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Generalized Path Algebras and Pointed Hopf Algebras

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Abstract Most of pointed Hopf algebras of dimension p^m with large coradical are shown to be generalized path algebras. By the theory of generalized path algebras, the representations, homological dimensions and radicals of these Hopf algebras are obtained. The relations between the radicals of path algebras and connectivity of directed graphs are given.

Keywords generalized path algebra; Hopf algebra; radical.

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1. Introduction

The concept of generalized path algebras was introduced by Coelho and Liu in [1]. Their structures and representations were studied by the authors in [2].

Pointed Hopf algebras of dimension p^m with large coradical were classified[3]. In this paper, using the theory of generalized path algebras we give the representations, homological dimensions and radicals of these Hopf algebras. In Section 1, we show that pointed Hopf algebra $H(C, n, c, c^*)$ is a generalized path algebra with relations and it is also a smash product. The left global dimensions of Taft algebras are infinity. The radicals and representations of $H(C, n, c, c^*)$ can be obtained by the theory of generalized path algebras in [2]. In Section 2, we give the relations between radicals of generalized matrix rings and Γ-rings. We obtain the explicit formulas for generalized matrix ring A and radical properties $r = r_b, r_l, r_j, r_n$:

$$r(A) = g.m.r(A) = \sum\{r(A_{ij}) \mid i, j \in I\}.$$

In Section 3, we give the relations between the radicals of path algebras and connectivity of directed graphs. That is, we obtain that every weak component of directed path D is a strong component if and only if every unilateral component of D is a strong component if and only if the Jacobson radical of path algebra $A(D)$ is zero if and only if the Baer radical of path algebra $A(D)$ is zero.

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2. Preliminaries

Let k be a field. We first recall the concepts of Γ_I-systems, generalized matrix rings (algebras) and generalized path algebras. Let I be a non-empty set. If for any $i, j, l, s \in I, A_{ij}$ is an additive group and there exists a map μ_{ijl} from $A_{ij} \times A_{jl}$ to A_{il} (written $\mu_{ijl}(x, y) = xy$) such that the following conditions hold:

(i) $(x + y)z = xz + yz$, $w(x + y) = wx + wy$;

(ii) $w(xz) = (wx)z$,

for any $x, y \in A_{ij}$, $z \in A_{jl}, w \in A_{li}$, then the set $\{A_{ij} \mid i, j \in I\}$ is a Γ_I-system with index I.

Let A be the external direct sum of $\{A_{ij} \mid i, j \in I\}$. We define the multiplication in A as

$$xy = \{\sum_k x_{ik}y_{kj}\}$$

for any $x = \{x_{ij}\}, y = \{y_{ij}\} \in A$. It is easy to check that A is a ring (possibly without the unity element). We call A a generalized matrix ring, or a g.m. ring for short, written as $A = \sum\{A_{ij} \mid i, j \in I\}$. For any non-empty subset S of A and $i, j \in I$, set $S_{ij} = \{a \in A_{ij} \mid$ there exists $x \in S$ such that $x_{ij} = a\}$. If B is an ideal of A and $B = \sum\{B_{ij} \mid i, j \in I\}$, then B is called a g.m. ideal. If for any $i, j \in I$, there exists $0 \neq e_{ii} \in A_{ii}$ such that $x_{ij}e_{jj} = e_{ii}x_{ij} = x_{ij}$ for any $x_{ij} \in A_{ij}$, then the set $\{e_{ii} \mid i \in I\}$ is called a generalized matrix unit of Γ_I-system $\{A_{ij} \mid i, j \in I\}$, or a generalized matrix unit of g.m. ring $A = \sum\{A_{ij} \mid i, j \in I\}$, or a g.m. unit for short. It is easy to show that if A has a g.m. unit $\{e_{ii} \mid i \in I\}$, then every ideal B of A is a g.m. ideal. Indeed, for any $x = \sum_{i,j\in I} x_{ij} \in B$ and $i_0, j_0 \in I$, since $e_{i_0i_0}xe_{j_0j_0} = x_{i_0j_0} \in B$, we have $B_{i_0j_0} \subseteq B$. Furthermore, if B is a g.m. ideal of A, then $\{A_{ij}/B_{ij} \mid i, j \in I\}$ is a Γ_I-system and $A/B \cong \sum\{A_{ij}/B_{ij} \mid i, j \in I\}$ as rings.

If for any $i, j, l, s \in I, A_{ij}$ is a vector space over field k and there exists a k-linear map μ_{ijl} from $A_{ij} \otimes A_{jl}$ into A_{il} (written $\mu_{ijl}(x, y) = xy$) such that $x(yz) = (xy)z$ for any $x \in A_{ij}$, $y \in A_{jl}, z \in A_{ls}$, then the set $\{A_{ij} \mid i, j \in I\}$ is a Γ_I- system with index I over field k. Similarly, we get an algebra $A = \sum\{A_{ij} \mid i, j \in I\}$, called a generalized matrix algebra, or a g.m. algebra for short.

