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Structure of Two-Groups Associative Rings*

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Introduction. T. S. Ravisanker and U. S. Shukla [1] introduced the notion of a weak Γ_N-ring A, more general than a ring and a Γ-ring in the sense of Nobusawa, and obtained analogical characterizations of the Jacobson radical for weak Γ_N-rings. It is clear that every Γ-ring A is a weak Γ'_N-ring A for some abelian group Γ' (Theorem 3.1 in [2]). Therefore auther gives consideration to such a fact that A and Γ are equal in the weak Γ_N-ring, moreover there is influence each other between A and Γ. And then, we shall rename a weak Γ_N-ring A to a twogroups associative ring (A, Γ). Auther obtained analogical characterizations of the Baer lower nil radical and of the Levitzki radical of the ring (A, Γ) in [3] and [4].

In this paper we shall begin the extension of the notions of modules for binary rings to two-groups assssciative ring, obtaining first some characterizations of primitive two-groups associative rings, and then the classical Chevalley-Jacobson density theorem of binary rings which will be extended to the two-groups associative rings. At the end, defining the Jacobson radical of a two-groups associative ring and getting the analogous results of corresponding part in rings theory.

We refer to [5] for all notions relevant to ring theory. For all other notions to the two-groups associative rings we refer to [3] and [4].

1. Preliminaries. Let A and Γ be additive abelian groups, and for all $a, b, c \in A$ and all $\alpha, \beta, \gamma \in \Gamma$, the following conditions are satisfied.

(1) $a\alpha b \in A$, $\alpha a \beta \in \Gamma$;

(2) $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$,

$(\alpha+\beta)a\gamma = \alpha a\gamma + \beta a\gamma$, $\alpha(a+b)\beta = \alpha a\beta + \alpha b\beta$, $\alpha a(\beta+\gamma) = \alpha a\beta + \alpha a\gamma$;

(3) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$, $(\alpha a\beta)b\gamma = \alpha(a\beta b)\gamma = \alpha a(\beta b\gamma)$.

then we say that A and Γ form a two-groups associative ring, and denoted by ring (A, Γ), or (A, Γ). In [1], the ring (A, Γ) is called a weak Γ_N-ring A. Clearly, A is a Γ-ring and Γ an A-ring, and their right operator rings are

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denoted by R and $\mathfrak{R}$ respectively. If S, S' are subsets of A and $\mathfrak{G}$, $\mathfrak{G}'$ subsets of Γ, we provide:

1. An ordered pair $\langle a, \alpha\rangle \in (S, \mathfrak{G})$ iff $a \in S$, and $\alpha \in \mathfrak{G}$ and we say that $\langle a, \alpha\rangle$ is an element of the set $(S\,\mathfrak{G})$.
2. $(S, \mathfrak{G}) \subseteq (S', \mathfrak{G}')$ iff $S \subseteq S'$ and $\mathfrak{G} \subseteq \mathfrak{G}'$.
3. $(S, \mathfrak{G}) + (S', \mathfrak{G}') = \{\langle a + a', \alpha + \alpha'\rangle \mid \langle a, \alpha\rangle \in (S, \mathfrak{G}),\ \langle a', \alpha'\rangle \in (S', \mathfrak{G}')\}$.
4. $(S, \mathfrak{G}) \cap (S', \mathfrak{G}') = (S \cap S', \mathfrak{G} \cap \mathfrak{G}')$.
5. $S\mathfrak{G}S' = \{\sum_{i\in\Lambda} s_i \alpha_i s'_i \mid s_i \in S,\ \alpha_i \in \mathfrak{G},\ s'_i \in S',\ \Lambda \text{ is finite}\}$,

   $\mathfrak{G}S\mathfrak{G}' = \{\sum_{j\in\Omega} \alpha_j s_j \alpha'_j \mid \alpha_j \in \mathfrak{G},\ s_j \in S,\ \alpha'_j \in \mathfrak{G}',\ \Omega \text{ is finite}\}$.

etc.

