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Left Δ-Product Structure of Left C-Wrpp Semigroups

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Abstract We study another structure of so-called left C-wrpp semigroups. In particular, the concept of left Δ-product is extended and enriched. The aim of this paper is to give a construction of left C-wrpp semigroups by a left regular band and a strong semilattice of left-$\mathcal{R}$ cancellative monoids. Properties of left C-wrpp semigroups endowed with left Δ-products are particularly investigated.

Keywords left C-wrpp semigroup; left regular band; left Δ-product.

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1. Introduction

Clifford semigroups play an important role in the theory of regular semigroups. Many authors have extensively investigated the generalizations of Clifford semigroups and have obtained plenty of results[1−10]. Clifford semigroups have been extended to left C-semigroups by Zhu, Guo and Shum[1], and Guo introduced the concept of left Δ-product of semigroups in paper[2]. Clifford semigroups have been extended to weakly left C-semigroups by Guo[3] in 1996. One of the most significant generalizations of Clifford semigroups was investigated by Fountain[4] when he introduced the concept of rpp monoids with central idempotents. The left C-rpp semigroups were studied by Guo[5], who obtained the semi-spined product structure of left C-rpp semigroups. Cao[8] obtained another structure of left C-rpp semigroups in terms of left Δ-product.

Tang[9] introduced Green's ** relations on a semigroup, and by using this new Green's relations, he defined the concepts of wrpp semigroups and C-wrpp semigroups. We have known that a C-wrpp semigroup can be expressed as a strong semilattice of a left-$\mathcal{R}$ cancellative monoids. C-wrpp semigroups have been extended to left C-wrpp semigroups by Du and Shum[10], and they obtained a description of curler structure of left C-wrpp semigroups.

In this paper, we generalize the concept of left Δ-product, and obtain left Δ-product structure of left C-wrpp semigroups. In Section 2, some basic results concerning left C-wrpp semigroups are recalled. In Section 3, a construction of left C-wrpp semigroups is given by a left regular band and a strong semilattice of left-$\mathcal{R}$ cancellative monoids, and some properties of left C-wrpp semigroups endowed with left Δ-products are obtained.

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2. Preliminaries

Throughout this paper, the terminologies and notations that are not defined can be found in [2] and [10].

The relation $\mathcal{L}^{**}$ on a semigroup S is defined by the rule that $a\mathcal{L}^{**}b$ if and only if $ax\mathcal{R}ay \Leftrightarrow bx\mathcal{R}by$ for any $x, y \in S^1$, where $\mathcal{R}$ is the usual Green's $\mathcal{R}$ relation on S. A semigroup S is called a wrpp semigroup if each $\mathcal{L}^{**}$-class of S contains at least one idempotent.

The following results are due to Du and Shum[10].

Definition 2.1 *A semigroup S is called a quasi-strong wrpp semigroup if for all $e \in E(L_a^{**})$, we have $a = ae$, where $E(L_a^{**})$ is the set of idempotents in L_a^{**}.*

Definition 2.2 *A quasi-strong wrpp semigroup S is called a strong wrpp semigroup if for all $a \in S$, there exists a unique idempotent a^+ satisfying $a\mathcal{L}^{**}a^+$ and $a = a^+a$.*

Note 2.3 It was noticed that so called strong wrpp semigroups are exactly adequate wrpp semigroups which were called by Du and Shum[10].

Definition 2.4 *A strong wrpp semigroup S satisfying $aS \subseteq L^+(a)$ for all $a \in S$ is called a left C-wrpp semigroup.*

Definition 2.5 *A monoid M is called a left-$\mathcal{R}$ cancellative monoid if for any $a, b, c \in M$, $(ca, cb) \in \mathcal{R}$ implies $(a, b) \in \mathcal{R}$. We call the direct product of a left zero I and a left-$\mathcal{R}$ cancellative monoid M a left-$\mathcal{R}$ cancellative stripe. We denote the left-$\mathcal{R}$ cancellative stripe by $I \times M$.*

Lemma 2.6 *Let S be a strong wrpp semigroup. Then the following conditions are equivalent:*

1) *S is a left C-wrpp semigroup;*
2) *$E(S)$ is a left regular band and $\mathcal{L}^{**}$ is a congruence on S;*
3) *S is semilattice of left-$\mathcal{R}$ cancellative stripes.*

3. Left $\triangle$-product of left C-wrpp semigroups

In this section, the concept of left $\triangle$-product of semigroups is introduced. We shall show that the structure of a left C-wrpp semigroup can be described by the left Δ-product of semigroups.

