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Finite Groups All of Whose Second Maximal Subgroups Are PSC^*-Groups

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Abstract This paper discusses the influence of minimal subgroups on the structure of finite groups and gives the structures of finite groups all of whose second maximal subgroups are PSC^*-groups.

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1. Introduction

All groups considered in this paper will be finite. For a group G, a subgroup H of G is said to be conjugate permutable if $HH^x = H^xH$ for any $x \in G$. This concept was introduced by Foguel in [10]. The conjugate permutable subgroups have many interesting properties. For example, for a finite group any conjugate permutable subgroup is subnormal[10, Corollary 1.1].

It is natural to introduce the dual concept of conjugate permutable subgroups, we have:

Definition 1.1 *Let G be a group. A subgroup H of G is said to be self-conjugate-permutable if $HH^x = H^xH$ implies $H^x = H$, where $x \in G$.*

Obviously, a subgroup H of G is normal if and only if H is conjugate-permutable and self-conjugate-permutable in G. It is easy to see that for a finite group G, all of whose maximal subgroups and Hall subgroups are self-conjugate-permutable.

A group is called a PN-group if its minimal subgroups are normal. The PN-groups were generalized by many authors[1,4,7]. In this paper, the generalization on PN-groups is continued. For convenience, we give the following definition.

Definition 1.2 *Let G be a group. G is called a PSC^*-group if every cyclic subgroup of G of order 2 or 4 is self-conjugate-permutable.*

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We notice that a maximal subgroup A_4 of A_5 has no subgroups of order 6. Thus there is a subgroup of order 2 which is not self-conjugate-permutable in A_4. So neither A_5 nor A_4 is a PSC^*-group. But all second maximal subgroups of A_5 are PSC^*-groups. It should be an interesting problem to find out all finite groups all of whose second maximal subgroups are PSC^*-groups.

The notation and terminology used in this paper are standard, as in [5] and [6].

2. Preliminaries

In this section we give some basic properties of our definition and collect some results that are needed in this paper.

Lemma 2.1[3] *Let G be a group. Suppose that H is self-conjugate-permutable in G, $K \le G$ and N a normal subgroup of G. We have:*

(1) If $H \le K$, then H is self-conjugate-permutable in K;

(2) Let $N \le K$. Then K/N is self-conjugate-permutable in G/N if and only if K is self-conjugate-permutable in G;

(3) If $(|K|, |N|) = 1$ and K is a p-subgroup of G, then K is self-conjugate-permutable in G if and only if KN is self-conjugate-permutable in G.

Lemma 2.2[3] *(1) A subgroup H of G is normal if and only if H is subnormal and self-conjugate-permutable in G.*

(2) A subgroup H of G is normal if and only if H is conjugate-permutable and self-conjugate-permutable in G.

Lemma 2.3[3] *Let G be a group. Suppose that $G = AB$, $A \le G$, $B \le G$. If H is self-conjugate-permutable in B and H is normalized by A, then H is self-conjugate-permutable in G.*

Lemma 2.4[6] *Let G be a minimal non-nilpotent group (A non-nilpotent group all of whose proper subgroups are nilpotent). Then:*

(1) $G = [G_p]G_q$ and G_q is a cyclic group;

(2) $G_p/\Phi(G_p)$ is a minimal normal subgroup of $G/\Phi(G_p)$;

(3) G_p has exponent p if $p > 2$ and exponent at most 4 if $p = 2$;

(4) G_p is an elementary abelian group if G_p is an abelian group; $Z(G_p) = \Phi(G_p) = G'_p$ if G_p is not an abelian group;

(5) $C_{G_p}(G_q) = G'_p$.

Lemma 2.5[2, Theorem B] *Let G be a nonsolvable group. Suppose that solvable subgroups of G are either 2-nilpotent or minimal non-nilpotent, then G is one of the following groups:*

(1) $PSL(2, 2^f)$, where $2^f - 1$ is a prime;

(2) $PSL(2, q)$, where q is an odd prime with $q > 3$ and $q \equiv 3$ or $5 \pmod 8$;

(3) $SL(2, q)$, where q is an odd prime with $q > 3$ and $q \equiv 3$ or $5 \pmod 8$.