Let I and $\mathfrak{I}$ be abelian subgroups of A and Γ respectively. We called $(I, \mathfrak{I})$ is subring of a ring (A, Γ) in case $(I\mathfrak{I}I, \mathfrak{I}I\mathfrak{I}) \subseteq (I, \mathfrak{I})$. A right (left) ideal of a ring (A, Γ) is a subring $(I, \mathfrak{I})$ of ring (A, Γ) such that $(I\Gamma A, \mathfrak{I}A\Gamma)$ $(I, \mathfrak{I})((A\Gamma I, \Gamma A\mathfrak{I}) \subseteq (I, \mathfrak{I}))$. If $(I, \mathfrak{I})$ is both a right and a left ideal such that $(A\mathfrak{I}A, \Gamma I\Gamma) \subseteq (I, \mathfrak{I})$, then we say that $(I, \mathfrak{I})$ is an ideal, or a two-sided ideal of ring (A, Γ).

Theorem 1.1 Let (f, φ) be a homomorphism from a ring (A, Γ) onto the ring (A', Γ'), then the kernel $(K, \mathfrak{K})$ of the homomorphism (f, φ) is an ideal of ring (A, Γ), and difference ring $(A/K, \Gamma/\mathfrak{K}) \cong (A', \Gamma')$. Conversely, if $(K, \mathfrak{K})$ is an ideal of a ring (A, Γ) and if f and φ are natural homomorphisms of the additive group A onto the difference group A/K and of the additive group Γ onto the difference group $\Gamma/\mathfrak{K}$ respectively, then (f, φ) is a homomorphism of the ring (A, Γ) onto the ring $(A/K, \Gamma/\mathfrak{K})$ with kernel $(K, \mathfrak{K})$. (see [3]).

Theorem 1.2 If $(I_i, \mathfrak{I}_i)$ is an ideal of a ring (A, Γ) for $i = 1, 2$ and $(I_1, \mathfrak{I}_1) \supseteq (I_2, \mathfrak{I}_2)$ then

$$(A, \Gamma)/(I_1, \mathfrak{I}_1) \cong [(A, \Gamma)/(I_2, \mathfrak{I}_2)]/[(I_1, \mathfrak{I}_1)/(I_2, \mathfrak{I}_2)]. \text{ (see [3])}.$$

Theorem 1.3 Let $(I, \mathfrak{I})$ be an ideal of a ring (A, Γ). If $(S, \mathfrak{G})$ is a subring of the ring (A, Γ) then $(I, \mathfrak{I}) \cap (S, \mathfrak{G})$ is an ideal of ring $(S, \mathfrak{G})$ and

$$(I + S, \mathfrak{I} + \mathfrak{G})/(I, \mathfrak{I}) \cong (S, \mathfrak{G})/(I, \mathfrak{I}) \cap (S, \mathfrak{G}). \text{ (see [3])}.$$

Theorem 1.4 A ring (A, Γ) is isomorphic to a subdirect sum of the ring (A_i, Γ_i), $i \in \Lambda$ if and only if the ring (A, Γ) contains a class of ideals $(I_i, \mathfrak{I}_i)$ such that $\bigcap_{i\in\Lambda} (I_i, \mathfrak{I}_i) = (0, 0)$ and $(A, \Gamma)/(I_i, \mathfrak{I}_i) \cong (A_i, \Gamma_i)$, $i \in \Lambda$. (see [3]).

2. Primitive two groups associative ring.

Let (A, Γ) be a two-groups associative ring, then an ordered pair $(\mathfrak{M}, M)$ of additive groups $\mathfrak{M}$ and M is said to be a right (A, Γ)-module if there are mappings $\mathfrak{M} \times A \to M$ and $M \times \Gamma \to \mathfrak{M}$ (sending (m', a) and (m, α) to $m'a$ and $m\alpha$ respectively) such that:

(1) $\langle (m_1+m_2)a,\ (m_1'+m_2')a\rangle = \langle m_1a+m_2a,\ m_1'a+m_2'a\rangle$

$\langle m(a+\beta),\ m'(a+b)\rangle = \langle ma+m\beta,\ m'a+m'b\rangle$

(2) $((ma)b)\gamma,\ ((m'a)\beta)c = \langle m(ab\gamma), m'(a\beta c)\rangle$

for all $\langle m', m\rangle$, $\langle m_1', m_2\rangle$, $\langle m_2', m_2\rangle \in (\mathfrak{M}, M)$ and all $\langle a, \alpha\rangle$, $\langle b, \beta\rangle$, $\langle c, \gamma\rangle \in (A, \Gamma)$.