We let Y be a semilattice and $T = [Y; T_\alpha, \theta_{\alpha,\beta}]$ a strong semilattice of left-$\mathcal{R}$ cancellative monoids T_α. Let I be a left regular band which is a semilattice of left zero bands I_α, denoted by $I = \cup_{\alpha\in Y} I_\alpha$. For every $\alpha \in Y$, we form the Cartesian product $S_\alpha = I_\alpha \times T_\alpha$.

Now, for any $\alpha, \beta \in Y$ with $\alpha \geq \beta$ and the left transformation semigroup $\mathcal{T}^*(I_\beta)$, we define a mapping

$$\Phi_{\alpha,\beta} : S_\alpha \to \mathcal{T}^*(I_\beta),\ a \mapsto \varphi^a_{\alpha,\beta},$$

where all $\Phi_{\alpha,\beta}$ satisfy the following conditions:

(Q_1) If $(i, u) \in S_\alpha$, $i' \in I_\alpha$, then $\varphi^{(i,u)}_{\alpha,\alpha} i' = i$;

(Q_2) For any $(i, u) \in S_\alpha, (j, v) \in S_\beta$, we consider the following situation respectively:

(a) $\varphi^{(i,u)}_{\alpha,\alpha\beta}\varphi^{(j,v)}_{\beta,\alpha\beta}$ is a constant mapping on $I_{\alpha\beta}$ and we denote the constant value by $\langle\varphi^{(i,u)}_{\alpha,\alpha\beta}\varphi^{(j,v)}_{\beta,\alpha\beta}\rangle$;

(b) If $\alpha,\beta,\delta\in Y$ with $\alpha\beta\geq\delta$ and $\langle\varphi^{(i,\,u)}_{\alpha,\,\alpha\beta}\varphi^{(j,\,v)}_{\beta,\,\alpha\beta}\rangle=k$, then we have

$$\varphi^{(k,\,uv)}_{\alpha\beta,\,\delta}=\varphi^{(i,\,u)}_{\alpha,\,\delta}\varphi^{(j,\,v)}_{\beta,\,\delta},$$

where uv is the product of u and v in T, i.e., $uv=u\theta_{\alpha,\,\alpha\beta}\cdot v\theta_{\beta,\,\alpha\beta}$;

(c) If $\varphi^{(g,\,w)}_{\gamma,\,\gamma\alpha}\varphi^{(i,\,u)}_{\alpha,\,\gamma\alpha}=\varphi^{(g,\,w)}_{\gamma,\,\gamma\beta}\varphi^{(j,\,v)}_{\beta,\,\gamma\beta}$ for any $(g,w)\in S_\gamma$, then $\varphi^{(g,\,1_\gamma)}_{\gamma,\,\gamma\alpha}\varphi^{(i,\,u)}_{\alpha,\,\gamma\alpha}=\varphi^{(g,\,1_\gamma)}_{\gamma,\,\gamma\beta}\varphi^{(j,\,v)}_{\beta,\,\gamma\beta}$, where 1_γ is the identity of the monoid T_γ.

We now form the set union $S=\cup_{\alpha\in Y}S_\alpha$ and define a multiplication "$\circ$" on S by

$$(i,u)(j,v)=(\langle\varphi^{(i,\,u)}_{\alpha,\,\alpha\beta}\varphi^{(j,\,v)}_{\beta,\,\alpha\beta}\rangle,uv). \tag{*}$$

After straightforward verification, we can verify that the multiplication "$\circ$" satisfies the associative laws, and hence $(S,\circ)$ is a semigroup. We denote this semigroup $(S,\circ)$ by $S=I\bigtriangleup_\Phi T$ and call it the left Δ-product of the semigroups I and T with respect to Y, under the structure mapping $\Phi_{\alpha,\,\beta}$.

We shall establish a construction theorem for left C-wrpp semigroups.