Proof The Lemma is a special example in [3]. Its proof does not depend on the classification of finite simple groups. $\square$

Lemma 2.6 *Let G be a group and $P \in \mathrm{Syl}_p(G)$, where $p \in \pi(G)$. If $H \trianglelefteq P$ and $H^g \trianglelefteq P$, then H and H^g are conjugate in $N_G(P)$.*

Proof Since $H^g \trianglelefteq P$, we have $H \trianglelefteq P^{g^{-1}}$. Also, as $H \trianglelefteq P$, it follows that $\langle P, P^{g^{-1}} \rangle \leq N_G(H)$. We notice that P and $P^{g^{-1}}$ are Sylow p-subgroups of $N_G(H)$. By Sylow's theorem, there exists $n \in N_G(H)$ such that $P^n = P^{g^{-1}}$ and hence $ng \in N_G(P)$. Moreover, $H^{ng} = H^g$, so H and H^g are conjugate in $N_G(P)$. $\square$

Lemma 2.7 *Let G be a non-nilpotent dihedral group of order $2n$ or $4n$, where n is odd. Then G is a PSC^*-group.*

Proof If $|G| = 2n$, n is odd, then $G = [C_n]C_2$. Assume that $C_2C_2^x = C_2^xC_2$ for $x \in G$. Since C_2 is a Sylow 2-subgroup, we have $C_2 = C_2^x$ and C_2 is self-conjugate-permutable in G. Hence G is a PSC^*-group.

If $|G| = 4n$, n is odd. Let K be a subgroup of G of order 2. Assume that $KK^x = K^xK$ for $x \in G$. Then there exists a Sylow 2-subgroup S of G such that $KK^x = K^xK \leq S$. By Lemma 2.6, K and K^x are conjugate in $N_G(S)$. Since G is non-nilpotent, it follows that $N_G(S) < G$ and $K = K^x$. Thus K is self-conjugate-permutable in G. Moreover, the subgroups of G of order 4 are Sylow subgroups of G and hence are self-conjugate-permutable. Thus G is also a PSC^*-group. $\square$

3. The main results and proofs

Theorem 3.1 *Let G be a PSC^*-group. Then G is 2-nilpotent.*

Proof Let $x \in G$ and $x^4 = 1$. Then there exists $P \in \mathrm{Syl}_2(G)$ such that $\langle x \rangle \leq P \leq N_G(P)$. By Lemma 2.1 (1), $\langle x \rangle$ is self-conjugate-permutable in $N_G(P)$. On the other hand, $\langle x \rangle \trianglelefteq\trianglelefteq P \trianglelefteq N_G(P)$, by Lemma 2.2, $\langle x \rangle \trianglelefteq N_G(P)$. Now applying [5, IX. 6.7] gives that G is 2-nilpotent. This completes the proof. $\square$

Corollary 3.2 *Let G be a group. Suppose every cyclic subgroup of second maximal subgroups of G of order 2 or 4 is self-conjugate-permutable in G, then G is 2-nilpotent.*

Proof Let $x \in G$ and $x^4 = 1$. If $\langle x \rangle = G_2$, then G is 2-nilpotent. If $\langle x \rangle < G_2$, then $\langle x \rangle$ is contained in some second maximal subgroup of G. By hypotheses, $\langle x \rangle$ is self-conjugate-permutable in G. By arbitrariness of $\langle x \rangle$, we have that G is a PSC^*-group. Theorem 3.1 implies that G is 2-nilpotent. $\square$

Theorem 3.3 *For a group G, if every maximal subgroup of G is a PSC^*-group, then one of the following results holds:*

(i) G is 2-nilpotent;

(ii) $G = [G_2]G_p$ is a minimal non-nilpotent group, where G_2 is an elementary abelian 2-group

and G_p is a cyclic group;

(iii) $G = [Q_8]Z_{3^n}$ is a minimal non-nilpotent group, where Q_8 is a quaternion of order 8, Z_{3^n} is a cyclic 3-group.