Henceforth the term module without modifier will always mean right module.

Clearly $\mathfrak{M}$ is both the right module of the A-ring Γ and its ri ht operator ring $\mathfrak{R}$, M is both the right module of the Γ-ring A and its right operator ring $\mathfrak{R}$, and denoted by $\mathfrak{M}_\Gamma$, $\mathfrak{M}_{\mathfrak{R}}$, M_A and M_R respectively (see [6]).

A submodule $(\mathfrak{N}, N)$ of an (A, Γ)-module $(\mathfrak{M}, M)$ is an ordered pair $(\mathfrak{N}, N)$ of subgroups of $(\mathfrak{M}, M)$ such that $(N\Gamma, \mathfrak{N}A) \subseteq (\mathfrak{N}, N)$. Clearly $\mathfrak{N}$ and N is a submodule of $\mathfrak{M}_\Gamma(\mathfrak{M}_{\mathfrak{R}})$ and of $M_A(M_R)$ respectively. $(\mathfrak{M}, M)$ is said to be an irreducible (A, Γ)-module if $(M\Gamma, \mathfrak{M}A) \neq (0,0)$ and if there is noproper submodule of $(\mathfrak{M}, M)$ other than $(0,0)$. Evidently $M\Gamma \neq (0)$ and $\mathfrak{M}A \neq (0)$ for the irreducible (A, Γ)-module $(\mathfrak{M}, M)$. Clearly we have

Lemma 2.1 $(\mathfrak{M}, M)$ is an irreducible (A, Γ)-module if and only if $(m\Gamma, m'A) = (\mathfrak{M}. M)$ for each $\langle m', m\rangle \in (\mathfrak{M}\setminus(0), M\setminus(0))$.

Proposition 2.1 If $(\mathfrak{M}, M)$ is an (A, Γ)-module, then following statements are equivalent:

(1) (A, Γ)-module $(\mathfrak{M}, M)$ is irreducible,

(2) $\mathfrak{M}_\Gamma$ and M_A is both irreducible,

(3) $\mathfrak{M}_{\mathfrak{R}}$ and M_R is both irreducible,

(4) if $\langle m', m\rangle \in (\mathfrak{M}\setminus(0), M\setminus(0))$ then $(m\Gamma, m'A) = (\mathfrak{M}, M)$,

(5) if $\langle m', m\rangle \in (\mathfrak{M}\setminus(0), M\setminus(0))$ then there is $\langle a, \alpha\rangle \in (A, \Gamma)$ such that $(m'a\Gamma, m\alpha A) = (\mathfrak{M}, M)$,

(6) if $\langle m', m\rangle \in (\mathfrak{M}\setminus(0), M\setminus(0))$ then $(m'\mathfrak{R}, mR) = (\mathfrak{M}, M)$.

Proof The equivalence of the above six statements is an immediate consequence of the above Lemma 2.1 and [6] Lemma 1.1.

The subset $\operatorname{Ann}(\mathfrak{M}, M) = \{\langle a, \alpha\rangle \in (A, \Gamma) \mid (M\alpha, \mathfrak{M}a) = (0,0)\}$ of ring (A, Γ) is called annihilator of the (A, Γ)-module $(\mathfrak{M}, M)$. It is easy to see that $\operatorname{Ann}(\mathfrak{M}, M)$ is an ideal of the ring (A, Γ). We say that $(\mathfrak{M}, M)$ is a faithful (A, Γ)-module if $\operatorname{Ann}(\mathfrak{M}, M) = (0,0)$. A ring (A, Γ) is said to be primitive if it has a faithful irreducible (A, Γ)-module.

Proposition 2.2 Let $(I, \mathfrak{J})$ be an ideal in a ring (A, Γ). If $(\mathfrak{M}, M)$ is a $(A, \Gamma)/(I, \mathfrak{J})$-module then $(\mathfrak{M}, M)$ can be considered as an (A, Γ)-module and $(I, \mathfrak{J}) \subseteq \operatorname{Ann}(\mathfrak{M}, M)$. Conversely, if $(\mathfrak{M}, M)$ is an (A, Γ)-module and $(I, \mathfrak{J}) \subseteq \operatorname{Ann}(\mathfrak{M}, M)$ then $(\mathfrak{M}, M)$ can be regarded as an $(A, \Gamma)/(I, \mathfrak{J})$-module.