Theorem 3.1 *Let T be a C-wrpp semigroup, i.e., $T=[Y;T_\alpha,\theta_{\alpha,\,\beta}]$ is a strong semilattice of left-$\mathcal{R}$ cancellative monoids T_α with structure homomorphism $\theta_{\alpha,\,\beta}$. Let left regular band $I=\cup_{\alpha\in Y}I_\alpha$ be a semilattice decomposition of left zero bands I_α. Then $I\Delta_\Phi T$, the left Δ-product of I and T, is a left C-wrpp semigroup. Conversely, any left C-wrpp semigroup can be constructed by using the left Δ-product of a left regular band and a strong semilattice of left-$\mathcal{R}$ cancellative monoids of the above form.*

In order to prove Theorem 3.1, we need the following lemma:

Lemma 3.2 *Let $I=\cup_{\alpha\in Y}I_\alpha$ be a semilattice of left zero bands I_α and $T=[Y;T_\alpha,\theta_{\alpha,\,\beta}]$ a strong semilattice of left- $\mathcal{R}$ cancellative monoids T_α. Let $S_\alpha=I_\alpha\times T_\alpha, S=I\Delta_\Phi T$. Then the following statements hold:*

1) $E(S)=\cup_{\alpha\in Y}(I_\alpha\times\{1_\alpha\})$, where 1_α denotes the identity of T_α;

2) $(i,a)\mathcal{R}(j,b)$ if and only if $a\mathcal{R}b$ and $i=j$ for any $(i,a),(j,b)\in S=\cup_{\alpha\in Y}(I_\alpha\times T_\alpha)$.

Proof 1) Let $(i,1_\alpha)\in I_\alpha\times\{1_\alpha\}$. by the multiplication given in (*) and by the condition (Q_1) described in above definition, we have

$$(i,1_\alpha)(i,1_\alpha)=(\langle\varphi^{(i,\,1_\alpha)}_{\alpha,\,\alpha}\varphi^{(i,\,1_\alpha)}_{\alpha,\,\alpha}\rangle,1_\alpha)=(i,1_\alpha).$$

On the other hand, if $(i,a)\in E(S)$, then there exists $\alpha\in Y$ such that $(i,a)\in E(S_\alpha)$ and $a^2=a\in T_\alpha$. Since $1_\alpha^2=1_\alpha\in T_\alpha$ and T_α is a monoid, we have $1_\alpha a=a$. Therefore, $a^2\mathcal{R}a1_\alpha$. Again, since T_α is a left-$\mathcal{R}$ cancellative monoid, so $a\mathcal{R}1_\alpha$. According to that $\mathcal{R}$ is a left congruence on T, we have $1_\alpha a\mathcal{R}1_\alpha$. Hence there exists $u\in T_\alpha$ such that $1_\alpha=1_\alpha au$. So we have $a=1_\alpha a1_\alpha=1_\alpha a1_\alpha au=1_\alpha a^2u=1_\alpha au=1_\alpha$. Therefore (1) holds.

2) Let $(i,a),(j,b)\in S$ such that $(i,a)\mathcal{R}(j,b)$. Then there exist $(k,u),(l,v)\in S^1$ such that $(i,a)=(j,b)(k,u),(j,b)=(i,a)(l,v)$. But this is equivalent to saying that there exists $\alpha\in Y$ such that $(i,a),(j,b)\in S_\alpha=I_\alpha\times T_\alpha$ and such that $i\mathcal{R}(I_\alpha)j$ and $a\mathcal{R}(T_\alpha)b$. The latter holds if

and only if $i = j$ and $a\mathcal{R}b$. So (2) holds.

Now we verify Theorem 3.1.

