to change of parameter in $\mathbb{C} \cup \{\infty\}$, homothety and rigid motion in $\mathbb{R}^3$, $\mathcal{M}_2$ has the following Weierstrass data:*

$$g(z) = Bz^2, \quad \eta = \frac{1}{z^3}\mathrm{d}z$$

where $B \in \mathbb{R}\backslash\{0\}$.

In this section, we will construct $\mathcal{M}_2$ precisely. We denote $X : M \to \mathbb{R}^3$ be such a minimal immersion, where $M = \mathbb{C}\backslash\{0\}$, and g, η is the Weierstrass data of X.

Based on WR-Theorem, we denote

$$\alpha_k = \phi_k \mathrm{d}z, \quad k = 1, 2, 3. \tag{9}$$

Then

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0, \quad \text{i.e., locally} \quad \phi_1^2 + \phi_2^2 + \phi_3^2 = 0 \tag{10}$$

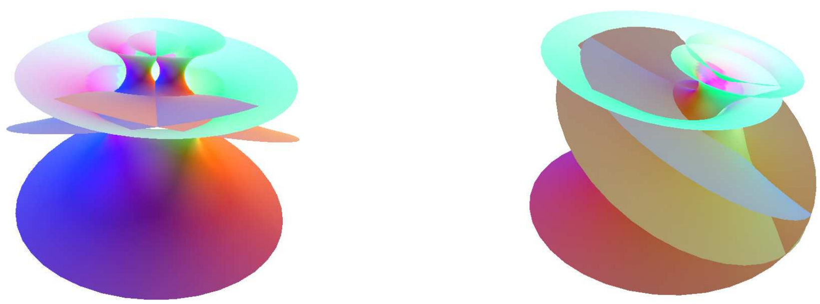
and $\alpha_1, \alpha_2, \alpha_3$ have not real period. Hence, in the generalized complex plane,

$$\mathrm{Re}\, s_{z=0}(\alpha_k) = -\mathrm{Re}\, s_{z=\infty}(\alpha_k), \quad k = 1, 2, 3, \tag{11}$$

and both of them are real numbers. Suppose that ϕ_k has the following form

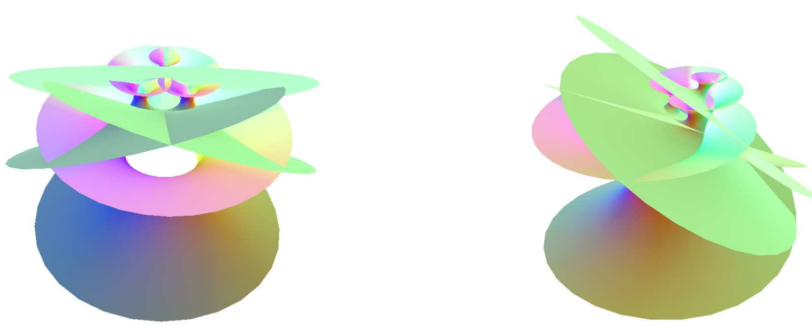
$$\phi_k = \sum_{j=-\infty}^{\infty} c_{kj} z^j, \quad k = 1, 2, 3. \tag{12}$$

where $c_{kj} \in \mathbb{C}$ are constant numbers.



(e) (f)

Figure 5 Surface $\mathcal{M}_3$



(g) (h)

Figure 6 Surface $\mathcal{M}_4$

We assume α_k, $k = 1, 2, 3$, have poles at p. Denote $\mathrm{Ord}_p \alpha_k$, $k = 1, 2, 3$, to be the pole's order of α_k at p. Meeks and Jorge [3] proved the following result:

Lemma 1 ([3]) *Let p_j be the point corresponding to an end of M. Denote by I_j the winding order of the end. Then*

$$I_j + 1 = \max\{\mathrm{Ord}_{p_j}\alpha_k, \quad k = 1, 2, 3\}. \tag{13}$$

Assume $I_1 = I_2 = 2$. From (13) we have $c_{kj} = 0$, for all $j > 1$ and $j < -3$, and exist non-zero numbers in $\{c_{k1}\}$ and in $\{c_{k,-3}\}$. So we can represent ϕ_k as follows

$$\phi_k = a_k z + b_k + c_k z^{-1} + d_k z^{-2} + e_k z^{-3} \tag{14}$$

where exist non-zero numbers in $\{a_k\}$ and in $\{e_k\}$.

