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Best Simultaneous Approximation in $L^{\Phi}(I, X)$

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Abstract Let X be a Banach space and Φ be an Orlicz function. Denote by $L^{\Phi}(I,X)$ the space of X-valued Φ-integrable functions on the unit interval I equipped with the Luxemburg norm. For $f_1, f_2, \ldots, f_m \in L^{\Phi}(I,X)$, a distance formula $\operatorname{dist}_{\Phi}(f_1, f_2, \ldots, f_m, L^{\Phi}(I,G))$ is presented, where G is a close subspace of X. Moreover, some existence and characterization results concerning the best simultaneous approximation of $L^{\Phi}(I,G)$ in $L^{\Phi}(I,X)$ are given.

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1. Introduction

A function $\Phi : (-\infty, +\infty) \mapsto [0, +\infty)$ is called an Orlicz function if it satisfies the following conditions:

(1) Φ is even, continuous, convex, and $\Phi(0) = 0$;

(2) $\Phi(x) > 0$ for all $x \neq 0$;

(3) $\lim_{x\to 0} \Phi(x)/x = 0$ and $\lim_{x\to\infty} \Phi(x)/x = \infty$.

We say that a function Φ satisfies the $\triangle_2$ condition if there are constants $k > 1$ and $x_0 > 0$ such that $\Phi(2x) \le k\Phi(x)$ for $x > x_0$. Examples of Orlicz functions that satisfy the Δ_2 conditions are widely available such as $\Phi(x) = |x|^p$, $1 \le p < \infty$, and $\Phi(x) = (1 + |x|)\log(1 + |x|) - |x|$.

Let X be a Banach space and let (I, μ) be a measure space, where I is a unit interval. For an Orlicz function Φ, let $L^{\Phi}(I,X)$ be the Orlicz-Bochner function space that consists of strongly measurable functions $f : I \to X$ with $\int_I \Phi(\alpha\|f(t)\|)\mathrm{d}\mu(t) < \infty$ with some $\alpha > 0$. It is known that $L^{\Phi}(I,X)$ is a Banach space under the Luxemburg norm

$$\|f\|_{\Phi} = \inf\Big\{\alpha > 0, \int_I \Phi(\|f(t)\|/\alpha)\mathrm{d}\mu(t) \le 1\Big\}. \tag{1.1}$$

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It should be remarked that if $\Phi(x)=|x|^p$, $1\le p<\infty$, the space $L^\Phi(I,X)$ is simply the $p-$Lebesgue Bochner function space $L^p(I,X)$ with

$$\|f\|_\Phi=\Phi^{-1}\int_I\Phi(\|f(t)\|)\mathrm{d}\mu(t)=\Big(\int_I\|f(t)\|^p\mathrm{d}\mu(t)\Big)^{1/p}=\|f\|_p.$$

For more information about $L^\Phi(I,X)$, we refer to [1,2]. Throughout this paper, X is a Banach space, G is a closed subspace of X, Φ is an Orlicz function, $p\ge 1$ and I is a unit interval.

Definition 1.1 *For $x_1,x_2,\dots,x_m\in X$, define* $\mathrm{dist}_p: X^m\mapsto \mathbf{R}$ *by*

$$\mathrm{dist}_p(x_1,x_2,\dots,x_m,G):=\inf_{y\in G}\Big[\sum_{i=1}^m\|x_i-y\|^p\Big]^{1/p}. \tag{1.2}$$

Similarly, for $f_1,f_2,\dots,f_m\in L^\Phi(I,X)$, we define $\mathrm{dist}_\Phi:(L^\Phi(I,X))^m\mapsto\mathbf{R}$

$$\mathrm{dist}_\Phi(f_1,f_2,\dots,f_m,L^\Phi(I,G)):=\inf_{g\in L^\Phi(I,G)}\Big\|[\sum_{i=1}^m\|f_i(\cdot)-g(\cdot)\|^p]^{1/p}\Big\|_\Phi. \tag{1.3}$$

Definition 1.2 *We say that $z\in G$ is a best simultaneous approximation from G of an m-tuple of elements $(x_1,x_2,\dots,x_m)\in X^m$, if*

$$\Big(\sum_{i=1}^m\|x_i-z\|^p\Big)^{1/p}=\mathrm{dist}_p(x_1,x_2,\dots,x_m,G).$$

We say that $h\in L^\Phi(I,G)$ is a best simultaneous approximation of an m-tuple of elements $(f_1,f_2,\dots,f_m)\in(L^\Phi(I,X))^m$, if

$$\Big\|[\sum_{i=1}^m\|f_i(\cdot)-h(\cdot)\|^p]^{1/p}\Big\|_\Phi=\mathrm{dist}_\Phi(f_1,f_2,\dots,f_m,L^\Phi(I,G)).$$

If each m-tuple of elements $(f_1,f_2,\dots,f_m)\in(L^\Phi(I,X))^m$ admits a best simultaneous approximation from $(L^\Phi(I,G))$, then $(L^\Phi(I,G))$ is said to be simultaneously proximinal in $L^\Phi(I,X)$.