Assume that D is a directed (or oriented) graph (D is possibly an infinite directed graph and also possibly not a simple graph) (or quiver). Let $I = D_0$ denote the vertex set of D and D_1 denote the set of arrows of D. Let Ω be a generalized matrix algebra over field k with g.m. unit $\{e_{ii} \mid i \in I\}$, the Jacobson radical $r(\Omega_{ii})$ of Ω_{ii} is zero and $\Omega_{ij} = 0$ for any $i \neq j \in I$. The sequence $x = a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2}a_{i_2}x_{i_2i_3}\cdots x_{i_{n-1}i_n}a_{i_n}$ is called a generalized path (or Ω-path) from i_0 to i_n via arrows $x_{i_0i_1}, x_{i_1i_2}, x_{i_2i_3}, \ldots, x_{i_{n-1}i_n}$, where $0 \neq a_{i_p} \in \Omega_{i_pi_p}$ for $p = 0, 1, 2, \ldots, n$. In this case, n is called the length of x, written as $l(x)$. For two Ω-paths $x = a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2}a_{i_2}x_{i_2i_3}\cdots x_{i_{n-1}i_n}a_{i_n}$ and $y = b_{j_0}y_{j_0j_1}b_{j_1}y_{j_1j_2}b_{j_2}y_{j_2j_3}\cdots y_{j_{m-1}j_m}b_{j_m}$ of D with $i_n = j_0$, we define the multiplication of x and y as

$$xy = a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2}a_{i_2}x_{i_2i_3}\cdots x_{i_{n-1}i_n}(a_{i_n}b_{j_0})y_{j_0j_1}y_{j_1j_2}b_{j_1}y_{j_2j_3}\cdots y_{j_{m-1}j_m}b_{j_m}. \qquad (*)$$

For any $i, j \in I$, let A'_{ij} denote the vector space over field k with basis being all Ω-paths from i

to j with length > 0. B_{ij} is the sub-space spanned by all elements of forms:

$$a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2}a_{i_2}\cdots x_{i_{s-1}i_s}(a_{i_s}^{(1)}+a_{i_s}^{(2)}+\cdots+a_{i_s}^{(m)})x_{i_si_{s+1}}\cdots x_{i_{n-1}i_n}a_{i_n}-$$
$$\sum_{l=1}^{m}a_{i_0}x_{i_0i_1}a_{i_1}x_{i_1i_2}a_{i_2}x_{i_2i_3}\cdots x_{i_{s-1}i_s}a_{i_s}^{(l)}x_{i_si_{s+1}}\cdots x_{i_{n-1}i_n}a_{i_n},$$

where $i_0=i, i_n=j, a_{i_s}^{(l)}\in\Omega_{i_si_s}, a_{i_p}\in\Omega_{i_pi_p}$, $x_{i_ti_{t+1}}$ is an arrow, $p=0,1,\ldots,n$, $t=0,1,\ldots,n-1$, $l=0,1,\ldots,m$, $0\le s\le n$, n and m are natural numbers. Let $A_{ij}=A'_{ij}/B_{ij}$ when $i\neq j$ and $A_{ii}=(A'_{ii}+\Omega_{ii})/B_{ii}$, written $[\alpha]=\alpha+B_{ij}$ for any generalized path α from i to j. We can get a k-linear map from $A_{ij}\otimes A_{jl}$ to A_{il} induced by $(*)$. We write a instead of $[a]$ when $a\in\Omega$. In fact, $[\Omega_{ii}]\cong\Omega_{ii}$ as algebras for any $i\in I$. Notice that we write $e_{ii}x_{ij}=x_{ij}e_{jj}=x_{ij}$ for any arrow x_{ij} from i to j. It is clear that $\{A_{ij}\mid i,j\in I\}$ is a Γ_I-system with g.m. unit $\{e_{ii}\mid i\in I\}$. The g.m. algebra $\sum\{A_{ij}\mid i,j\in I\}$ is called the generalized path algebra, or Ω-path algebra, written as $k(D,\Omega)$ (see, [4, Chapter 3] and [1]). Let J denote the ideal generated by all arrows in D of $k(D,\Omega)$. If ρ is a non-empty subset of $k(D,\Omega)$ and the ideal (ρ) generated by ρ satisfies $J^t\subseteq(\rho)\subseteq J^2$, then $k(D,\Omega)/(\rho)$ is called generalized path algebra with relations. If $J^t\subseteq(\rho)\subseteq J$, then $k(D,\Omega)/(\rho)$ is called generalized path algebra with weak relations. If $\Omega_{ii}=ke_{ii}$ for any $i\in I$, then $k(D,\Omega)$ is called a path algebra, written as kD. For any $i,j\in I$ and $A=kD$, if $u=\sum_{s=1}^{n}k_sp_s\in A_{ij}$, then the length $l(u)$ is defined as the maximal length $l(p_s)$ for $s=1,2,\ldots,n$, where $p_1,p_2,\ldots,p_s$ are different paths in A_{ij}. Furthermore, sometimes, we call $k(D,\Omega)$ a generalized path algebra although the Jacobson radical $r(\Omega)\neq 0$.

2. Application in Hopf algebras

Let C be an abelian group and G a group. Let C^* denote the character group of C, $\mathbf{N}$ the set of natural numbers and $\mathbf{Z}^+$ the set of positive integrals. Assume $c_i\in C$, $c_i^*\in C^*, n=(n_1,n_2,\ldots,n_t)\in(\mathbf{Z}^+)^t$ and $a=(a_1,a_2,\ldots,a_t)\in\{0,1\}^t$; $b_{ij}\in k$ for $i,j=1,2,\ldots,t$. Throughout this section, D denotes the following quiver (or directed graph): vertex set D_0 has only one element and arrow set $D_1=\{X_1,X_2,\ldots,X_t\}$. $\Omega=kC$.

Let $k_t=k\{X_1,X_2,\ldots,X_t\}/(\rho)$ with $\rho=\{X_jX_i-c_j^*(c_i)X_iX_j\mid i,j=1,2,\ldots,t \text{ and } i\neq j\}$. That is, k_t is the path algebra $k(D,\rho)$ with relation ρ.

We first recall $A_t(C,c,c^*,a,b)$ and $H(C,n,c,c^*,a,b)$, which was defined in [3, Definitions 5.6.8 and 5.6.15].