In general, the period problems played an important role in construction of minimal surfaces. From (11) we know that α_k has no real period if and only if $\mathrm{Re}\, s_{z=0}(\alpha_k) \in \mathbb{R}$, $k = 1, 2, 3$.

Lemma 2 *α_k has no real period if and only if every $c_k \in \mathbb{R}$, for $k = 1, 2, 3$.*

Since $z^2, z, 1, \ldots, z^{-6}$ are liner independent, it follows from (9) that

$$\begin{aligned}
&a_1^2 + a_2^2 + a_3^2 = 0,\\
&a_1 b_1 + a_2 b_2 + a_3 b_3 = 0,\\
&2a_1 c_1 + b_1^2 + 2a_2 c_2 + b_2^2 + 2a_3 c_3 + b_3^2 = 0,\\
&a_1 d_1 + b_1 c_1 + a_2 d_2 + b_2 c_2 + a_3 d_3 + b_3 c_3 = 0,\\
&2a_1 e_1 + 2b_1 d_1 + c_1^2 + 2a_2 e_2 + 2b_2 d_2 + c_2^2 + 2a_3 e_3 + 2b_3 d_3 + c_3^2 = 0,\\
&b_1 e_1 + c_1 d_1 + b_2 e_2 + c_2 d_2 + b_3 e_3 + c_3 d_3 = 0,\\
&2c_1 e_1 + d_1^2 + 2c_2 e_2 + d_2^2 + 2c_3 e_3 + d_3^2 = 0,\\
&d_1 e_1 + d_2 e_2 + d_3 e_3 = 0,\\
&e_1^2 + e_2^2 + e_3^2 = 0.
\end{aligned} \tag{15}$$

Form (8), the total curvature of X is

$$C(M) = 2\pi(\chi(M) - (I_1 + I_2)) = -8\pi$$

and from (5), $g(z)$ covers $S^2(1)$ two times. So we can set

$$g(z) = g_1 z^2 + g_2 z + g_3 \tag{16}$$

where $g_1 \neq 0$.

If $\alpha_1 \equiv i\alpha_2$, then from (10) we have $\alpha_3 = 0$ and the resulting minimal surface is a plane. We suppose that $\alpha_1 \neq i\alpha_2$. From (1), we have

$$\eta(z) = \alpha_1 - i\alpha_2, \quad g(z) = \frac{\alpha_3}{\alpha_1 - i\alpha_2}. \tag{17}$$

From (16) and (17), we have

$$g_1 z^2 + g_2 z + g_3 = \frac{p_3(z)}{p_1(z) - ip_2(z)}, \tag{18}$$

where $p_k(z) = a_k z^4 + b_k z^3 + c_k z^2 + d_k z + e_k$, $k = 1, 2, 3$. It follows from (18) that

$$\begin{aligned} &a_2 = -ia_1, \\ &b_2 = -ib_1, \\ &g_1(c_1 - ic_2) = a_3, \\ &g_2(c_1 - ic_2) + g_1(d_1 - id_2) = b_3, \\ &g_3(c_1 - ic_2) + g_2(d_1 - id_2) + g_1(e_1 - ie_2) = c_3, \\ &g_3(d_1 - id_2) + g_2(e_1 - ie_2) = d_3, \\ &g_3(e_1 - ie_2) = e_3. \end{aligned} \tag{19}$$