Proof We first show that the direct part of theorem. To show that $S = I\Delta_\Phi T$ is a left C-wrpp semigroup, according to Lemma 2.6, we need to show that S is a strong wrpp semigroup and $E(S)$ is a left regular band, and $\mathcal{L}^{**}$ is a congruence of S. By Lemma 3.2(1), we know that $E(S) = \cup_{\alpha\in Y}(I_\alpha \times \{1_\alpha\})$ is a left regular band. If $S_\alpha = I_\alpha \times T_\alpha$ is an $\mathcal{L}^{**}$-class, then $\mathcal{L}^{**}$ is a congruence of S. According to the multiplication of S_α, we have $ae_\alpha = a$ for any $e_\alpha = (i, 1_\alpha) \in E(S_\alpha), (i, a) \in S_\alpha$, and $(i, 1_\alpha)$ is a unique idempotent such that $(i, 1_\alpha)(i, a) = (i, a)$. Since for any $(k, 1_\alpha) \in E(S_\alpha)$, we have $(k, 1_\alpha)(i, a) = (k, a)$, $k = i$ is obtained. Thus S is a strong wrpp semigroup. We can deduce that S is a left C-wrpp semigroup. For this purpose, we only prove that S_α is an $\mathcal{L}^{**}$-class of S. Let $(i, a), (j, b) \in S_\alpha = I_\alpha \times T_\alpha$. If for any $(k, u), (l, v) \in S^1$, we have $(i, a)(k, u)\mathcal{R}(i, a)(l, v)$, then $(\langle\varphi^{(i,a)}_{\alpha,\alpha\gamma}\varphi^{(k,u)}_{\gamma,\alpha\gamma}\rangle, au)\mathcal{R}(\langle\varphi^{(i,a)}_{\alpha,\alpha\lambda}\varphi^{(l,v)}_{\lambda,\alpha\lambda}\rangle, av)$, and we obtain $\langle\varphi^{(i,a)}_{\alpha,\alpha\gamma}\varphi^{(k,u)}_{\gamma,\alpha\gamma}\rangle = \langle\varphi^{(i,a)}_{\alpha,\alpha\lambda}\varphi^{(l,v)}_{\lambda,\alpha\lambda}\rangle$ and $au\mathcal{R}av$. Since $T = [Y; T_\alpha, \theta_{\alpha,\beta}]$ is a C-wrpp semigroup, T_α is an $\mathcal{L}^{**}$-class. Now, $au\mathcal{R}av$ if and only if $bu\mathcal{R}bv$. Again by condition (c) of (Q_2), we can deduce that $\langle\varphi^{(j,b)}_{\alpha,\alpha\gamma}\varphi^{(k,u)}_{\gamma,\alpha\gamma}\rangle = \langle\varphi^{(j,b)}_{\alpha,\alpha\lambda}\varphi^{(l,v)}_{\lambda,\alpha\lambda}\rangle$. Consequently, we have $(j, b)(k, u)\mathcal{R}(j, b)(l, v)$. From this result and its dual, we immediately obtain $(i, a)\mathcal{L}^{**}(j, b)$. On the other hand, if for any $(i, a) \in S_\alpha, (j, b) \in S_\beta$, and such that $(i, a)\mathcal{L}^{**}(j, b)$, then we have $(j, b)(i, 1_\alpha)\mathcal{R}(j, b)$ since $(i, a)(i, 1_\alpha) = (i, a)$. So $\langle\varphi^{(j,b)}_{\beta,\beta\alpha}\varphi^{(i,a)}_{\alpha,\beta\alpha}\rangle = j$. Thus we get $\alpha \geq \beta$. Similarly, we also can show that $\alpha \leq \beta$. Consequently, $\alpha = \beta$. Hence, S_α is an $\mathcal{L}^{**}$-class of S. Therefore, according to Lemma 2.6, S is a left C-wrpp semigroup.

Next we show the converse part of this theorem. We first assume that S is an arbitrary left C-wrpp semigroup and what we will do is to construct a left Δ-product $I\Delta_\Phi T$ which is isomorphic to S. In fact, by Lemma 2.6(3), there exists a semilattice Y of semigroups $S_\alpha = I_\alpha \times T_\alpha$, where I_α is a left zero band and T_α is a left-$\mathcal{R}$ cancellative monoid. Let $I = \cup_{\alpha\in Y}I_\alpha$ and $T = \cup_{\alpha\in Y}T_\alpha$.