The proof is established by the quite similar fashion to that for an ordinary ring (cf. [5]) and so we omit it.

By Proposition 2.1 and 2.2 we have

Lemma 2.2 (A,Γ)-module $(\mathfrak{M},M)$ is faithful irreducible if and only if $\mathfrak{M}_\Gamma$ and M_A are both faithful irreducible.

From Lemma 2.2 and [6] Lemma 1.3, we immediately have the following.

Proposition 2.3 Let $(\mathfrak{M},M)$ be an (A,Γ)-module, then the following statements are equivalent:

(1) $(\mathfrak{M},M)$ is faithful irreduible.

(2) $\mathfrak{M}_\Gamma$ and M_A is both faithful irreducible,

(3) $\mathfrak{M}_{\mathfrak{R}}$ and M_R is both faithful irreducible and, moreover $((0,0):(A,\Gamma))_r = (0,0)$ where $((0,0):(A,\Gamma))_r = \{\langle x,\xi\rangle \in (A,\Gamma) \mid (A\Gamma x, \Gamma A\xi) = (0,0)\}$.

Thus, we have

Theorem 2.1 Let (A,Γ) be a two-groups associativelv ring then the following statements are equivalent:

(1) ring (A,Γ) is primitive,

(2) Γ-ring A and A-ring Γ is both primitive,

(3) $\mathfrak{R}$ and R is both primitive and, moreover $((0,0):(A,\Gamma))_r = (0,0)$(as above).

If $(I,\mathfrak{J})$ is a right ideal of the ring (A,Γ), we call $(I,\mathfrak{J})$ maximal in case $(I,\mathfrak{J})\neq(A,\Gamma)$ and, for any right ideal $(G,\mathfrak{G})$ such that $(G,\mathfrak{G})\gneqq(I,\mathfrak{J})$ (i.e. $G\supsetneqq I$ and $\mathfrak{G}\supsetneqq\mathfrak{J}$), we have $(G,\mathfrak{G})=(A,\Gamma)$.

We call $(I,\mathfrak{J})$ regular in case there is $\langle e,\varepsilon\rangle\in(A,\Gamma)$ such that

$$\langle x-e\varepsilon x,\ \xi-\varepsilon e\xi\rangle\in(I,\mathfrak{J}) \text{ for every } \langle x,\xi\rangle\in(A,\Gamma).$$

By the definition of regular right ideal of Γ-ring (see [1]), the right ideal $(I,\mathfrak{J})$ of a ring (A,Γ) is regular if and only if the right ideal I of the Γ-ring A and the right ideal $\mathfrak{J}$ of the A-ring Γ are regular.

Proposition 2.4 If (A,Γ)-module $(\mathfrak{M},M)$ is irreducible then $\mathrm{Ann}\langle m', m\rangle = \{\langle x,\xi\rangle\in(A,\Gamma) \mid \langle m\xi, m'x\rangle = \langle 0,0\rangle\}$ is a maximal regular right ideal of the ring (A,Γ) for each $\langle m',m\rangle\in(\mathfrak{M}\setminus(0), M\setminus(0))$. Conversely, if $(I,\mathfrak{J})$ is a maximal regular right ideal of ring (A,Γ) then there is an irreducible (A,Γ)-module $(\mathfrak{M},M)$ such that $(I,\mathfrak{J})=\mathrm{Ann}\langle m',m\rangle$ for some $\langle m',m\rangle\in(\mathfrak{M}\setminus(0),M\setminus(0))$.