By (15) and (19), we have

$$a_3 = 0. \tag{20}$$

According to Lemma 2, we have $c_1, c_2 \in \mathbb{R}$, and from (19), we have

$$c_1 = c_2 = 0. \tag{21}$$

By (15), (19) and (20), we have

$$b_3 = 0, \qquad d_2 = -id_1. \tag{22}$$

From (19) and (20), $a_1 \neq 0$. For convenient, let $a_1 = 1$. Now, (15) can be represented as

$$\begin{aligned} &2(e_1 - ie_2) + c_3^2 = 0, \\ &b_1(e_1 - ie_2) + c_3 d_3 = 0, \\ &2c_3 e_3 + d_3^2 = 0, \\ &d_1(e_1 - ie_2) + d_3 e_3 = 0, \\ &e_1^2 + e_2^2 + e_3^2 = 0. \end{aligned} \tag{23}$$

There are seven variables in (23), we want to express five of them by the other two ones. Let c_3, d_3 be the uncertain ones. With the aid of computer, we can give a solution to (23), and all of them are rational forms.

$$b_1 = \frac{2d_3}{c_3},\ d_1 = -\frac{d_3^3}{c_3^3},\ e_1 = \frac{-c_3^6 + d_3^4}{4c_3^4},\ e_2 = -\frac{i(c_3^6 + d_3^4)}{4c_3^4},\ e_3 = -\frac{d_3^2}{2c_3}. \tag{24}$$

From (17), the Weierstrass data of X can be represented by

$$g(z) = -\frac{2z^2}{c_3} - \frac{2d_3 z}{c_3^2} + \frac{d_3^2}{c_3^3}, \quad \eta(z) = -\frac{c_3^2}{2z^3}\mathrm{d}z. \tag{25}$$

Since $g(z)$ and $\eta(z)$ have poles at 0 and ∞, we can easily conclude that X is complete. As we know, X is regular if and only if $\{\alpha_k, k = 1, 2, 3\}$ have no real periods on M. From Lemma 2, we know that X is regular if and only if $c_3 \in \mathbb{R}$, which has no relations with d_3.

Now, with different choice of $c_3 \in \mathbb{R}$, $c_3 \neq 0$, and $d_3 \in \mathbb{C}$, we have a family of complete oriented minimal immersion $X = (x_1, x_2, x_3) : \mathbb{C}\backslash\{\infty\} \to \mathbb{R}^3$, where

$$x_k = \mathrm{Re}\int_{p_0}^{p} \phi_k(z)\mathrm{d}z, \quad k = 1, 2, 3,$$

and

$$\phi_1 = 2t + z - \frac{t^3}{z^2} + \frac{t^4 - s^2}{4z^3}, \quad i\phi_2 = 2t + z - \frac{t^3}{z^2} + \frac{t^4 + s^2}{4z^3}, \quad \phi_3 = \frac{s}{z} + \frac{st}{z^2} - \frac{st^2}{2z^3}, \tag{26}$$

where $s = c_3$, $t = d_3/c_3$.

It is easy to verify that X has two ends, and the winding order of each end is 2. Therefore we got $\mathcal{M}_2$.

Let $z = u + iv$, where $u,\ v \in \mathbb{R}$. From (3) and (4), with direct calculation, we have

$$\mathrm{d}s^2 = \frac{\lambda^2}{4^2(u^2+v^2)^3}(\mathrm{d}u^2 + \mathrm{d}v^2), \tag{27}$$

$$K = \frac{4^5 s^2}{-\lambda^4}(u^2+v^2)^3(u^2+v^2+tu+t^2/4), \tag{28}$$

where $\lambda = 4(u^2+v^2)^2 + 8tu(u^2+v^2) + 8t^2v^2 - 4t^3u + s^2 + t^4$.

Observing Figure 2 carefully, we find that it looks like the catenoid. In fact, Figure 5 is generated from Figure 2 via symmetrization, and we have the following result.