Definition 2.1 *$A_t(C,c,c^*,a,b)$,A_t for short,is called the Hopf algebra generated by the element $g\in C$ and $X_j, j=1,2,\ldots,t$, where*

(i) the elements of C are commuting grouplikes;

(ii) X_j is $(1,c_j)$-primitive;

(iii) $x_jg=c_j^*(g)gx_j$;

(iv) $X_jX_i=c_j^*(c_i)X_iX_j+b_{ij}(c_ic_j-1)$ for $i,j=1,2,\ldots,t$ and $i\neq j$;

(v) $c_i^*(c_j)c_j^*(c_i)=1$ for $j\neq i$;

(vi) If $b_{ij}\neq 0$, then $c_i^*c_j^*=1$;

(vii) *If* $c_i c_j = 1$, *then* $b_{ij} = 0$.

The antipode of A_t *is given by* $S(g) = g^{-1}$ *for* $g \in G$ *and* $S(X_j) = -c_j^{-1} X_j$.

Moreover,

(viii) $c_i^*(c_i)$ *is a primitive* n_i*-th root of unity for any* i;

(ix) *If* $a_i = 1$, *then* $(C_i^*)^{n_i} = 1$;

(x) *If* $(c_i)^{n_i} = 1$, *then* $a_i = 0$;

(xi) $b_{ij} = -c_i^*(c_j) b_{ji}$ *for any* i, j.

Let Hopf algebra $H(C, n, c, c^*, a, b) = A_t / J(a)$ *where* $J(a)$ *is an ideal of* A_t, *generated by*

$$\{X_1^{n_1} - a_1(c_1^{n_1} - 1), X_2^{n_2} - a_2(c_2^{n_2} - 1), \ldots, X_t^{n_t} - a_t(c_t^{n_t} - 1)\}.$$

If $a = 0$ and $b = 0$, we denote $A_t(C, c, c^*, a, b)$ and $H(C, n, c, c^*, a, b)$ by $A_t(C, c, c^*)$ and $H(C, n, c, c^*)$, respectively. If C is a cyclic group generated by c with order n and $t = 1$, then $H(C, n, c, c^*)$ is called a Taft algebra, written as $H_{n^2}(\lambda)$, where $\lambda = c^*(c)$.

Since we only consider the algebra structures of $A_t(C, c, c^*, a, b)$ and $H(C, n, c, c^*, a, b)$ in this section, we use the two signs "$\cong$" and "$=$" to denote isomorphism and equation as algebras, respectively.

Theorem 2.2 Under the notations above we have the following:

(i) $A_t(C, c, c^*) = k_t \# kC$.

(ii) $A_t(C, c, c^*) = k(D, \Omega)/(\rho)$, *with* $\rho = \{X_j X_i - c_j^*(c_i) X_i X_j, X_i h - c_i^*(h) h X_i \mid h \in C, i, j = 1, 2, \ldots, t$ *and* $i \neq j\}$.

(iii) $H(C, n, c, c^*) = k(D, \rho) \# kC$ *with* $\rho = \{X_j X_i - c_j^*(c_i) X_i X_j, X_i^{n_i} \mid i, j = 1, 2, \ldots, t$ *and* $i \neq j\}$.

(iv) $H(C, n, c, c^*) = k(D, \Omega, \rho)$ *with* $\rho = \{X_i^{n_i}, X_j X_i - c_j^*(c_i) X_i X_j, X_i h - c_i^*(h) h X_i \mid h \in C, i, j = 1, 2, \ldots, t$ *and* $i \neq j\}$.

Proof It is well-known. □

A representation of (D, Ω) is a set $(V, f) =: \{V, f_i \mid V$ is a unitary kC-module, $f_i : V \to V$ is a k-linear map, $i = 1, 2, \ldots, n\}$. A morphism $h : (V, f) \to (V', f')$ between two representations of (D, Ω) is a k-linear map $h : V \to V'$ such that $h f_i = f_j' h$ for $i, j = 1, 2, \ldots, t$. Let $\mathrm{Rep}(D, \Omega)$ denote the category of representations of (D, Ω).

By [2, Theorem 2.9], we have

Corollary 2.3 *Let* $\rho = \{X_j X_i - c_j^*(c_i) X_i X_j, X_i h - C_i^*(h) h X_i \mid i, j = 1, 2, \ldots, t$ *and* $i \neq j\}$. *Then*

(i) $\mathrm{Rep}k(D, \Omega, \rho)$ *and* ${}_{A_t(C,c,c^*)}\mathcal{M}$ *are equivalent.*

(ii) $\mathrm{f.d.Rep}(D, \Omega, \rho)$ *and* $\mathrm{f.d.}_{A_t(C,c,c^*)}\mathcal{M}$ *are equivalent. Here,* $\mathrm{f.d.Rep}(D, \Omega, \rho)$ *and* $\mathrm{f.d.}_{k(D,\Omega,\rho)}\mathcal{M}$ *denote the full subcategories of finite dimensional objects in the corresponding categories, respectively.*

Notice that although $A_t(C, c, c^*) = k(D, \Omega)/(\rho)$ is not a generalized path algebra with weak relations, we may use the conclusion in [2, Theorem 2.9].

Corollary 2.4 Let $\rho = \{X_jX_i - c_j^*(c_i)X_iX_j, x_i^{n_i}, X_ih - c_i^*(h)hX_i \mid i,j = 1,2,\ldots,t$ *and* $i \neq j\}$. *Then*

(i) $\mathrm{Rep}k(D,\Omega,\rho)$ *and* ${}_{H(C,n,c,c^*)}\mathcal{M}$ *are equivalent.*

(ii) $\mathrm{f.d.Rep}(D,\Omega,\rho)$ *and* $\mathrm{f.d.}_{H(C,n,c,c^*)}\mathcal{M}$ *are equivalent.*

Let $lgd(R)$ and $wd(R)$ denote the left global dimension and weak dimension of algebra R, respectively.