In order to prove that left Δ-product $I\Delta_\Phi T$ is isomorphic to S, we have to go through the following steps:

Firstly, we verify that I is a left regular band. For this purpose, we shall show that I is isomorphic to $E(S)$ which is the set of all idempotents of S. Now, we define an operation "$\circ$" as follows:

For any $i \in I_\alpha$, $j \in I_\beta$,

$$ij = k \quad \text{if and only if } (i, 1_\alpha) \circ (j, 1_\beta) = (k, 1_{\alpha\beta}) \tag{1}$$

where $(i, 1_\alpha) \in I_\alpha \times T_\alpha$ and 1_α is the identity of T_α. Thus, I forms a regular band with respect to the above operation. Since S is a left C-wrpp semigroup, the mapping $\eta : E(S) \to I = \cup_{\alpha\in Y}I_\alpha$ defined by $(i, 1_\alpha) \mapsto i$ is clearly bijective. Then, we easily see that I is isomorphic to $E(S)$, and hence, I is a semilattice of left zero bands I_α. We call I the left regular band component of the left C-wrpp semigroup S.

Next, we shall claim that T is a strong semilattice of left-$\mathcal{R}$ cancellative monoids T_α.

(i) $(j, 1_\beta)(i, 1_\alpha)$, for $\alpha \geq \beta$.

Notice that $\alpha \geq \beta$ and I_β is a left zero band, by Eq.(1), we have

$$\begin{aligned}(ji, 1_\beta) &= (j, 1_\beta)(1, 1_\alpha) = [(j, 1_\beta)(j, 1_\beta)](i, 1_\alpha)\\ &= (j, 1_\beta)(ji, 1_\beta) = (j, 1_\beta).\end{aligned}$$

So, for any $\alpha,\ \beta \in Y$ with $i \in I_\alpha,\ j \in I_\beta$, we have

$$(j, 1_\beta)(i, 1_\alpha) = (j, 1_\beta). \tag{2}$$

(ii) $(j, 1_\beta)(1, a)$, where $\alpha \geq \beta$ and $(i, a) \in I_\alpha \times T_\alpha$.

Let $(j, 1_\beta)(i, a) = (j_1, a^*_{ij})$. Notice that $(j, 1_\beta)[(j, 1_\beta)(i, a)] = (j, 1_\beta)(i, a)$, I_β is a left zero band and $I_\beta \times T_\beta$ is a direct product. We obtain $j_1 = j$ and

$$\begin{aligned}(j^{'}, a^*_{ij}) &= (j^{'}, 1_\beta)(i, a) = [(j^{'}, 1_\beta)(j, 1_\beta)](i, a)\\ &= (j^{'}, 1_\beta)(j, a^*_{ij}) = (j^{'}, a^*_{ij}).\end{aligned}$$

This shows that a^*_{ij} does not depend on the choice of j in I_β. Also, by Eq.(2), we have

$$\begin{aligned}(j, a^*_{ij}) &= (j, 1_\beta)(i^{'}, a) = (j, 1_\beta)[(i^{'}, 1_\alpha)(i, a)]\\ &= (j, 1_\beta)(i, a) = (j, a^*_{ij}).\end{aligned}$$

Therefore, a^*_{ij} does not depend on the choice of i in I_α either. So, for any $\alpha,\ \beta \in Y$ with $\alpha \geq \beta$ and $i \in I_\alpha, j \in I_\beta$, we have

$$(j, 1_\beta)(i, a) = (j, a^*). \tag{3}$$

By Eq.(3), we define a mapping

$$\theta_{\alpha,\,\beta} : T_\alpha \to T_\beta,\ a \mapsto a^* = a\theta_{\alpha,\,\beta},\ (\alpha \geq \beta).$$

By routing checking, all the mapping

$$\{\theta_{\alpha,\beta} | \alpha, \beta \in Y, \alpha \geq \beta\}$$

are indeed the structure homomorphisms of a strong semilattice Y of monoids. Thus, $T = [Y; T_\alpha, \theta_{\alpha,\,\beta}]$ is a strong semilattice of T_α.