Proposition 1 *Considering the surface $\mathcal{M}_2$. If d_3 is a real or a purely imaginary number, then $\mathcal{M}_2$ is symmetric with the symmetry plane $x_2 = c \in \mathbb{R}$, where c is a constant.*

Proof Let $z = u + iv$. From (26), with direct calculation, we have

Case 1 d_3 is a real number. It follows that

$$\begin{aligned}
x_1 &= \Big(\frac{1}{2} + \frac{s^2 - t^4}{8(u^2+v^2)^2}\Big)(u^2 - v^2) + u\Big(2t + \frac{t^3}{u^2+v^2}\Big) + C_1,\\
x_2 &= \Big(1 + \frac{s^2 + 4t^4}{4(u^2+v^2)^2}\Big)uv + v\Big(2t - \frac{t^3}{u^2+v^2}\Big) + C_2,\\
x_3 &= \frac{s}{2}\log(u^2+v^2) + \frac{st^2(u^2-v^2)}{4(u^2+v^2)^2} - \frac{stu}{u^2+v^2} + C_3,
\end{aligned}$$

where $C_k \in \mathbb{R}$, $k = 1, 2, 3$, are constant numbers.

Case 2 d_3 is a purely imaginary number. It follows that

$$\begin{aligned}
x_1 &= \Big(\frac{1}{2} + \frac{s^2 - t^4}{8(u^2+v^2)^2}\Big)(u^2 - v^2) + iv\Big(2t - \frac{t^3}{u^2+v^2}\Big) + C_4,\\
x_2 &= \Big(1 + \frac{s^2 + 4t^4}{4(u^2+v^2)^2}\Big)uv - iu\Big(2t + \frac{t^3}{u^2+v^2}\Big) + C_5,\\
x_3 &= \frac{s}{2}\log(u^2+v^2) + \frac{st^2(u^2-v^2)}{4(u^2+v^2)^2} + \frac{istv}{u^2+v^2} + C_6,
\end{aligned}$$

where $C_i \in \mathbb{R}$, $i = 4, 5, 6$, are constant numbers. Clearly, in the first case, we have

$$\begin{aligned}
&x_1(u+iv) = x_1(u-iv),\\
&x_2(u+iv) + x_2(u-iv) = 2C_2,\\
&x_3(u+iv) = x_3(u-iv)
\end{aligned}$$

and in the second case, we have

$$x_1(u+iv) = x_1(-u+iv),$$
$$x_2(u+iv) + x_2(-u+iv) = 2C_4,$$
$$x_3(u+iv) = x_3(-u+iv).$$

So the symmetry plane of $\mathcal{M}_2$ is $x_2 = C_2$ (or C_4). This proved the proposition. □

Obviously, $\mathcal{M}_2$ is symmetric (See Figures 4–6), and the ends of $\mathcal{M}_2$ is different from Enneper type ones.

4. The Weierstrass data of $\mathcal{M}_n (n \geqslant 1)$

With the same method of construction as $\mathcal{M}_2$, we can get the Weierstrass data of $\mathcal{M}_n$ easily, $1 \le n \le 5$. We just list them below, where $a \in \mathbb{R}$, $a \neq 0$, $b_k \in \mathbb{C}$ for $1 \le k \le 4$.

	η	$g(z)$
$\mathcal{M}_1$	$-\frac{a^2}{2z^2}\mathrm{d}z$	$-\frac{2}{a}z$
$\mathcal{M}_2$	$-\frac{a^2}{2z^3}\mathrm{d}z$	$-\frac{2}{a}z^2 - \frac{2b_1}{a^2}z + \frac{b_1^2}{a^3}$
$\mathcal{M}_3$	$-\frac{a^2}{2z^4}\mathrm{d}z$	$-\frac{2}{a}z^3 - \frac{2b_2}{a^2}z^2 - \frac{2b_1}{a^2}z + \frac{2b_1b_2}{a^3}$
$\mathcal{M}_4$	$-\frac{a^2}{2z^5}\mathrm{d}z$	$-\frac{2}{a}z^4 - \frac{2b_3}{a^2}z^3 - \frac{2b_2}{a^2}z^2 - \frac{2b_1}{a^2}z + \frac{2b_1b_3+b_2^2}{a^3}$
$\mathcal{M}_5$	$-\frac{a^2}{2z^6}\mathrm{d}z$	$-\frac{2}{a}z^5 - \frac{2b_4}{a^2}z^4 - \frac{2b_3}{a^2}z^3 - \frac{2b_2}{a^2}z^2 - \frac{2b_1}{a^2}z + \frac{2b_1b_4+2b_2b_3}{a^3}$