Corollary 2.5 *Let k be a field whose characteristic is not divided by the order of finite group C. Let $\rho = \{X_jX_i - c_j^*(c_i)X_iX_j, X_i^{n_i} \mid i,j = 1,2,\ldots,t$ and $i \neq j\}$. Then*

(i) $r_b(A_t(C,c,c^*)) = 0$;

(ii) $lgd(A_t(C,c,c^*)) = lgd(k_t)$;

(iii) $wd(A_t(C,c,c^*)) = wd(k_t)$;

(iv) $lgd(H(C,n,c,c^*)) = lgd(k(D,\rho))$;

(v) $wd(H(C,n,c,c^*)) = wd(k(D,\rho))$.

Proof (i) It is seen that

$$\begin{aligned} r_b(A_t(C,c,c^*)) &= r_b(k_t\#kC) \text{ by Theorem 2.2(i)} \\ &= r_b(k_t)\#kC \ \text{ by [5, Theorem 2.6]} \\ &= 0 \ \text{ since } r_b(k_t) = 0. \end{aligned}$$

(ii)–(v) By Theorem 2.2 (i) (iii), $A_t(C,c,c^*)$ and $H(C,n,c,C^*)$ are smash products. Using [6, Theorem 2.2], we can complete the proof. □

Corollary 2.6 *Let k be a field whose characteristic is not divided by the order of finite group C. Let $\rho = \{X_jX_i - c_j^*(c_i)X_iX_j, X_i^{n_i}, X_ih - C_i^*(h)hX_i \mid i,j = 1,2,\ldots,t$ and $i \neq j\}$. Then*

(i) $r_j(H(C,n,c,c^*)) = J/(\rho)$, *where J is the ideal of $H(C,n,c,c^*)$, generated by $X_i's$;*

(ii) $lgd(H_{n^2}(\lambda)) = \infty$.

Proof (i) The conclusion follows from Theorem 2.2 and [2, Lemma 3.6].

(ii) The Taft algebra is finite dimensional Hopf algebra and hence self-injective, also the Taft algebra is not semi-simple and hence of infinite global dimension. □

If we replace the abelian group C by group G in Definition 2.1, we obtain two algebra $A_t(G,c,c^*,a,b)$ and $H(G,n,c,c^*,a,b)$, where $c_i \in Z(G)$ for $i = 1,2,\ldots,t$. It is clear that Theorem 2.2 and Corollaries 2.3–2.6 hold for $A_t(G,c,c^*,a,b)$ and $H(G,n,c,c^*,a,b)$ since we only consider their algebra structures.

Theorem 2.7 *Let k be an algebraically closed field of characteristic zero and H a pointed Hopf algebra with dimension p^m where p is prime. If $m \leq 3$ or the dimension of the coradical of H is more than p^{m-2}, then H is one of the following:*

(i) *a group algebra;*

(ii) $H(C,n,c,c^*)$;

(iii) $H(C,n,c,c^*,a,b)$;

(iv) $H(G,n,c,c^*)$ *with* $t=1$;

(v) $H(G,n,c,c^*,a,b)$ *with* $t=1$.

Proof We assume H is not a group algebra. If the dimension of the coradical of H is more than p^{m-2}, then $G(H)=G$ is a group of order p^{m-1}. By [3, Theorem 7.8.2], $H=H(G,n,c,c^*)$ with $t=1, n=p, c=g\in Z(G)$, or $H=H(G,n,c,c^*,a,0)$ with $t=1, n=p, c=g\in Z(G)$ and $a=1$.

If H is a Hopf algebra of dimension p^3 with $G(H)=(g)$ a cyclic group of order p, then it follows from [3, Theorem 7.9.6] that $H\cong H(C,n,c,c^*)$ with $C=(g)$ of order p, $t=2$, $n=(p,p), c=(g,g^i), c^*=(c^*,(c^*)^{-i})$ for $1\le i\le p-1$, or $H\cong H(C,n,c,c^*,a,b)$ with $C=(g)$ of order p, $t=2$, $n=(p,p), c=(g,g), c^*=(c^*,(c^*)^{-1}), a=0, b_{12}=1$. $\square$

Example 2.8 Recall the duality theorem (see [3, Corollary 6.5.6 and Theorem 6.5.11]) for co-Frobenius Hopf algebra H:

$$(R\#H^{*rat})\#H\cong M_H^f(R) \text{ and } (R\#H)\#H^{*rat}\cong M_H^f(R) \quad \text{(as algebras)},$$

where $M_H^f(R)=\sum\{B_{ij}\mid i,j\in I\}$ is a g.m. algebra and I is the basis of H with $B_{ij}=R$ for $i,j\in I$. Note for the Baer radical, Levitzki radical, Jacobson radical and von Neumann regular radical $r(M_H^f(R))=M_H^f(r(R))$ of $M_H^f(R)$. Consequently, $r((R\#H^{*rat})\#H)\cong M_H^f(r(R))$ and $r((R\#H)\#H^{*rat})\cong M_H^f(r(R))$. In particular, the Heseberg algebra $H\#H^{*rat}\cong M_H^f(k)$ for infinite co-Frobenius Hopf algebra H. Therefore

$$r(H\#H^{*rat})\cong r(M_H^f(k))=M_H^f(r(k))=\begin{cases}0 & \text{when } r=r_b,r_l,r_j\\ M_H^f(k) & \text{when } r=r_n.\end{cases}$$

3. The radicals of generalized matrix rings

Since every generalized path algebra is a generalized matrix ring, we study the radicals of generalized matrix rings in this section.