Proof Since $(\mathfrak{M},M)$ is irreducible, we must have that $(m\Gamma, m'A)=(\mathfrak{M},M)$ for every $\langle m',m\rangle\in(\mathfrak{M}\setminus(0),M\setminus(0))$. Therefore there is $\langle e,\varepsilon\rangle\in(A,\Gamma)$ such that $\langle x-e\varepsilon x, \xi-\varepsilon e\xi\rangle\in\mathrm{Ann}\langle m',m\rangle$. Clearly, the right ideal is regular and $\mathrm{Ann}\langle m',m\rangle \subsetneqq (A,\Gamma)$. Suppose $\mathrm{Ann}\langle m',m\rangle \lneqq (G,\mathfrak{G})\subseteq(A,\Gamma)$and $(G,\mathfrak{G})$ is a right ideal; then $(m'G\Gamma, m'G)$ is a nonzero submodule of (A,Γ)-module $(\mathfrak{M},M)$ and so $m'G=M$. Use the same method, we can get $m\mathfrak{G}=\mathfrak{M}$. Thorefore $(m\mathfrak{G}, m'G)=(\mathfrak{M},M)=(M\Gamma, \mathfrak{M}A)$. It follows that for each $\langle x,\xi\rangle\in(A,\Gamma)$, there exists a $\langle b,\beta\rangle\in(A,\Gamma)$ such that $\langle x-b, \xi-\beta\rangle \in \mathrm{Ann}\langle m', m\rangle$. Since $\mathrm{Ann}\langle m', m\rangle \lneqq (G, \mathfrak{G})$, $(A, \Gamma)=(G, \mathfrak{G})$ and $\mathrm{Ann}\langle m', m\rangle$ is maximal.

Conversely, let $(I, \mathfrak{J})$ be a maximal regular right ideal then there exists $\langle e, \varepsilon\rangle \in (A, \Gamma)$ such that $\langle x-e\varepsilon x, \xi-\varepsilon e\xi\rangle \in (I, \mathfrak{J})$ for each $\langle x, \xi\rangle \in (A, \Gamma)$; define $\mathfrak{M}=\Gamma/\mathfrak{J}$ and $M=A/I$. For $(\mathfrak{M}, M)$ consider the map

$$\langle (m+I)a,\ (m'+\mathfrak{J})\alpha\rangle = \langle \varepsilon m a+\mathfrak{J},\ e m'\alpha+I\rangle$$

for all $\langle m'+\mathfrak{J}, m+I\rangle \in (\mathfrak{M}, M)$ and all $\langle a, \alpha\rangle \in (A, \Gamma)$, and one can easily verify that $(\mathfrak{M}, M)$ is an (A, Γ)-module. It follows that (A, Γ)-module is irreducible and $(I, \mathfrak{J})=\mathrm{Ann}\langle \varepsilon+\mathfrak{J}, e+I\rangle$ for $\langle \varepsilon+\mathfrak{J}, e+I\rangle \in (\mathfrak{M}, M)$.

By proposition 2.4, we have immediately

Theorem 2.2 The ring (A, Γ) is primitive if and only if there is a maximal regular right ideal $(I, \mathfrak{J})$ such that $((I, \mathfrak{J}):(A, \Gamma))_r=(0,0)$ where $((I, \mathfrak{J}):(A, \Gamma))_r=\{\langle x, \xi\rangle \in (A, \Gamma) \mid (A\Gamma x, \Gamma A\xi) \subseteq (I, \mathfrak{J})\}$.

3. The density Let $(\mathfrak{M}, M)$ be an (A, Γ)-module. The additive abelian groups $\mathrm{Hom}(\mathfrak{M}, M)$ and $\mathrm{Hom}(M, \mathfrak{M})$ is a two-groups associative ring with usual homomorphic composition, and denoted by $(\mathrm{Hom}(\mathfrak{M}, M), \mathrm{Hom}(M, \mathfrak{M}))$. For each $\langle a, \alpha\rangle \in (A, \Gamma)$ consider the map $\langle a_r, \alpha_r\rangle : \langle m a_r, m'\alpha_r\rangle = \langle m a, m'\alpha\rangle$ for every $\langle m', m\rangle \in (\mathfrak{M}, M)$, and $(A_1, \Gamma_1)=\{\langle a_r, \alpha_r\rangle \mid \langle a, \alpha\rangle \in (A, \Gamma)\}$ is a two-groups associative ring and it is subring of the ring $(\mathrm{Hom}(\mathfrak{M}, M), \mathrm{Hom}(M, \mathfrak{M}))$.