The problem of best simultaneous approximation has been studied by many authors in [3–5]. Most of these works have dealt with the characterization of best simultaneous approximation in spaces of continuous functions with values in a Banach space X. Results on best simultaneous approximation in general Banach spaces may be found in [6,7]. Related results on $L^p(I,X)$ were given in [8,9]. In [8], it was shown that if G is a reflexive subspace of a Banach space X, then $L^p(I,G)$ is simultaneously proximinal in $L^p(I,X)$. The aim of this work is to prove that for a closed separable subspace G of X and an Orlicz function Φ satisfying Δ_2 condition, $L^\Phi(I,G))$ is simultaneously proximinal in $L^\Phi(I,X)$ if and only if G is simultaneously proximinal in X.

2. Some lemmas

For $x=(x_1,x_2,\dots,x_m),y=(y_1,y_2,\dots,y_m)\in X^m$, we set $d(x,y)=[\sum_{i=1}^m\|x_i-y_i\|^p]^{1/p}$. It is easy to verify that $\{X^m;d\}$ is a complete metric space.

Lemma 2.1 *The function* $\mathrm{dist}_p(x_1,\dots,x_m,G)$ *is continuous in* $\{X^m;d\}$.

Proof Let $x=(x_1,x_2,\dots,x_m)$, $y=(y_1,y_2,\dots,y_m)\in X^m$. By (1.2) and triangle inequality, we have that

$$\begin{aligned}\mathrm{dist}_p(x_1,x_2,\dots,x_m,G) &\le \inf_{z\in G}\Big[\sum_{i=1}^{m}(\|x_i-y_i\|+\|y_i-z\|)^p\Big]^{1/p}\\ &\le \inf_{z\in G}\Big([\sum_{i=1}^{m}\|x_i-y_i\|^p]^{1/p}+[\sum_{i=1}^{m}\|y_i-z\|^p]^{1/p}\Big)=d(x,y)+\mathrm{dist}_p(y_1,\dots,y_m,G). \quad (2.1)\end{aligned}$$

Similarly, we obtain the following inequality

$$\mathrm{dist}_p(y_1,y_2,\dots,y_m,G)\le d(x,y)+\mathrm{dist}_p(x_1,\dots,x_m,G). \quad (2.2)$$

From (2.1), (2.2), the proof is completed. □

Lemma 2.2 *Let Φ be an Orlicz function satisfying Δ_2 condition. Suppose $f_1,f_2,\dots,f_m\in L^{\Phi}(I,X)$. Then*

$$\mathrm{dist}_{\Phi}(f_1,f_2,\dots,f_m,L^{\Phi}(I,G))=\|\mathrm{dist}_p(f_1(\cdot),f_2(\cdot),\dots,f_m(\cdot),G)\|_{\Phi}. \quad (2.3)$$

Proof Let $f_1,\dots,f_m\in L^{\Phi}(I,X)$. Then for each $i=1,2,\dots,m$, there exists a sequence $\{f_{i,n}\}$ of simple functions in $L^{\Phi}(I,X)$ such that

$$\|f_{i,n}(t)-f_i(t)\|\to 0, \quad \text{as} \quad n\to\infty, \quad (2.4)$$

for almost all t in I. From (2.4), Lemma 2.1 and Jensen' inequality, it follows that

$$\lim_{n\to\infty}|\mathrm{dist}_p(f_{1,n}(t),\dots,f_{m,n}(t),G)-\mathrm{dist}_p(f_1(t),\dots,f_m(t),G)|=0.$$