Finally, we show that the mapping $\Phi_{\alpha,\,\beta}$ for left Δ-product satisfies the conditions (Q_1) and (Q_2) as stated above so that $I\Delta_\Phi T$ is the required left Δ-product. For this purpose, we let $\alpha, \beta \in Y$ with $\alpha \geq \beta$. Then for any $(i, a) \in S_\alpha = I_\alpha \times T_\alpha, (j, 1_\beta) \in E(S_\beta)$, we have $(i, a)(j, 1_\beta) = (k, a^{'})$ for some $a^{'} \in T_\beta$ and $k \in I_\beta$. This leads to

$$(k, 1_\beta)(i, a)(j, 1_\beta) = (k, a\theta_{\alpha,\,\beta})(j, 1_\beta) = (k, a\theta_{\alpha,\,\beta}),$$

and hence, we have

$$(i, a)(j, 1_\beta) = (k, a\theta_{\alpha,\,\beta}) \in S_\beta. \tag{4}$$

Now by Eq.(4), we can easily deduce a mapping $\Phi_{\alpha,\,\beta}$ which maps S_α to the left transformation semigroup $\mathcal{T}^*(I_\beta)$, say,

$$\Phi_{\alpha,\,\beta} : (i, a) \mapsto \varphi^{(i,a)}_{\alpha,\,\beta},$$

where $\varphi^{(i,a)}_{\alpha,\,\beta}$ is defined by

$$(i,a)(j,1_\beta)=(\varphi^{(i,a)}_{\alpha,\,\beta}j,a\theta_{\alpha,\,\beta}). \tag{5}$$

We now verify that the conditions (Q_1) and (Q_2) for left Δ-product are satisfied. We consider the following cases:

(i) To show that $\Phi_{\alpha,\,\beta}$ satisfies the condition (Q_1) in above definition of left Δ-product, we let $(i,a)\in S_\alpha$ and $i^{'}\in I_\alpha$. Then, by Eq.(5), we have

$$(i,a)(i^{'},1_\alpha)=(\varphi^{(i,\,a)}_{\alpha,\,\alpha}i^{'},a).$$

Since $S_\alpha=I_\alpha\times T_\alpha$ and I_α is a left zero band, we have $(i,a)(i^{'},1_\alpha)=(i,a)$. This implies that $\varphi^{(i,\,a)}_{\alpha,\,\alpha}i^{'}=i$. Therefore, the condition (Q_1) is satisfied.

(ii) To show that $\Phi_{\alpha,\,\beta}$ satisfies the condition (Q_2), we let $(i,a)\in S_\alpha$ and $(j,b)\in S_\beta$ for any $\alpha,\beta\in Y$. Then, by Eq.(5), for any $(\lambda,1_{\alpha\beta})\in E(S_{\alpha\beta})$, we have

$$\begin{aligned}(i,a)[(j,b)(\lambda,1_{\alpha\beta})]=&(i,a)(\varphi^{(j,b)}_{\beta,\,\alpha\beta}\lambda,b\theta_{\beta,\,\alpha\beta})\\=&(i,a)(\varphi^{(j,\,b)}_{\beta,\,\alpha\beta}\lambda,1_{\alpha\beta})(\varphi^{(j,\,b)}_{\beta,\,\alpha\beta}\lambda,b\theta_{\beta,\,\alpha\beta})\\=&(\varphi^{(i,\,a)}_{\alpha,\,\alpha\beta}\varphi^{(j,\,b)}_{\beta,\,\alpha\beta}\lambda,a\theta_{\beta,\,\alpha\beta})(\varphi^{(j,\,b)}_{\beta,\,\alpha\beta}\lambda,b\theta_{\beta,\,\alpha\beta})\\=&(\varphi^{(i,\,a)}_{\alpha,\,\alpha\beta}\varphi^{(j,\,b)}_{\beta,\,\alpha\beta}\lambda,a\theta_{\alpha,\,\alpha\beta}b\theta_{\beta,\,\alpha\beta}).\end{aligned} \tag{6}$$

On the other hand, we also have

$$[(i,a)(j,b)](\lambda,1_{\alpha\beta})=(\bar{k},\bar{a})(\lambda,1_{\alpha\beta})=(\bar{k},\bar{a}). \tag{7}$$

By comparing Eq.(6) with (7), we have

$$\bar{k}=\varphi^{(i,\,a)}_{\alpha,\,\alpha\beta}\varphi^{(j,\,b)}_{\beta,\,\alpha\beta}\lambda,$$

which implies that $\varphi^{(i,\,a)}_{\alpha,\,\alpha\beta}\varphi^{(j,\,b)}_{\beta,\,\alpha\beta}$ is a constant value mapping on $I_{\alpha\beta}$. Thus, condition (a) of (Q_2) is satisfied. By using similar arguments, we can also show that (b) of (Q_2) is satisfied.