The map $\langle a, \alpha\rangle \mapsto \langle a_r, \alpha_r\rangle$ is easily seen to be a homomorphism from ring (A, Γ) to ring $(\mathrm{Hom}(\mathfrak{M}, M), \mathrm{Hom}(M, \mathfrak{M}))$, and ring (A_1, Γ_1) is the image of ring (A, Γ). It follows that $(A_1, \Gamma_1) \cong (A, \Gamma)/\mathrm{Ann}(\mathfrak{M}, M)$. Therefore we have

Lemma 3.1 If (A, Γ)-module $(\mathfrak{M}, M)$ is faithful, then ring (A, Γ) is isomorphic to the subring (A_1, Γ_1) of the ring $(\mathrm{Hom}(\mathfrak{M}, M), \mathrm{Hom}(M, \mathfrak{M}))$.

For the remainder of this section we assume that $(\mathfrak{M}, M)$ is a faithful irreducible (A, Γ)-module. By Lemma 3.1, any element of ring (A, Γ) can be considered as an element of ring $(\mathrm{Hom}(\mathfrak{M}, M), \mathrm{Hom}(M, \mathfrak{M}))$. By Theorem 2.1 and the classical theory, the right operator ring $\mathfrak{R}$ and R are dense rings of linear transformations of $\mathfrak{M}$ and M respectively. By Shur's lemma, the underlying division rings are given by

$$D_1=\{d_1 \in \mathrm{Hom}(\mathfrak{M}, \mathfrak{M}) \mid (m'd)a\alpha=(m'a\alpha)\mathrm{d}_1 \text{ for all } a\in A,\ \alpha\in\Gamma \text{ and all } m'\in\mathfrak{M}\},$$

and

$$D_2=\{d_2 \in \mathrm{Hom}(M, M) \mid (md_2)\alpha a=(m\alpha a)d_2 \text{ for all } a\in A,\ \alpha\in\Gamma \text{ and all } m\in M\}.$$

Lemma 3.2 $D_1 \cong D_2$ (as ring).

Proof Let $\langle m', m\rangle \in (\mathfrak{M}\setminus(0); M\setminus(0))$. For each $d_i \in D_i$ $(i=1,2)$ and each $\langle a, \alpha\rangle \in (A, \Gamma)$ define the maps d_1' and d_2' as follows: $(m'a)d_1'=(m'd_1)$ and $(m\alpha)d_2'=(md_2)\alpha$. It is easy to show that mappings $d_1 \mapsto d_1'$ and $d_2 \mapsto d_2'$ are actually inverse ring isomorphisms.

We can now regard both $\mathfrak{M}$ and M as vector spaces over D where D is a division ring, $D=D_1=D_2$. Now we give the Chevalley-Jacobson dense theorem of the two-groups associative ring (A,Γ).

Theorem 3.1 Let $(\mathfrak{M},M)$ be a faithful, irreducible (A,Γ)-module. If $x_1, x_2,\cdots,x_m$ are D-linearly independent in $\mathfrak{M}$ and suppose $y_1, y_2,\cdots,y_n$ are D-linearly independent in M, then there is $\langle a,\alpha\rangle\in(A,\Gamma)$ such that $\langle y_j\alpha, x_i a\rangle=\langle u_j,v_i\rangle$ for $i=1,2,\cdots,m$ and $j=1.2.\cdots,n$, where $\langle u_j,v_i\rangle\in(\mathfrak{M},M)$ for $i=1,2,\cdots,m$ and $j=1,2,\cdots,n$. That is, ring (A,Γ) is a dense ring of linear transformations of $(\mathfrak{M},M)$.

The proof is established by the quite similar fashion to that for an ordinary ring (cf. [5]) and so we omit it.

Let $D_{m,n}$ be the set of all rectangular matrices of type $m\times n$ over a division ring D then $(D_{m,n},D_{n,m})$ is clearly a two-groups associative ring under the usual matrix addition and multiplication, and call (m,n)-matrix ring.

By Theorem 3.1, we have immediately

Theorem 3.2 Let ring (A,Γ) be primitive. Then for some division ring D either ring (A,Γ) isomorphic to (m,n)-matrix ring $(D_{m,n},D_{n,m})$, or given any integer m and n there exists subring $(S,\mathfrak{S})$ of ring (A,Γ) which map homomorphically onto (m,n)-matrix ring $(D_{m,n},D_{n,m})$.