Table 1 The Weierstrass data of $\mathcal{M}_n$, $n = 1, 2, 3, 4, 5$

As we have seen, for $\mathcal{M}_n$, the number of uncertain variables is n. In fact, much like in (19), the number of equations is $2n+1$, but with $3n+1$ variables. As a sequence, we have

Proposition 2 $\mathcal{M}_n (n > 1)$ *has the following Weierstrass data:*

$$\begin{cases} \eta = -\dfrac{a^2}{2z^{n+1}}\mathrm{d}z, \\ g(z) = -\dfrac{2}{a}z^n - \displaystyle\sum_{j=1}^{n-1}\frac{2b_j}{a^2}z^j + \frac{1}{a^3}\sum_{j=1}^{n-1}b_jb_{n-j} \end{cases} \tag{29}$$

where $a \in \mathbb{R}$, $a \neq 0$, and $b_1, b_2, \ldots, b_{n-1} \in \mathbb{C}$, and the total curvature of $\mathcal{M}_n$ is $-4n\pi$.

Proof The information about the poles and zeros of $g(z)$ and η is as follows:

z	0	∞
$g(z)$	0^n	∞^n
η	∞^{n+1}	0^{n-1}

Table 2 The degree of poles and zeros of $g(z)$ and η

So we can take $M=\mathbb{C}\backslash\{0\}$.

From (1) and (29), with direct calculation, we get the following:

	$\cdots$	z^{n-1}	$\cdots$	z^{-1}	$\cdots$	$z^{-(n+1)}$	$\cdots$
ϕ_1	0	1	$\cdots$	0	$\cdots$	$-\frac{a^2}{4}\Big(1-(\frac{1}{a^3}\sum_{j=1}^{n-1}b_jb_{n-j})^2\Big)$	0
ϕ_2	0	$-i$	$\cdots$	0	$\cdots$	$-\frac{ia^2}{4}\Big(1+(\frac{1}{a^3}\sum_{j=1}^{n-1}b_jb_{n-j})^2\Big)$	0
ϕ_3	0	0	$\cdots$	a	$\cdots$	$-\frac{1}{2a}\sum_{j=1}^{n-1}b_jb_{n-j}$	0

Table 3 The related coefficients of ϕ_k, $k=1,2,3$.

We want to apply Theorem 2 to obtain a minimal immersion

$$X=(x_1,x_2,x_3):M\to\mathbb{R}^3$$

where

$$x_k=\operatorname{Re}\int_{z_0}^{z}\phi_k\mathrm{d}z,\quad k=1,2,3.$$

For this purpose, we must show that $\phi_k\mathrm{d}z$ $(k=1,2,3)$ has no real periods on M. As we seen in Table 3,

$$\operatorname{Re}s_{z=0}(\phi_k)=-\operatorname{Re}s_{z=\infty}(\phi_k)\in\mathbb{R},\quad k=1,2,3.$$

It is equivalent to $\phi_k\mathrm{d}z$ have no real periods on M. Obviously, at least one of

$$-\frac{a^2}{4}\Big(1-(\frac{1}{a^3}\sum_{j=1}^{n-1}b_jb_{n-j})^2\Big)\quad\text{and}\quad-\frac{ia^2}{4}\Big(1+(\frac{1}{a^3}\sum_{j=1}^{n-1}b_jb_{n-j})^2\Big)$$

is non-zero number, so that

$$\max\{\mathrm{Ord}_0\phi_k\mathrm{d}z\}=\max\{\mathrm{Ord}_\infty\phi_k\mathrm{d}z\}=n+1,\quad k=1,2,3.$$

From Lemma 1 we know that the winding order of the two ends are both n.

Since $g(z)$ and η have poles at 0 and ∞, we easily conclude that X is complete.