Proof Assume that G is not 2-nilpotent. By Theorem 3.1 and hypotheses, every proper subgroup of G is 2-nilpotent. Therefore G is a minimal non-nilpotent group and $G = [G_2]G_p$ by Lemma 2.4.

(1) If G_2 is abelian, then Lemma 2.4 implies that (ii) holds.

(2) If G_2 is not abelian, then Lemma 2.4 implies that $\exp(G_2) \leq 4$. Let $x \in G_2$. Then $x^4 = 1$. Since $G_2 < G$, we have G_2 is contained in some maximal subgroup of G. By hypotheses and Lemma 2.1, $\langle x \rangle$ is self-conjugate-permutable in G_2. Moreover, G_2 is a 2-group, so $\langle x \rangle \trianglelefteq\trianglelefteq G_2$. By Lemma 2.2, $\langle x \rangle \trianglelefteq G_2$. Arbitrariness of $\langle x \rangle$ implies that all subgroups of G_2 are normal. So G_2 is a Hamiltion group. By [6, III, 7.12], $G_2 = Q_8 \times A$, where Q_8 is a quaternion of order 8, A is an elementary abelian 2-group or 1. By Lemma 2.4, $A \leq Z(G_2) = G_2' \leq Q_8$, so $A \leq Q_8 \cap A = 1$. Therefore $G = [Q_8]Z_p$. We notice $\mathrm{Aut}(Q_8) \cong S_4$ and G_p acts on Q_8 by conjugate, it follows that $24||G|$. Moreover, p is odd, it follows that $p = 3^n$. Thus G_p is a cyclic 3-group. This proves (iii). $\square$

Theorem 3.4 *Let G be a group. If all cyclic subgroups of the third maximal subgroup of G of order 2 or 4 are self-conjugate-permutable in G, then one of the following results holds:*

(i) G is 2-nilpotent;

(ii) $G = A_4$;

(iii) $G = [Q_8]Z_3$ is a minimal non-nilpotent group, where Q_8 is a quaternion of order 8, Z_3 is a cyclic group of order 3.

Proof If all cyclic subgroups of G of order 2 or 4 are self-conjugate-permutable in G, then Theorem 3.1 implies that G is 2-nilpotent. This proves (i).

Assume that G is non-2-nilpotent. By Corollary 3.2, all maximal subgroups of G are 2-nilpotent. So G is a minimal non-2-nilpotent group and $G = [G_2]G_p$ by Lemma 2.4.

Case 1 If G_2 is abelian, then Lemma 2.4 implies that G_2 is an elementary abelian 2-group. Let $|G_2| = 2^n$. If $n > 2$. Let $x \in G$ and $o(x) = 2$. Then $\langle x \rangle$ is contained in some third maximal subgroup of G, by hypotheses, $\langle x \rangle$ is self-conjugate-permutable in G. Moreover, $\langle x \rangle \trianglelefteq G_2 \trianglelefteq G$, by Lemma 2.2, we have $\langle x \rangle \trianglelefteq G$. Also, as G_2 is a minimal normal subgroup of G by Lemma 2.4, we have $G_2 = \langle x \rangle$, this is a contradiction. Hence G_2 is an elementary abelian group of order 4. Since $|\mathrm{Aut}(G_2)| = (2+1)2(2-1)^2 = 6$, we have $p = 3$, which implies $G \cong A_4$. This proves (ii).