Furthermore for each n, the function: $t\mapsto \mathrm{dist}_p(f_{1,n}(t),\dots,f_{m,n}(t),G)$ is a simple function. Thus $\mathrm{dist}_p(f_1(t),\dots,f_m(t),G)$ is measurable. From (1.2), it follows that

$$\mathrm{dist}_p(f_1(t),\dots,f_m(t),G)\le\Big[\sum_{i=1}^{m}\|f_i(t)-z\|^p\Big]^{1/p},$$

for all z in G. Or

$$\mathrm{dist}_p(f_1(t),\dots,f_m(t),G)\le\Big(\sum_{i=1}^{m}\|f_i(t)-g(t)\|^p\Big)^{1/p}, \quad (2.5)$$

for all $g\in L^{\Phi}(I,G)$. From (2.5) and (1.1), it follows that

$$\|\mathrm{dist}_p(f_1(\cdot),\dots,f_m(\cdot),G)\|_{\Phi}\le\Big\|(\sum_{i=1}^{m}\|f_i(\cdot)-g(\cdot)\|^p)^{1/p}\Big\|_{\Phi}, \quad (2.6)$$

for all $g\in L^{\Phi}(I,G)$. Hence $\mathrm{dist}_p(f_1(t),\dots,f_m(t),G)\in L^{\Phi}(I,X)$. By (2.6) and (1.3), we obtain

$$\|\mathrm{dist}_p(f_1(\cdot),f_2(\cdot),\dots,f_m(\cdot),G)\|_{\Phi}\le\mathrm{dist}_{\Phi}(f_1,f_2,\dots,f_m,L^{\Phi}(I,G)). \quad (2.7)$$

Let us show the reverse inequality. By assumption, simple functions are dense in $L^{\Phi}(I,X)$. For fixed $\epsilon>0$, and $f_1,f_2,\dots,f_m\in L^{\Phi}(I,X)$, there exist simple functions $F_i\in L^{\Phi}(I,X)$ such that

$$\|f_i-F_i\|_{\Phi}\le\frac{\epsilon}{4m}, \quad i=1,2,\dots,m. \quad (2.8)$$

Then we assume that

$$F_i(t)=\sum_{k=1}^{n}\chi_{A_k}(t)y_k^i,\quad i=1,2,\dots,m, \tag{2.9}$$

where χ_{A_k} are the characteristic functions of the measurable sets A_k in I and $y_k^i\in X$. We can assume that $\sum_{k=1}^{n}\chi_{A_k}=1,\mu(A_k)>0$ and $\Phi(1)\le 1$. Given $\epsilon>0$ for each $k=1,2,\dots,n$, select $g_k\in G$ such that

$$\Big(\sum_{i=1}^{m}\|y_k^i-g_k\|^p\Big)^{1/p}<\mathrm{dist}_p(y_k^1,\dots,y_k^m,G)+\frac{\epsilon}{4}. \tag{2.10}$$

Set

$$g(t)=\sum_{k=1}^{n}g_k\chi_{A_k}(t),\ \ H(t)=\mathrm{dist}_p(f_1(t),\dots,f_m(t),G)+\Big[\sum_{i=1}^{m}\|f_i(t)-F_i(t)\|^p\Big]^{1/p}+\frac{\epsilon}{4}. \tag{2.11}$$

Clearly $H\in L^\Phi(I,X)$. From (2.8)–(2.11), it follows that

$$\begin{aligned}\int_I\Phi\Big(\frac{(\sum_{i=1}^{m}\|F_i(t)-g(t)\|^p)^{1/p}}{\|H\|_\Phi+\epsilon/4}\Big)\mathrm{d}\mu(t)&=\sum_{k=1}^{n}\int_{A_k}\Phi\Big(\frac{(\sum_{i=1}^{m}\|y_k^i-g_k\|^p)^{1/p}}{\|H\|_\Phi+\epsilon/4}\Big)\mathrm{d}\mu(t)\\&\le\sum_{k=1}^{n}\int_{A_k}\Phi\Big(\frac{\mathrm{dist}_p(y_k^1,\dots,y_k^m,G)+\epsilon/4}{\|H\|_\Phi+\epsilon/4}\Big)\mathrm{d}\mu(t)\\&=\int_I\Phi\Big(\frac{\mathrm{dist}_p(F_1(t),\dots,F_m(t),G)+\epsilon/4}{\|H\|_\Phi+\epsilon/4}\Big)\mathrm{d}\mu(t)\\&\le\int_I\Phi\Big(\frac{\mathrm{dist}_p(f_1(t),\dots,f_m(t),G)+(\sum_{i=1}^{m}\|f_i(t)-F_i(t)\|^p)^{1/p}+\epsilon/4}{\|H\|_\Phi+\epsilon/4}\Big)\mathrm{d}\mu(t)\\&=\int_I\Phi\Big(\frac{H(t)}{\|H\|_\Phi+\epsilon/4}\Big)\mathrm{d}\mu(t)\le 1.\end{aligned} \tag{2.12}$$