Let $xE(i,j)$ denote the generalized matrix having a lone x as its (i,j)-entry and all other entries are zero. If B is a non empty subset of generalized matrix ring A and $s,t\in I$, we call the set $\{x\in A_{st}$ —there exists $y\in B$ such that $y_{st}=x\}$ the projection on (s,t) of B, written as B_{st}.

Let g.m.r(A) denote the maximal g.m. ideal of $r(A)$ for a radical property r of rings[8]. Let r_b,r_l,r_k,r_j,r_n denote the Baer radical, Levitzki radical, nil radical, Jacobson radical and von Neumann regular radical of rings and Γ-rings, respectively. Let $r(A_{ij})$ denote r radical of A_{ji}-ring A_{ij} for any $i,j\in I$.

Now we study the von Neumann radical $r_n(A)$ of generalized matrix ring

$$A=\sum\{A_{ij}\mid i,j\in I\}.$$

Definition 3.1 *If for all* $s,t\in I$, *there exists* $0\ne d_{st}\in A_{st}$ *such that* $x_{is}d_{st}\ne 0$ *and* $d_{st}y_{tj}\ne 0$ *for any* $i,j\in I, x_{is}\in A_{is}, y_{tj}\in A_{tj}$, *then we say that* A *has a left g.m. non-zero divisor.*

Similarly, we can define the right g.m. non-zero divisor of A. The “divisor” here is not the well-known one in ring theory.

Lemma 3.2 *(i) If B is an ideal of A, then $\bar{B} = \sum\{B_{ij} \mid i,j \in I\}$ is the g.m. ideal generated by B in A.*

(ii) If D is a g.m. ideal of A and $D \subseteq \sum\{r_n(A_{ij}) \mid i,j \in I\}$, then D is an r_n -g.m. ideal of A.

(iii) Let B_{st} be an r_n-ideal of A_{ts}-ring A_{st} and $D_{ij} = A_{is}B_{st}A_{tj}$ for any $i,j \in I$. If A has left and right g.m. non-zero divisors, then D is an r_n-g.m. ideal of A.

(iv) If A has left and right g.m. non-zero divisors and g.m. $r_n(A) = 0$, then $r_n(A_{ij}) = 0$ for any $i,j \in I$.

Proof (i) It is trivial.

(ii) For any $x \in D$, there exists a finite subset J of I such that $x_{ij} = 0$ for any $i,j \notin J$. Without loss of generality, we can assume that $J = \{1,2,\ldots,n\}$ and $J' = \{1,2,\ldots,n,n+1\} \subseteq I$. Let $J' \times J' = \{(u,v) \mid u,v = 1,2,\ldots,n+1\}$ with the dictionary order. We now show that there exist two sequences $\{y_{t_2t_1} \in A_{t_2t_1} \mid (t_1,t_2) \in J' \times J'\}$ and $\{x^{(t)} \in D \mid (t_1,t_2) \in J' \times J'\}$ with $x^{(1,1)} = x$ and

$$x^{(t+1)} = x^{(t)} - x^{(t)}(y_{(t_2t_1)}E(t_2,t_1))x^{(t)} \tag{1}$$

such that $x_s^{(t)} = 0$ for any $s,t \in J' \times J'$ with $s \prec t$ by induction. Since $x_{11}^{(11)} = x_{11}$ is a von Neumann regular element, there exists $y_{11} \in A_{11}$ such that $x_{11} = x_{11}y_{11}x_{11}$. One sees that $x_{11}^{(12)} = x_{11}^{(11)} - x_{11}^{(11)}y_{11}x_{11}^{(11)} = 0$. For $t = (t_1,t_2) \in B$, we assume that there exists $y_{s_2s_1} \in A_{s_2s_1}$ and $x_s^{(t)} = 0$ for any $s = (s_1,s_2) \prec (t_1,t_2)$. Since $x_{t_1t_2}^{(t_1t_2)}$ is a von Neumann regular element, there exists $y_{t_2t_1} \in A_{t_2t_1}$ such that $x_{t_1t_2}^{(t_1t_2)} = x_{t_1t_2}^{(t_1t_2)}y_{t_2t_1}x_{t_1t_2}^{(t_1t_2)}$. By (1), we have $x_{t_1t_2}^{(t+1)} = 0$. For $s = (s_1,s_2) \prec t = (t_1,t_2)$, we have either $s_1 = t_1$, $s_2 < t_2$ or $s_1 < t_1$. This implies that $(t_1,s_2) \prec (t_1,t_2)$ or $(s_1,t_2) \prec (t_1,t_2)$. Thus $x_{s_1s_2}^{(t+1)} = 0$ by (1). Since $x^{(n,n)+1} = 0 \in r_n(A)$, we have that x is von Neumann regular by [9, Lemma 1].

(iii) Let d_{ij} and d'_{ij} in A_{ij} denote the left and right g.m. non-zero divisors of A for any $i,j \in I$, respectively. For any $x_{ij} \in D_{ij}$, there exists $u_{ts} \in A_{ts}$ such that $d_{st}d_{ti}x_{ij}d'_{js}d'_{st} = d_{st}d_{ti}x_{ij}d'_{js}d'_{st}u_{ts}d_{st}d_{ti}x_{ij}d'_{js}d'_{st}$ and $x_{ij} = x_{ij}d'_{js}d'_{st}u_{ts}d_{st}d_{ti}x_{ij}$ since $d_{st}d_{ti}x_{ij}d'_{js}d'_{st} \in B_{st}$. This implies $D_{ij} \subseteq r_n(A_{ij})$. Considering part (ii), we complete the proof.