Now we know that X is $\mathcal{M}_n$, and from (8), the total curvature of $\mathcal{M}_n$ is

$$C(\mathcal{M}_n)=2\pi(0-\sum_{i=1}^{2}n)=-4n\pi.$$

This proved the proposition. □

Now, summarize the above result, we have

Therorem 5 *There exists a family of complete oriented minimal surfaces in $\mathbb{R}^3$ with finite total curvature $-4n\pi$, each of which has 0 genus and two ends, and both of the ends have winding order n, where $n\in\mathbb{N}$.*

Remark 2 Note that the form of ϕ_k, the coefficient of η can be reduced to 1, and with change of parameters, the Weierstrass date of $\mathcal{M}_n$ can be represented by

$$\eta = \frac{1}{z^{n+1}}\mathrm{d}z, \qquad g(z) = az^n + \sum_{j=1}^{n-1} b_j z^j - \frac{1}{2a}\sum_{j=1}^{n-1} b_j b_{n-j} \tag{30}$$

where $a \in \mathbb{R}$, $a \neq 0$, $b_1, b_2, \ldots, b_{n-1} \in \mathbb{C}$.

Similar to Proposition 1, we have the following

Proposition 3 *Considering the Weierstrass data (30). If $b_k \in \mathbb{R}$, $1 \leqslant k \leqslant n-1$, then $\mathcal{M}_n$ is symmetric with the symmetry plane $x_2 = c \in \mathbb{R}$, where c is a constant.*

Proof Let $z = u + iv$, $f_m(z) = z^m$, $m \in \mathbb{Z}$, and $u, v \in \mathbb{R}$. $f_m(z)$ have the following properties:

$$\mathrm{Re}(f_m(z)) = \begin{cases} \displaystyle\sum_{k=0}^{2k\leq m} C_m^{2k}(-1)^k v^{2k} u^{m-2k}, & m > 0; \\ \displaystyle\frac{1}{(u^2+v^2)^m} \sum_{k=0}^{2k\leq m} C_m^{2k}(-1)^k v^{2k} u^{m-2k}, & m < 0. \end{cases}$$

$$\mathrm{Im}(f_m(z)) = \begin{cases} \displaystyle\sum_{k=0}^{2k< m} C_m^{2k+1}(-1)^k v^{2k+1} u^{m-2k-1}, & m > 0; \\ \displaystyle\frac{-1}{(u^2+v^2)^m} \sum_{k=0}^{2k< m} C_m^{2k}(-1)^k v^{2k} u^{m-2k-1}, & m < 0. \end{cases}$$

It is easy to see that

$$\mathrm{Re}(f_m(u+iv)) = \mathrm{Re}(f_m(u-iv)), \quad \mathrm{Re}(f_m(u+iv)) + \mathrm{Re}(f_m(u-iv)) = 0.$$

From (1) and (30), after direct calculation, we know that x_k's have the following form:

$$\begin{aligned} x_1 &= \sum_{m=-n}^{n} r_{1m}\mathrm{Re}(f_m(z)) + C_1, \\ x_2 &= \sum_{m=-n}^{n} r_{2m}\mathrm{Im}(f_m(z)) + C_2, \\ x_3 &= \sum_{m=-n}^{-1} r_{3m}\mathrm{Re}(f_m(z)) + \log\sqrt{u^2+v^2} + C_3, \end{aligned}$$

where the coefficients of $\mathrm{Re}(f_m(z))$ and $\mathrm{Im}(f_m(z))$ are all real numbers, and C_k, $k = 1, 2, 3$, are constant numbers. Using the properties of $f_m(z)$, we have

$$x_1(u,v) = x_1(u,-v), \quad x_2(u,v) + x_2(u,-v) = 2C_2, \quad x_3(u,v) = x_3(u,-v).$$

Now we can see, the symmetry plane of $\mathcal{M}_n$ is $x_2 = C_2$. This proved the proposition. □

Remark 3 In general, the image of $\mathcal{M}_n (n > 2)$ is not symmetric (See Figure 3 (a)).

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