Case 2 If G_2 is not abelian. We claim: all subgroups of G_2 are normal in G_2. If not, there is $x \in G$ and $x^4 = 1$ such that $\langle x \rangle < N_{G_2}(\langle x \rangle) < G_2 < G$. So $\langle x \rangle$ is contained in some third maximal subgroup of G. By hypotheses, $\langle x \rangle$ is self-conjugate-permutable in G, Lemma 2.1 implies that $\langle x \rangle$ is self-conjugate-permutable in G_2 and $\langle x \rangle \trianglelefteq G_2$ by Lemma 2.2. This is a contradiction. So the claim holds. By proof of Theorem 3.3 (iii), we obtain that $G = [Q_8]Z_{3^n}$, where Q_8 is a quaternion of order 8, Z_{3^n} is a cyclic 3-group. If $n > 1$. Let $x \in G$, $x^4 = 1$. Then $\langle x\rangle < Q_8 < Q_8\langle Z_3\rangle < G$. So $\langle x\rangle$ is contained in some third maximal subgroup of G. By hypotheses, $\langle x\rangle$ is self-conjugate-permutable in G. On the other hand, $\langle x\rangle \trianglelefteq Q_8 \trianglelefteq G$, Lemma 2.2 implies that $\langle x\rangle \trianglelefteq G$. So $\langle x\rangle\Phi(Q_8)/\Phi(Q_8) = \langle x\rangle/\Phi(Q_8) \trianglelefteq G/\Phi(Q_8)$. By Lemma 2.4, $Q_8/\Phi(Q_8)$ is minimal normal in $G/\Phi(Q_8)$, so we have $Q_8 = \langle x\rangle$. This is a contradiction. Thus $n = 1$ and this proves (iii). □

Theorem 3.5 *Let G be a non-abelian simple group and all of whose second maximal subgroups are PSC^*-groups. Then G is one of the following groups:*

(i) $PSL(2, 2^f)$, *where $2^f - 1$ is a prime;*

(ii) $PSL(2, p)$, *where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$ and $p \equiv 3$ or $5 \pmod 8$;*

(iii) $PSL(2, 3^f)$, *where f is an odd prime, $3^f \equiv 3$ or $5 \pmod 8$.*

Proof Let M be a maximal subgroup of G. Then all maximal subgroups of M are PSC^*-groups by hypotheses. It follows from Theorem 3.3 that M is solvable. Hence all proper subgroups of G are solvable. Applying Thompson's theorem, it follows that G is isomorphic to one of the following five kinds of simple groups:

(1) $PSL(3, 3)$;

(2) $PSL(2, 2^f)$, where f is a prime;

(3) $PSL(2, p)$, where p is a prime with $p > 3$ and $p^2 - 1 \not\equiv 0 \pmod 5$;

(4) $PSL(2, 3^f)$, where f is an odd prime;

(5) The Suzuki group $Sz(2^f)$, where f is an odd prime.

We claim:

(a) $G \not\cong PSL(3, 3)$;

If $G \cong PSL(3, 3)$. Let $x \in Z(G_2)$ and $o(x) = 2$. By [9, Lemma 5.1], $C_G(\langle x\rangle) \cong GL(2, 3)$. Since $SL(2, 3)$ is a proper subgroup of $GL(2, 3)$, by hypotheses, $SL(2, 3)$ is a PSC^*-group. So a cyclic subgroup $\langle y\rangle$ of $SL(2, 3)$ of order 4 is self-conjugate-permutable in $SL(2, 3)$. On the other hand, every cyclic subgroup of $SL(2, 3)$ of order 4 is subnormal in $SL(2, 3)$. Applying Lemma 2.2, $\langle y\rangle \trianglelefteq SL(2, 3)$. But $SL(2, 3)$ has no normal subgroups of order 4. This is a contradiction.

(b) (b-1) $G \not\cong PSL(2, 2^f)$, where $2^f - 1$ is not a prime.

In fact, if $G \cong PSL(2, 2^f)$, where $2^f - 1$ is not a prime, then G possesses a *Frobenius* group N and a normalizer of Sylow 2-subgroup and $N = [G_2]C$ is also a minimal nonabelian group, where G_2 is an elementary abelian group and C is a cyclic group of order $(2^f - 1)$. Since $(2^f - 1)$ is not a prime, it follows that $\langle c\rangle G_2 < N$, where $\langle c\rangle < C$. So $\langle c\rangle G_2$ is contained in some second maximal subgroup, by hypotheses, $\langle c\rangle G_2$ is a PSC^*-group. Let $\langle y\rangle$ be a subgroup of G_2 of order 2. Then $\langle y\rangle$ is self-conjugate-permutable in $\langle c\rangle G_2$. On the other hand, $\langle y\rangle \lhd \lhd \langle c\rangle G_2$, by Lemma 2.2, $\langle y\rangle \trianglelefteq \langle c\rangle G_2$. Therefore $\langle y\rangle\langle c\rangle = \langle y\rangle \times \langle c\rangle$ and $\langle c\rangle \le C_N(\langle y\rangle)$. By [8, p38, Theorem 7.6], $C_N(\langle y\rangle) \le G_2$ and $\langle c\rangle \le G_2$. This is a contradiction. Thus $G \not\cong PSL(2, 2^f)$, where $2^f - 1$ is not a prime.