(iv) If there exist $s,t \in I$ such that $r_n(A_{st}) \neq 0$, let $B_{st} = r_n(A_{st})$ and $D_{ij} = A_{is}B_{st}A_{tj}$ for any $i,j \in I$. By part (iii), we have that $D = 0$ and $B_{st} = 0$. This is a contradiction. $\Box$

Theorem 3.3 *If A has left and right g.m. non-zero divisors, then $r_n(A) = g.m.r_n(A) = \sum\{r_n(A_{ij}) \mid i,j \in I\}$.*

Proof Let $B = r_n(A)$. For any $i,j \in I$ and $x_{ij} \in B_{ij}$, there exists $y \in B$ such that $y_{ij} = x_{ij}$. Let d_{ii} and d'_{jj} be left and right non-zero divisors in A_{ii} and A_{jj}, respectively. Since $(d_{ii}E(i,i))y(d'_{jj}E(j,j)) = (d_{ii}x_{ij}d'_{jj})E(i,j) \in B$, we have that there exists $z \in B$ such that $(d_{ii}x_{ij}d'_{jj})E(i,j) = (d_{ii}x_{ij}d'_{jj})E(i,j)z(d_{ii}x_{ij}d'_{jj})E(i,j)$. By simple computation, we have $d_{ii}x_{ij}d'_{jj} = (d_{ii}x_{ij}d'_{jj})z_{ji}(d_{ii}x_{ij}d'_{jj})$ and $x_{ij} = x_{ij}d'_{jj}z_{ji}d_{ii}x_{ij}$. Thus x_{ij} is von Neumann regular.

This implies $B_{ij} \subseteq r_n(A_{ij})$ and $r_n(A) \subseteq \sum\{r_n(A_{ij}) \mid i,j \in I\}$.

Let $N = g.m.r_n(A)$. Since $g.m.r_n(A//N) = 0$, we have that A_{ij}/N_{ij} is an r_n-semisimple A_{ji}/N_{ji}-ring for any $i,j \in I$ by Lemma 1.2 (iv). It is clear that A_{ij}/N_{ij} is an r_n-semisimple A_{ji}-ring . This implies $r_n(A_{ij}) \subseteq N_{ij}$ for any $i,j \in I$. Consequently, $\sum\{r_n(A_{ij}) \mid i,j \in I\} \subseteq g.m.r_n(A)$. □

If for any $s \in I$, there exists $u_{ss} \in A_{ss}$ such that $xe_{ss} = x$ for any $i \in I$ and $x \in A_{is}$, then we say that A has a right g.m. unit and u_{ss} is a right g.m. unit in A_{ss}. Similarly, we can define a left g.m. unit of A and g.m. unit of A. In fact, if A has left and right g.m. units, then every ideal of A is a g.m. ideal, so $r_n(A) = g.m.r_n(A) \subseteq \sum\{r_n(A_{ij}) \mid i,j \in I\}$ by the proof of Theorem 3.3.

It is clear that if R is a ring and M is a Γ-ring with $R = M = \Gamma$, then $r_n(R) = r_n(M)$. We also have that $r(R) = r(M)$ for $r = r_b, r_k, r_l, r_j$ (see [10, Theorem 5.2], [11, Theorem 10.1], [8, Theorem 3.3] and [12, Theorem 5.1]).

Theorem 3.4 *Let $r = r_b, r_l, r_j, r_n$. Then*

(i) $r(A) = g.m.r(A) = \sum\{r(A_{ij}) \mid i,j \in I\}$.

(ii) $r(A) = \sum\{r(A_{ii}) \mid i \in I\}$ when $A_{ij} = 0$ for any $i \neq j$, i.e., r radical of the direct sum of rings is equal to the direct sum of r radicals of these rings.

(iii) $r(A)$ is graded by G when the index set I of A is an abelian group G. Moreover the grading is canonical.

Here A has left and right g.m. non-zero divisors when $r = r_n$ in (i), (ii) and (iii).

Proof (i)It is obtained from the following:

$$r_b(A) = g.m.r_b(A) = \sum\{r_b(A_{ij}) \mid i,j \in I\} \text{ (by [8, Theorem 3.7])}$$
$$r_l(A) = g.m.r_l(A) = \sum\{r_l(A_{ij}) \mid i,j \in I\} \quad \text{(by [12, Theorems 1.3 and 2.5])}$$
$$r_j(A) = g.m.r_j(A) = \sum\{r_j(A_{ij}) \mid i,j \in I\} \quad \text{(by [12, Theorems 3.10 and 1.3]).}$$

(ii) Since the radicals of ring A_{ii} and A_{ii}-ring A_{ii} are the same, we have (ii).

(iii) The conclusion follows from (i) and [2, Lemma 2.1]. □

Let $M_I^f(R)$ denote the generalized matrix ring $A = \sum\{A_{ij} \mid A_{ij} = R, i,j \in I\}$ with infinite index set I, which is called an infinite matrix ring over ring R. In this case, $M_I^f(k)$ is called an infinite matrix algebra over field k. Let $M_{m\times n}(R)$ denote the ring of all $(m \times n)$ matrices over ring R.

Example 3.5 (i) Let $V = \sum_{g\in G} \oplus V_g$ be a vector space over field k graded by abelian group G with $\dim V_g = n_g < \infty$. Let I denote a basis of V. Then $\sum\{A_{ij} \mid A_{ij} = \mathrm{Hom}(V_j, V_i), i,j \in G\} = \sum\{A_{ij} \mid A_{ij} = M_{n_i\times n_j}(k), i,j \in G\}$ as generalized matrix algebras. However, $\sum\{A_{ij} \mid A_{ij} = \mathrm{Hom}(V_j, V_i), i,j \in G\} = \sum\{A_{ij} \mid A_{ij} = M_{n_i\times n_j}(k), i,j \in G\} = \{f \in \mathrm{End}_k V \mid \ker f$ has finite codimension $\} = M_I^f(k) = \sum\{A_{ij} \mid A_{ij} = k, i,j \in I\}$ as algebras. Then $r(M_I^f(k)) = \sum\{r(A_{ij}) \mid A_{ij} = k, i,j \in I\} = r(\{f \in \mathrm{End}_k V \mid \ker f$ has finite codimension $\}) = 0$ for $r = r_b, r_l, r_j$. It is clear that generalized matrix algebra $A = \sum\{A_{ij} \mid A_{ij} = \mathrm{Hom}(V_j, V_i), i,j \in G\}$ has left and right g.m. non-zero divisors if and only if $n_i = n_j$ for any $i,j \in G$. Consequently, $r_n(M_I^f(k)) = \sum\{r_n(A_{ij}) \mid A_{ij} = k, i, j \in I\} = M_I^f(k)$. That is, $\{f \in \mathrm{End}_k V \mid \ker f$ has finite codimension $\}$ is a von Neumann regular algebra.