By Theorem 2.1 and Theorem 3.2, we have immediately

Theorem 3.3 Let ring (A,Γ) be primitive. Then for some division ring D either $R\cong D_n$ and $\mathfrak{R}\cong D_m$, or given any integer m and n there exists subring R_n of R and $\mathfrak{R}_m$ of $\mathfrak{R}$ which maps homomorphically onto D_n and D_m respectively, where D_i is the ring of all $i\times i$ matrices over D.

4. The Jacobson radical An ideal $(I,\mathfrak{I})$ of a ring (A,Γ) is called primitive if ring $(A,\Gamma)/(I,\mathfrak{I})$ is primitive. Since a primitive ring (A,Γ) is necessarily $\neq(0,0)$, $(I,\mathfrak{I})$ is a proper ideal.

Definition 4.1 The Jacobson radical of ring (A,Γ), written as $J(A,\Gamma)$, is the intersection of all primitive ideals of ring (A,Γ). If a ring (A,Γ) has no primitive ideals we put $J(A,\Gamma)=(A,\Gamma)$. A ring (A,Γ) is said to be semi-simple if $J(A,\Gamma)=(0,0)$. A ring (A,Γ) is said to be radical ring if $J(A,\Gamma)=(A,\Gamma)$.

Proposition 4.1 Let $(I,\mathfrak{I})$ be an ideal of a ring (A,Γ). Then the $(I,\mathfrak{I})$ is primitive if and only if $(I,\mathfrak{I})=\mathrm{Ann}(\mathfrak{M},M)$, where $(\mathfrak{M},M)$ is some irreducible (A,Γ)-module.

Proof Let $(I,\mathfrak{I})$ ba a primitive ideal of ring (A,Γ), then the ring $(A,\Gamma)/(I,\mathfrak{I})=(\overline{A},\overline{\Gamma})$ is primitive. Therefore there exists a faithful, irreducible $(\overline{A},\overline{\Gamma})$-module $(\mathfrak{M},M)$. By Proposition 2.2, $(\mathfrak{M},M)$ is a irreducible (A,Γ)-module and $(I,\mathfrak{I})\subseteq\mathrm{Ann}(\mathfrak{M},M)$ under the action of $\langle ma,m'\alpha\rangle=\langle m\overline{a},m'\overline{\alpha}\rangle$. If $\langle a,\alpha\rangle\in\mathrm{Ann}(\mathfrak{M},$

M) then $(Ma, \mathfrak{M}a) = (0,0)$, and $(M\bar{a}, \mathfrak{M}\bar{a}) = (0,0)$. Since $(\bar{A}, \bar{\Gamma})$-module $(\mathfrak{M}, M)$ is faithful, we have $\langle \bar{a}, \bar{a} \rangle = \langle \bar{0}, \bar{0} \rangle$. Thus $\langle a, a \rangle \in (I, \mathfrak{J})$. It follows that $(I, \mathfrak{J}) = \mathrm{Ann}(\mathfrak{M}, m)$.

Conversely, let $(I, \mathfrak{J}) = \mathrm{Ann}(\mathfrak{M}, M)$ where $(\mathfrak{M}, M)$ is a irreducible (A, Γ)-module, then $(\mathfrak{M}, M)$ is a faithful irreducible module of the ring $(A, \Gamma)/(I, \mathfrak{J})$. It follows that the ideal $(I, \mathfrak{J})$ is primitive.

The proofs of following three theorems are minor modification of the proofs of the corresponding theorems ring theory (cf. [5]). and we omit it.

Theorom 4.1 If a ring (A, Γ) has no irreducible (A, Γ)-modules then $J(A, \Gamma) = (A, \Gamma)$; If a ring (A, Γ) has irreducible (A, Γ)-modules then $J(A, \Gamma) = \bigcap_{a \in \Omega} \mathrm{Ann}(\mathfrak{M}_a, M_a)$ where Ω is the set of all irreducible (A, Γ)-modules.

Theorom 4.2 The ring $(A, \Gamma)/J(A, \Gamma)$ is semi-simple.

Theorom 4.3 A ring (A, Γ) is semi-simple if and only if it is a subdirect sum of primitive ring (A_a, Γ_a), $a \in \Omega$.

By Proposition 2.4, we have

Theorem 4.4 $J(A, \Gamma) = \bigcap_{a \in \Omega} (I_a, \mathfrak{J}_a)$ where Ω is the set of all maximal regular right ideals of the ring (A, Γ).