(b-2) $G \cong PSL(2, 2^f)$ satisfies the hypotheses, where $2^f - 1$ is a prime.

By [6, II, 8.27], $PSL(2, 2^f)$ has only three kinds of maximal subgroups, where $2^f - 1$ is a prime:

1^o minimal nonabelian groups of order $2^f(2^f - 1)$;

2^o dihedral groups of order $2(2^f - 1)$;

3^o dihedral groups of order $2(2^f + 1)$.

For minimal non-abelian groups of order $2^f(2^f - 1)$, whose maximal subgroups are abelian groups. So they are PSC^*-groups.

Remark dihedral groups of order $2(2^f - 1)$ and $2(2^f + 1)$, whose Sylow 2-subgroups are of order 2. By Lemma 2.7, their maximal subgroups are PSC^*-groups. This proves (i).

(c) (c-1) $G \ncong PSL(2, p)$, where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$ and $p \not\equiv 3$ and $5 \pmod 8$.

In fact, suppose that $G \cong PSL(2, p)$, where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$. By [6, II, 8.27], $A_4 < PSL(2, p)$. Since A_4 has no subgroups of order 6, we have A_4 isn't a PSC^*-group. By hypotheses, A_4 is a maximal subgroup of $PSL(2, p)$. We claim: Sylow 2-subgroups of $PSL(2, p)$ are subgroups of order 4. If not, since $K_4 \trianglelefteq A_4 < PSL(2, p)$, we have $A_4 < N_G(K_4)$. It follows from maximality of A_4 that $N_G(K_4) = G$ and so $K_4 \trianglelefteq PSL(2, p)$, a contradiction. Thus the claim holds and $p \equiv 3$ or $5 \pmod 8$.

(c-2) $G \cong PSL(2, p)$ satisfies the hypotheses, where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$ and $p \equiv 3$ or $5 \pmod 8$.

By [6, II, 8.27], G has only three kinds of maximal subgroups:

1^o dihedral groups of order $p + 1$ or $p - 1$;

2^o A_4;

3^o Frobenius group N and a normalizer of Sylow 2-group and $N = [P]C$ is also a minimal nonabelian group, where P is an elementary abelian group and C is a cyclic group of order $(p - 1)/2$.

By Lemma 2.7, all maximal subgroups of 1^o are PSC^*-groups. Clearly, all maximal subgroups of A_4 are PSC^*-groups. All maximal subgroups of 3^o are abelian. So they satisfy the hypotheses. This proves (ii).

(d) (d-1) $G \ncong PSL(2, 3^f)$, where f is an odd prime and $3^f \not\equiv 3$ and $5 \pmod 8$.

In fact, it follows from the proof of (c-1) that $3^f \equiv 3$ or $5 \pmod 8$.

(d-2) $G \cong PSL(2, 3^f)$ satisfies the hypotheses, where f is an odd prime, $3^f \equiv 3$ or $5 \pmod 8$.

The proof of (d-2) is similar to that of (c-2). This proves (iii).

(e) $G \ncong Sz(2^f)$, where f is an odd prime.