(ii) By Theorem 3.4, $r(M_I^f(R)) = M_I^f(r(R))$ for $r = r_b, r_l, r_j$. If R is a ring with left and right non-zero divisors, then $r_n(M_I^f(R)) = M_I^f(r_n(R))$. Obviously, if R has left and right units, then R is a ring with left and right non-zero divisors ($R \neq 0$), so $r_n(M_I^f(R)) = M_I^f(r_n(R))$.

4. Application in path algebras

Lemma 4.1 *Let r denote r_b, r_k, r_1, r_j and $s, t \in I$. If $A_{st} \neq 0$, then*

(i) $r(A_{st}) = 0$ if and only if $A_{ts} \neq 0$.

(ii) $r(A_{st}) = A_{st}$ if and only if $A_{ts} = 0$.

Proof If $r(A_{st}) = 0$, then $r_b(A_{st}) = 0$ and $A_{ts} \neq 0$. Conversely, if $A_{ts} \neq 0$ and $r_j(A_{st}) \neq 0$, then $r_j(A_{st}) = k$ or there exists $y \in r_j(A_{st})$ with $l(y) > 0$. Since y is a right quasi-regular element of A_{ts}-ring A_{st}, for $x \in A_{ts}$, there exists $u \in A_{ts}A_{st}$ such that

$$y(xy)u + y(xy) = -yu. \tag{2}$$

If $l(u) > 0$, then the right hand side of (2) is shorter than the left hand side of (2), we get a contradiction. If $l(u) = 0$, then the left hand side of (2) is either equal to zero, or longer than the right hand side of (2). We get a contradiction. This implies that $r_j(A_{st}) = 0$ and $r(A_{st}) = 0$. □

Lemma 4.2 *$r_n(A_{st}) = 0$ for any $s \neq t$.*

Proof For any $0 \neq x \in A_{st}$, if x is a von Neumann regular element, then there exists $y \in A_{ts}$ such that $x = xyx$. Considering the length of both sides we get a contradiction. Consequently, $r_n(A_{st}) = 0$. □

Theorem 4.3 *(i) As radicals, $r(A) = g.m.r(A) = \sum\{r(A_{ij}) \mid i, j \in I\} = kR(D)$, where r denotes r_b, r_1, r_k and r_j.*

(ii) $r_n(A) = g.m.r_n(A) = \oplus\{N_{ii} \mid i \in I\}$, where

$$N_{ii} = \begin{cases} ke_{ii} = r_n(A_{ii}), & \text{when } A_{si} = A_{is} = 0 \text{ for any } s \in I \text{ with } i \neq s. \\ 0, & \text{otherwise}. \end{cases}$$

Proof (i) By Lemma 4.1, $\sum\{r(A_{ij}) \mid i, j \in I\} \subseteq kR(D)$. Let $x \in kR(D)$ be a regular path from i to j. Then $A_{ij} \neq 0$ and $A_{ji} = 0$, which implies $r(A_{ij}) = A_{ij}$ and $x \in r(A_{ij})$. This has proved $\sum\{r(A_{ij}) \mid i, j \in I\} = kR(D)$. It follows that $r_b(A) = r_j(A) = kR(D)$ from [13, Proposition 5] or Theorem 3.4. Thus $r(A) = kR(D)$. By [12, Theorem 1.3] or Theorem 3.4, $r(A) = g.m.r(A)$. We complete the proof.

(ii) Since A has a g.m. unit $e_{ss} \in A_{ss}$ for any $s \in I$, we have that $r_n(A) = g.m.r_n(A)$. By Lemma 4.2 and the proof of Theorem 3.3, we have $r_n(A) = g.m.r_n(A) \subseteq \sum\{r_n(A_{ij}) \mid i, j \in I\} = \oplus\{r_n(A_{ii}) \mid i \in I\}$. Let $N = r_n(A)$. If $N_{ss} \neq 0$, then $N_{ss} = r_n(A_{ss}) = ke_{ss}$ by the proof of Lemma 4.2. For any $t \in I$ with $t \neq s$, since $A_{ts}N_{ss} = A_{ts} \subseteq N_{ts} = 0$, we have $A_{ts} = 0$. Similarly, $A_{st} = 0$. □

Next we give the relations between the radicals of path algebras and connectivity of directed graphs.