To see that $\Phi_{\alpha,\,\beta}$ satisfies condition (c) of (Q_2), we recall that S is a left C-wrpp semigroup. Thus, if $ax\mathcal{R}ay$ for any $a\in S$ and $x,y\in S^1$, then there exists an idempotent $e\in S$ such that $ex\mathcal{R}ey$ and $a=ae$. By writing $a=(i,u)\in S_\alpha$ and $e=(i,1_\alpha)\in E(S)$, we can verify that $\Phi_{\alpha,\,\beta}$ satisfies condition (c) of (Q_2).

Therefore, $I\Delta_\Phi T$ is indeed a left Δ-product of I and T.

It remains to show that the left C-wrpp semigroup S is isomorphic to $I\Delta_\Phi T$. To this end, it suffices to prove that the multiplication on S is compatible with the multiplication on $I\Delta_\Phi T$. Since for any $(i,a)\in S_\alpha,(j,b)\in S_\beta$, we clearly have

$$(i,a)(j,b)=(\bar{k},\bar{a})\in S_{\alpha\beta}.$$

Hence, for any $(k,1_{\alpha\beta})\in E(S_{\alpha\beta})$, we have

$$(i,a)(j,b)=(i,a)(j,b)(k,1_{\alpha\beta}).$$

Then, by using the same arguments as step (ii) which are used to verify that the conditions (Q_1)

and (Q_2) are satisfied for left Δ-product, we have

$$(i,a)(j,b) = (\langle \varphi^{(i,a)}_{\alpha,\alpha\beta}\varphi^{(j,b)}_{\beta,\alpha\beta}\rangle, ab).$$

This shows that multiplication on S coincides with the multiplication on the left Δ-product $I\Delta_\Phi T$. Therefore, $S \cong I\Delta_\Phi T$. The proof is completed. □

In what follows, we give some properties on left C-wrpp semigroups endowed with left Δ-product structure.

Theorem 3.3 *Let $S = I\Delta_\Phi T = \cup_{\alpha\in Y}(I_\alpha \times T_\alpha)$ be a left C-wrpp semigroup, where $I = \cup_{\alpha\in Y} I_\alpha$ is the semilattice decomposition of the left regular band I into the left zero bands I_α on the semilattice Y, and $T = [Y; T_\alpha, \theta_{\alpha,\beta}]$ is a strong semilattice of the left-$\mathcal{R}$ cancellative monoids T_α. Then $x\mathcal{L}^{**}_S y$ if and only if $\alpha = \beta$ for any $x = (i,a) \in S_\alpha$ and $y = (j,b) \in S_\beta$.*

Proof It is easy to see that the result holds since each S_α is an $\mathcal{L}^{**}$-class of the semigroup S.

Theorem 3.4 *Let $S = I\Delta_\Phi T$ be a left C-wrpp semigroup. Then the following statements are equivalent:*

1) If $(i,a) \in S_\alpha$, and $(j,b) \in S_\beta$ for $\alpha,\beta \in Y$ and $\alpha \geq \beta$, then $\langle \varphi^{(i,a)}_{\alpha,\alpha\beta}\varphi^{(j,b)}_{\beta,\alpha\beta}\rangle = ij$;

2) $I\Delta_\Phi T$ and the spined product of the semigroup $I = \cup_{\alpha\in Y} I_\alpha$ and the semigroup $T = \cup_{\alpha\in Y} T_\alpha$ are equivalent.

Proof Let $S = I\Delta_\Phi T$ be a left C-wrpp semigroup. Recall the definition of $\Phi_{\alpha,\beta}$, we define multiplication operation "$\circ$" as follows:

$$\forall (i,a) \in I_\alpha \times T_\alpha, (j,b) \in I_\beta \times T_\beta, (i,a)\circ(j,b) = (\langle \varphi^{(i,a)}_{\alpha,\alpha\beta}\varphi^{(j,b)}_{\beta,\alpha\beta}\rangle,\ a\theta_{\alpha,\alpha\beta}b\theta_{\beta,\alpha\beta}).$$

Then above left Δ-product of semigroup is a spined product if and only if for any $\alpha,\beta \in Y, (i,a) \in I_\alpha \times T_\alpha, (j,b) \in I_\beta \times T_\beta$ and $\alpha \geq \beta$, we have $\langle \varphi^{(i,a)}_{\alpha,\alpha\beta}\varphi^{(j,b)}_{\beta,\alpha\beta}\rangle = ij$. So 1), 2) are equivalent.