In fact, if $G \cong Sz(2^f)$, where f is an odd prime. By [8, p41, Theorem 8.2], G possesses a Frobenius group N and $N = [P]C$, where P is a non-abelian kernel of order 4^f and C is a cyclic complement of order $2^f - 1$. Therefore $Z(P)C < N < G$. By hypotheses, $Z(P)C$ is a PSC^*-group. Let $\langle y \rangle$ be a subgroup of $Z(P)$ of order 2. So $\langle y \rangle$ is self-conjugate-permutable in $Z(P)C$. On the other hand, $\langle y \rangle \trianglelefteq Z(P) \trianglelefteq Z(P)C$. By Lemma 2.2, $\langle y \rangle \trianglelefteq Z(P)C$. So $\langle y \rangle C = \langle y \rangle \times C$ and $C \leq C_N(\langle y \rangle)$. By [8, p38, Theorem 7.6], $C_N(\langle y \rangle) \leq P$ and $C \leq P$. This is a contradiction. $\square$

Theorem 3.6 *Let G be a finite group all of whose second maximal subgroups are PSC-groups. Then G is either a solvable group or one of the following groups:*

(i) $PSL(2, 2^f)$*, where* $2^f - 1$ *is a prime;*

(ii) $PSL(2, p)$*, where* p *is a prime with* $p > 3$*,* $p^2 - 1 \not\equiv 0 \pmod 5$ *and* $p \equiv 3$ *or* $5 \pmod 8$*;*

(iii) $PSL(2, 3^f)$*, where* f *is an odd prime,* $3^f \equiv 3$ *or* $5 \pmod 8$*.*

(iv) $SL(2, p)$*, where* p *is a prime with* $p > 3$*,* $p^2 - 1 \not\equiv 0 \pmod 5$ *and* $p \equiv 3$ *or* $5 \pmod 8$*;*

(v) $SL(2, 3^f)$*, where* f *is an odd prime,* $3^f \equiv 3$ *or* $5 \pmod 8$*.*

Proof Suppose that G is a nonsolvable group. Let M be a maximal subgroup of G. Then all maximal subgroups of M are PSC^*-groups by hypotheses. It follows from Theorem 3.3 that M is either 2-nilpotent or minimal non-2-nilpotent. Hence all proper subgroups of G are either 2-nilpotent or minimal non-2-nilpotent. Applying Lemma 2.5, G is isomorphic to one of the following three kinds of groups:

(1) $PSL(2, 2^f)$, where $2^f - 1$ is a prime;

(2) $PSL(2, q)$, where q is an odd prime with $q > 3$ and $q \equiv 3$ or $5 \pmod 8$;

(3) $SL(2, q)$, where q is an odd prime with $q > 3$ and $q \equiv 3$ or $5 \pmod 8$.

We claim:

(i) $G \cong PSL(2, 2^f)$ satisfies the hypotheses, where $2^f - 1$ is a prime.

In fact, the proof of (i) is similar to that of (b-2) of Theorem 3.5. We can obtain all second maximal subgroups of $PSL(2, 2^f)$ are PSC^*-groups. This proves (i).

(ii) (ii-1) $G \not\cong PSL(2, p)$, where p is a prime with $p > 3$, $p^2 - 1 \equiv 0 \pmod 5$ and $p \equiv 3$ or $5 \pmod 8$.

In fact, suppose that $G \cong PSL(2, q)$, where $q > 3$ and $q \equiv 3$ or $5 \pmod 8$. Let $q = p^f$, where p is a prime.

Let $p > 3$. If $f > 1$, then $PSL(2, p^f)$ contains a nonsolvable proper subgroup $PSL(2, p)$. Moreover, all proper subgroups of G are solvable, this is a contradiction. Thus $f = 1$.

If $p^2 - 1 \equiv 0 \pmod 5$. By [6, II, 8.27], $PSL(2, p)$ contains a nonsolvable subgroup A_5. This is also a contradiction. So $p^2 - 1 \not\equiv 0 \pmod 5$.

(ii-2) $G \cong PSL(2, p)$ satisfies the hypotheses, where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$ and $p \equiv 3$ or $5 \pmod 8$.

The proof of (ii-2) is similar to that of (c-2) of Theorem 3.5. So all second maximal subgroups of $PSL(2, 2^f)$ are PSC^*-groups. This proves (ii).

(iii) (iii-1) $G \not\cong PSL(2, 3^f)$, where f is an even or a composite and $3^f \equiv 3$ or $5 \pmod 8$.

In fact, suppose that $G \cong PSL(2, q)$, where $q > 3$ and $q \equiv 3$ or $5 \pmod 8$. Let $q = p^f$, where p is a prime.