Theorem 4.4 *Directed graph D is strong connected if and only if $A = A(D)$ is a prime algebra.*

Proof If D is strong connected and A is not prime, then there exist $u, v, s, t \in I$ and $0 \neq x \in A_{uv}, 0 \neq y \in A_{st}$ such that $xAy = 0$, i.e., $xA_{vs}y = 0$, which contradicts the strong connectivity of D. Consequently, A is prime. Conversely, if A is prime, then $e_{ii}A_{ij}e_{jj} \neq 0$ for any $i, j \in I$, which implies that D is strong connected. □

Theorem 4.5 *Every weak component of D has at least two vertexes if and only if $r_n(A) = 0$.*

Proof The conclusion follows from Theorem 4.3 (ii). □

Lemma 4.6 *Every directed graph D is the union of all of its unilateral components.*

Proof For any path $x \in A_{st}$, set

$$\mathcal{K} = \{E \mid E \text{ is a unilateral connected subgraph of } D \text{ with } x \in E\}.$$

By Zorn's Lemma, we have that there exists a maximal Q in $\mathcal{K}$. □

Theorem 4.7 *Let r denote r_b, r_k, r_l and r_j, respectively. The following conditions are equivalent.*

(i) Every weak component of D is a strong component.
(ii) Every unilateral component of D is a strong component.
(iii) Weak component, unilateral component and strong component of D are the same.
(iv) D is the union of strong components of D.
(v) D has no regular path.
(vi) $A_{ij} = 0$ if and only if $A_{ji} = 0$ for any $i, j \in I$.
(vii) A is a direct sum of prime algebras.
(viii) A is semiprime.
(ix) A_{ij} is a semiprime A_{ji}-ring for $i, j \in I$.
(x) $r(A_{ij}) = 0$ for any $i, j \in I$.
(xi) $r(A) = 0$.

Proof By Theorem 4.3, (v), (vi), (viii), (ix), (x) and (xi) are equivalent.

(i)⇒(vi) If i and j belong to the same weak component, then $A_{ij} \neq 0$ and $A_{ji} \neq 0$. If i and j do not belong to the same weak component, then obviously $A_{ij} = 0$ and $A_{ji} = 0$.

(ii)⇒(vi) If $A_{ij} \neq 0$, then there exists a path $x \in A_{ij}$. By Lemma 4.6, x belongs to a certain unilateral component of D. Consequently, x belongs to a certain strong component of D. This implies $A_{ji} \neq 0$.

(vi)⇒(ii) If i and j belong to the same unilateral component of D, then $A_{ij} \neq 0$ or $A_{ji} \neq 0$. Consequently $A_{ij} \neq 0$ and $A_{ji} \neq 0$, which implies that i and j belong to the same strong component of D. Therefore (ii) holds.

(iv)$\Rightarrow$(vi) If $A_{ij} \neq 0$, then there exists a path $x \in A_{ij}$ and x belongs to a certain strong component of D. This implies $A_{ji} \neq 0$.

(vi)$\Rightarrow$(iv) For any arrow $x \in A_{ij}$, we only need show that x belongs to a certain strong component of D. By Lemma 4.6, there exists a certain unilateral component C of D such that $x \in C$. Since (ii) and (vi) are equivalent, we have that C is a strong component of D.

(iv)$\Rightarrow$(i) If i and j belong to the same weak component, then there exists a semi-path $x = x_{ii_1}x_{i_1i_2}\ldots, x_{i_nj}$. If i and j belong to different strong components, then we can assume that i_s is the first vertex, which does not belong to the strong component containing i. Consequently, $A_{i_{s-1}i} \neq 0$ and $A_{ii_{s-1}} \neq 0$, and either $A_{ii_s} \neq 0$ or $A_{i_si} \neq 0$. Since (iv) and (vi) are equivalent, we have that $A_{ii_s} \neq 0$ and $A_{i_si} \neq 0$. We get a contradiction. This shows that i and j belong to the same strong components.

(iii)$\Rightarrow$(i) It is obvious.

(i)$\Rightarrow$(iii) Since (i) and (ii) are equivalent, we have (iii).

(vii)$\Rightarrow$(viii) It follows from Theorem 3.4 (ii).

(iv)$\Rightarrow$(vii) Let $\{D^{(\alpha)} \mid \alpha \in \Omega\}$ be all of the strong component of D and $D = \cup\{D^{(\alpha)} \mid \alpha \in \Omega\}$. Thus $A(D) = \oplus\{A(D^{(\alpha)}) \mid \alpha \in \Omega\}$. However, for any $\alpha \in \Omega$, $A(D^{(\alpha)})$ is a prime algebra by Theorem 4.4. We complete the proof. $\square$

We easily obtain the following by the preceding conclusion for $r = r_b, r_l, r_k, r_j$. D has no cycle if and only if $r(A_{ij}) = A_{ij}$ for any $i \neq j \in I$; s and t $(s \neq t)$ are not contained in the same cycle if and only if $r(A_{st}) = A_{st}$; s and t ($s \neq t$) are contained in the same cycle if and only if $r(A_{st}) = 0$.

We give an example to show whether the condition in Theorem 3.3 is a necessary one.

Example 4.8 (i) Let D be a directed graph with vertex set $I = \{1, 2\}$ and only one arrow $x_{12} \in A_{12}$. Obviously, $A_{12} = kx_{12}, A_{11} = ke_{11}, A_{22} = ke_{22}, A_{21} = 0$, $r_n(A_{ii}) = ke_{ii}$ and $r_n(A_{ij}) = 0$ for any $i, j \in I$ with $i \neq j$. By Theorem 4.3 (ii), $r_n(A) = 0 \neq \sum\{r_n(A_{ij}) \mid i, j \in I\}$. It is clear that A has no left g.m. non-zero divisor since $A_{21} = 0$. Consequently, it is possible that Theorem 3.3 does not hold if its condition is dropped.

(ii) Let $I = \{1, 2\}$ and $A_{ij} = M_{i\times j}(k)$ for any $i, j \in I$. It is clear that A has no left g.m. non-zero divisor in A_{12} since, for any non-zero $x \in A_{12}$, there exists a non-zero $y \in A_{21}$ such that $xy = 0$. However, $r_n(A) = \sum\{r_n(A_{ij}) \mid i, j \in I\} = M_{3\times 3}(k)$. Consequently, the condition in Theorem 3.3 is not a necessary condition.

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