When ring (A, Γ) is regarded as the Γ-ring A, let $\mathfrak{G}$ be the free abelian group generated by $M x \Gamma$ where M is an irreducible module of the Γ-ring A. Then $\mathfrak{F} = \{ \sum_i n_i (m_i, a_i) \in \mathfrak{G} \mid \sum_i n_i m_i a_i a = 0$ for all $a \in A\}$ is a subgroup of $\mathfrak{G}$. Let $\mathfrak{M} = \mathfrak{G}/\mathfrak{F}$, the factor group, and denote the coset $(m, a) + \mathfrak{F}$ by $[m, a]$. It can be verified easily that $[m, a] + [m, \beta] = [m, a + \beta]$ and $[m_1, a] + [m_2, a] = [m_1 + m_2, a]$ for all $a, \beta \in \Gamma$ and all $m, m_1 m_2 \in M$. We define mappings $\mathfrak{M} \times A \to M$ and $M \times \Gamma \to \mathfrak{M}$ (sending $([m, a], a)$ and (m, a) to maa and ma respectively). It can be verified easily that $(\mathfrak{M}, M)$ is an (A, Γ)-module and it irreducible, moreover $\mathrm{Ann}(\mathfrak{M}, M) = (\mathrm{Ann}(M), \mathrm{Ann}(\mathfrak{M}))$ where $\mathrm{Ann}(\mathfrak{M}) = \{a \in \Gamma \mid \mathfrak{M} A a = (0)\}$ and $\mathrm{Ann}(M) = \{a \in A \mid M \Gamma a = (0)\}$. Therefore, by symmetry we have

Proposition 4.2 Let (A, Γ) be a two-groups associative ring. If the Γ-ring A (A-ring Γ) has an irreducible module M ($\mathfrak{M}$) then there exists an irreducible (A, Γ)-module $(\mathfrak{M}, M)$ such that

$$\mathrm{Ann}(\mathfrak{M}, M) = (\mathrm{Ann}(M), \mathrm{Ann}(\mathfrak{M})).$$

The Jacobson radical of a Γ-ring A, written as $J(A)$, is the set of all elements of A which annihilate all the irreducible module of the Γ-ring A (see [1] or [8]). Hence, by Proposition 4.2 we have

Theorem 4.5 $J(A, \Gamma) = (J(A), J(\Gamma))$ for the ring (A, Γ).

This theorem shows that if we regard the weak Γ_N-ring A as the Γ-ring

A and the A-ring Γ then $(J(A), J(\Gamma))$ is an ideal of the ring (A, Γ), moreover $J(A)$ is a weak Γ'_N-ring where $\Gamma' = J(\Gamma)$.

The left Jaobson radical can be defined similarly and it is naturally asked if the right Jacobson radical coincides with the left one for ring (A, Γ).

From Theorem 4.5 and Kyuno's result in [7], we have

Theorem 4.6 The right Jacobson radical and the left one coincide on a two-groups associative ring.

For following other notions and notations to the Γ-ring we refer to [7]. Use Theorem 4.5 and Theorem 3.1 in [8], we have

Theorem 4.7 $J(A, \Gamma) = (J(R)^*, J(\mathfrak{R})^*)$.

From Theorem 4.5 and Theorem 3 in [7], we have

Theorem 4.8 Let $J(A, \Gamma)$ is Jacobson radical of a ring (A, Γ), then

(1) $J_1(A, \Gamma) = (J_1(A), J_1(\Gamma))$,

(2) $J_1(A, \Gamma) = (J_1(L)^+, J_1(\mathscr{L})^+)$.

Use Theorem 4.5 and Theorem 3.7 in [1], we have

Theorem 4.9 Let A be an ordinary associative ring and its Jacobson radical $J^*(A)$. Then A can be considered as a two-groups associative ring (A, A). Moreover $J(A, A) = (J^*(A), J^*(A))$.

By Theorem 4.5 and Proposition 2.8 in [1], we have

Theorem 4.10 If $(I, \mathfrak{J})$ is an ideal of ring (A, Γ) then $J(I, \mathfrak{J}) = (I, \mathfrak{J}) \cap J(A, \Gamma)$.

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