Let $p = 3$. Since $PSL(2, 9)$ contains a nonsolvable proper subgroup A_5, f cannot be an even. So f is an odd number.

If f is an odd composite, let $f = mn$, where m is a prime with $m < f$. By [6, II, 8.27], we know $PSL(2, p^f)$ contains a nonsolvable proper subgroup $PSL(2, p^m)$, which is a contradiction. Thus f is an odd prime.

(iii-2) $G \cong PSL(2, 3^f)$ satisfies the hypotheses, where f is an odd prime, $3^f \equiv 3$ or $5 \pmod 8$. The proof of (iii-2) is similar to that of (d-2) of Theorem 3.5. This proves (iii).

(iv) (iv-1) $G \ncong SL(2,p)$, where p is a prime with $p > 3$, $p^2 - 1 \equiv 0 \pmod 5$ and $p \equiv 3$ or $5 \pmod 8$.

In fact, $SL(2,p)$ has only a subgroup of order 2, let it be $\langle u\rangle$ and $Z = \langle u\rangle$. If $G \cong SL(2,p)$, then $G/Z \cong PSL(2,p)$. Similarly to the proof of (ii-1) of (ii), we obtain $p^2 - 1 \not\equiv 0 \pmod 5$.

(iv-2) $G \cong SL(2,p)$ satisfies the hypotheses, where p is an odd prime with $p > 3$, $p^2 - 1 \equiv 0 \pmod 5$ and $p \equiv 3$ or $5 \pmod 8$.

In fact, if $G \cong SL(2,p)$, then $G/Z \cong PSL(2,p)$. Since the subgroup $\langle u\rangle$ of $SL(2,p)$ of order 2 is unique, $\langle u\rangle \trianglelefteq SL(2,p)$ and $\langle u\rangle$ is self-conjugate-permutable. Moreover, Sylow 2-groups of $SL(2,p)$ are isomorphic to Q_8. Let $\langle v\rangle$ be an arbitrary cyclic subgroup of $SL(2,p)$ of order 4. Then $Z < \langle v\rangle$. Let V be a second maximal subgroup of $SL(2,p)$ and contain $\langle v\rangle$. Therefore $\langle v\rangle/Z$ is self-conjugate-permutable in V/Z. Hence $\langle v\rangle$ is self-conjugate-permutable in V by Lemma 2.1(2). It follows that all second maximal subgroups of $SL(2,p)$ are PSC^*-groups. This proves (iv).

(v) Similarly to the proof of (iii) and (iv), we can obtain (v). $\square$

References

[1] LI Shirong. *Finite groups in which the cyclic subgroups of orders 2 and 4 of the second maximal subgroups are quasinormal* [J]. Acta Math. Sinica, 1994, **37**(3): 317–323. (in Chinese)

[2] LI Shirong, ZHAO Yaoqing. *Some finite nonsolvable groups characterized by their solvable subgroups* [J]. Acta Math. Sinica (N.S.), 1988, **4**(1): 5–13.

[3] LI Shirong, SHEN Zhencai, KONG Xianghong. *Finite groups with self-conjugate-permutable subgroups* [J]. Joural of Pure and Applied Algebra. In Press.

[4] REN Yongcai. *Finite simple groups all of whose 2-maximal subgroups are PQN-groups* [J]. Acta Math. Sinica, 1990, **33**(6): 798–803. (in Chinese)

[5] XU Mingyao. *Introduction to the Theory of Finite Groups* [M]. Beijing: Science Press, 1999.

[6] HUPPERT B. *Endliche Gruppen* [M]. Springer-Verlag, Berlin-New York, 1967.

[7] SASTRY N. *On minimal non-PN-groups* [J]. J. Algebra, 1982, **65**: 104–109.

[8] GORENSIEIN D. *Finite Groups* [M]. NewYork, 1968.

[9] HUPPERT B, BLACKBURN N. *Finite Groups II* [M]. Springer-Verlag, Berlin-New York, 1982.

[10] FOGUEL T. *Conjugate-permutable subgroups* [J]. J. Algebra, 1997, **191**(1): 235–239.