From (2.12), (2.11) and (1.1), it follows that

$$\begin{aligned}\Big\|(\sum_{i=1}^{m}\|F_i(\cdot)-g(\cdot)\|^p)^{1/p}\Big\|_\Phi&\le\|\mathrm{dist}_p(f_1(\cdot),\dots,f_m(\cdot),G)\|_\Phi+\sum_{i=1}^{m}\|f_i-F_i\|_\Phi+\epsilon/2\\&\le\|\mathrm{dist}_p(f_1(\cdot),\dots,f_m(\cdot),G)\|_\Phi+3\epsilon/4.\end{aligned} \tag{2.13}$$

By (2.13) and (2.8), we have that

$$\begin{aligned}\mathrm{dist}_\Phi(f_1,\dots,f_m,L^\Phi(I,G))&\le\mathrm{dist}_\Phi(F_1,\dots,F_m,L^\Phi(I,G))+\Big\|(\sum_{i=1}^{m}\|f_i(\cdot)-F_i(\cdot)\|^p)^{\frac{1}{p}}\Big\|_\Phi\\&\le\Big\|(\sum_{i=1}^{m}\|F_i(\cdot)-g(\cdot)\|^p)^{1/p}\Big\|_\Phi+\sum_{i=1}^{m}\|f_i-F_i\|_\Phi\\&\le\|\mathrm{dist}_p(f_1(\cdot),\dots,f_m(\cdot),G)\|_\Phi+\epsilon,\end{aligned}$$

which implies that

$$\mathrm{dist}_\Phi(f_1,f_2,\dots,f_m,L^\Phi(I,G))\le\|\mathrm{dist}_p(f_1(\cdot),f_2(\cdot),\dots,f_m(\cdot),G)\|_\Phi. \tag{2.14}$$

From (2.7) and (2.14), (2.3) follows.

Lemma 2.3 ([10]) *Let G be a closed separable subspace of X. Suppose that G is simultaneously* *proximinal in X and $f_1, f_2, \ldots, f_m : I \mapsto X$ are measurable functions. Then there is a measurable function $g : I \mapsto G$ such that $g(t)$ is a best simultaneous approximation of $f_1(t), \ldots, f_m(t)$ for almost all t.*

3. Main results and their proof

As an application of Lemma 2.2, we have the following theorem.

Theorem 3.1 *Let Φ be an Orlicz function satisfying Δ_2 condition. An element $g \in L^{\Phi}(I,G)$ is a best simultaneous approximation of $f_1, \ldots, f_m \in L^{\Phi}(I,X)$ if and only if $g(t)$ is best simultaneous approximation of $f_1(t), \ldots, f_m(t) \in X$ for almost all $t \in I$.*

By Theorem 3.1, we obtain the following corollary.

Corollary 3.2 *Let Φ be an Orlicz function satisfying Δ_2 condition. If $g \in L^{\Phi}(I,G)$ is a best simultaneous approximation of $f_1, \ldots, f_m \in L^{\Phi}(I,X)$, then for every measurable subset A of I and every $h \in L^{\Phi}(I,G)$,*

$$\Big\|\big(\sum_{i=1}^{m} \|f_i(\cdot) - g(\cdot)\|^p\big)^{1/p}\Big\|_{A,\Phi} \le \Big\|\big(\sum_{i=1}^{m} \|f_i(\cdot) - h(\cdot)\|^p\big)^{1/p}\Big\|_{A,\Phi},$$

where $\|f\|_{A,\Phi} = \inf\{\alpha > 0, \int_A \Phi(\|f(t)\|/\alpha)\mathrm{d}\mu(t) \le 1\}$.

Theorem 3.3 *If G is simultaneously proximinal in X, then every m-tuple of simple functions $f_1, f_2, \ldots, f_m \in L^{\Phi}(I,X)$ admits a best simultaneous approximation in $L^{\Phi}(I,G)$.*