Theorem 3.5 *Let $S = I\Delta_\Phi T = \cup_{\alpha\in Y}(I_\alpha \times T_\alpha)$ be a left Δ-product. Then the following statements are equivalent:*

1) S is a strong semilattice of the semigroups $I_\alpha \times T_\alpha$;

2) For any $a \in T_\alpha, \alpha,\beta \in Y$, if $\alpha \geq \beta$, then $\varphi^a_{\alpha,\beta}$ is constant;

3) $E(S)$ is a left normal band of S.

Proof 1)⇔3) is similar to the proof of [5, Theroem 4.2], and we omit it. Next we only show that 1) ⇔ 2).

1) ⇒ 2) If $S = [Y; S_\alpha, \psi_{\alpha,\beta}]$ is a strong semilattice of left-$\mathcal{R}$ cancellative semigroups $I_\alpha \times T_\alpha$ with structure homomorphisms $\psi_{\alpha,\beta}$, for any $\alpha,\beta \in Y, \alpha \geq \beta$, and $a = (i,u) \in S_\alpha, (j,1_\beta) \in S_\beta \cap E$, we have $(i,u)\psi_{\alpha,\beta} \in S_\beta$, written as (k,v), and

$$(i,u)(j,1_\beta) = (i,u)\psi_{\alpha,\beta}(j,1_\beta)\psi_{\alpha,\beta} = (k,v)(j,1_\beta) = (k,v).$$

But

$$(i,u)(j,1_\beta) = (\varphi^{(i,u)}_{\alpha,\beta} j, u\theta_{\alpha,\beta}).$$

So $\varphi_{\alpha,\beta}^{(i,u)} j = k$ for any $j \in I_\beta$. Thus $\varphi_{\alpha,\beta}^{a}$ is a constant mapping.

2) $\Rightarrow$ 1) Let $\alpha, \beta \in Y$ with $\alpha \geq \beta$, and $a = (i,u) \in S_\alpha$. If $\varphi_{\alpha,\beta}^{(i,u)}$ is constant, for any $(j, 1_\beta) \in S_\beta \cap E(S)$, then $\Psi_{\alpha,\beta}$ is defined by the following rule:

$$\Psi_{\alpha,\beta}: \quad I_\alpha \times T_\alpha \to I_\beta \times T_\beta, \quad (i,u) \mapsto (i,u)\Psi_{\alpha,\beta} = (i,u)(j), 1_\beta) = (\varphi_{\alpha,\beta}^{(i,u)} j, u\theta_{\alpha,\beta}).$$

$\Psi_{\alpha,\beta}$ is clearly a semigroup homomorphism.

If $\alpha = \beta$, and $(i,u) \in S_\alpha, (i', 1_\alpha) \in S_\alpha \cap E(S)$, then

$$(i,u)\Psi_{\alpha,\alpha} = (i,u)(i', 1_\alpha) = (i,u).$$

Thus each $\Psi_{\alpha,\alpha}$ is an identity mapping on $I_\alpha \times T_\alpha$.

If $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$, and $(i,u) \in S_\alpha$, for any $(j, 1_\beta) \in S_\beta \cap E(S)$, $(k, 1_\gamma) \in S_\gamma \cap E(S)$, we have

$$\begin{aligned} (i,u)\Psi_{\alpha,\beta}\Psi_{\beta,\gamma} &= [(i,u)(j,1_\beta)](k,1_\gamma) = (i,u)[(j,1_\beta)(k,1_\gamma)] \\ &= (i,u)(\overline{k}, 1_\gamma) = (i,u)\Psi_{\alpha,\gamma}, \end{aligned}$$

since $(j,1_\beta)(k,1_\gamma) = (\overline{k}, 1_\gamma)$. So $S = [Y; I_\alpha \times T_\alpha, \Psi_{\alpha,\beta}]$ forms a strong semilattice of $I_\alpha \times T_\alpha$. $\Box$

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