Proof Let $f_1, \ldots, f_m$ be an m-tuple of simple functions in $L^{\Phi}(I,X)$. Without loss of generality, we can assume that $f_i = \sum_{k=1}^{n} \chi_{A_k} y_k^i$, where A_k's are disjoint measurable sets such that $\bigcup_{k=1}^{n} A_k = I$. By assumption, we know that for each $1 \le k \le n$ there exists a best simultaneous approximation g_k in G of the m-tuple of elements $y_k^1, \ldots, y_k^m$. Set $g(t) = \sum_{k=1}^{n} \chi_{A_k}(t) g_k$. Then for any $\alpha > 0$ and $h \in L^{\Phi}(I,G)$ we have

$$\begin{aligned}\int_I \Phi\Big(\frac{(\sum_{i=1}^{m} \|f_i(t) - h(t)\|^p)^{1/p}}{\alpha}\Big)\mathrm{d}\mu(t) &= \sum_{k=1}^{n} \int_{A_k} \Phi\Big(\frac{(\sum_{i=1}^{m} \|y_k^i - h(t)\|^p)^{1/p}}{\alpha}\Big)\mathrm{d}\mu(t) \\ &\ge \sum_{k=1}^{n} \int_{A_k} \Phi\Big(\frac{(\sum_{i=1}^{m} \|y_k^i - g_k\|^p)^{1/p}}{\alpha}\Big)\mathrm{d}\mu(t) \\ &= \int_I \Phi\Big(\frac{(\sum_{i=1}^{m} \|f_i(t) - g(t)\|^p)^{1/p}}{\alpha}\Big)\mathrm{d}\mu(t). \qquad (3.1)\end{aligned}$$

From (3.1) and (1.1), it follows that

$$\Big\|\big(\sum_{i=1}^{m} \|f_i(\cdot) - h(\cdot)\|^p\big)^{1/p}\Big\|_{\Phi} \ge \Big\|\big(\sum_{i=1}^{m} \|f_i(\cdot) - g(\cdot)\|^p\big)^{1/p}\Big\|_{\Phi},$$

for all $h \in L^{\Phi}(I,G)$. Hence g is a best simultaneous approximation of these simple functions.

Theorem 3.4 *Let Φ be an Orlicz function satisfying Δ_2 condition. If $L^{\Phi}(I,G)$ is simultaneously proximinal in $L^{\Phi}(I,X)$, then G is simultaneously proximinal in X.*

Proof Let $x_1, x_2, \ldots, x_m \in X$. Set $f_i = 1 \bigotimes x_i$, $i = 1, \ldots, m$, where 1 is the constant function 1. Clearly for each $i = 1, \ldots, m$, $f_i \in L^\Phi(I, X)$. By assumption there exists $g \in L^\Phi(I, G)$ such that

$$\Big\|(\sum_{i=1}^m \|f_i(\cdot) - g(\cdot)\|^p)^{1/p}\Big\|_\Phi \le \Big\|(\sum_{i=1}^m \|f_i(\cdot) - h(\cdot)\|^p)^{1/p}\Big\|_\Phi,$$

for any $h \in L^\Phi(I, G)$. By lemma 2.2

$$\Big(\sum_{i=1}^m \|f_i(t) - g(t)\|^p\Big)^{1/p} \le \Big(\sum_{i=1}^m \|f_i(t) - h(t)\|^p\Big)^{1/p},$$

a.e., in I. Or

$$\Big(\sum_{i=1}^m \|x_i - g(t)\|^p\Big)^{1/p} \le \Big(\sum_{i=1}^m \|x_i - h(t)\|^p\Big)^{1/p}.$$

Let $h(t)$ run over all functions $1 \bigotimes z$, for $z \in G$, we obtain

$$\Big(\sum_{i=1}^m \|x_i - g(t)\|^p\Big)^{1/p} \le \inf_{z \in G} \Big(\sum_{i=1}^m \|x_i - z\|^p\Big)^{1/p},$$

a.e., in I. Hence there exists $t_0 \in I$ such that

$$\mathrm{dist}_p(x_1, \ldots, x_m, G) = \Big(\sum_{i=1}^m \|x_i - g(t_0)\|^p\Big)^{1/p}.$$

Theorem 3.5 *Let G be a closed separable subspace of X and Φ be an Orlicz function satisfying Δ_2 condition. Then $L^\Phi(I, G)$ is simultaneously proximinal in $L^\Phi(I, X)$ if and only if G is simultaneously proximinal in X.*

Proof Necessity is in Theorem 3.3. Let us show sufficiency. Suppose that G is simultaneously proximinal in X, and let $f_1, f_2, \ldots, f_m$ be functions in $L^\Phi(I, X)$. Lemma 2.3 ensures that there exists a measurable function g defined on I with values in X such that $g(t)$ is a best simultaneous approximation of $f_1(t), f_2(t), \ldots, f_m(t)$ in G for almost all t. It follows from Theorem 3.1 that g is a best simultaneous approximation of $f_1, f_2, \ldots, f_m$ in $L^\Phi(I, G)$.

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