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Jordan Maps on Standard Operator Algebras

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Abstract Let A be a standard operator algebra on a Banach space of dimension > 1 and B be an arbitrary algebra over Q the field of rational numbers. Suppose that $M : A \longrightarrow B$ and $M^* : B \longrightarrow A$ are surjective maps such that

$$\begin{cases} M(r(aM^*(x) + M^*(x)a)) = r(M(a)x + xM(a)), \\ M^*(r(M(a)x + xM(a))) = r(aM^*(x) + M^*(x)a) \end{cases}$$

for all $a \in A, x \in B$, where r is a fixed nonzero rational number. Then both M and M^* are additive.

Keywords Jordan maps; standard operator algebras; additivity.

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Let A and B be two associative algebras over the field Q of rational numbers, and let r be a fixed nonzero rational number. Let $M : A \longrightarrow B$ and $M^* : B \longrightarrow A$ be two maps. The ordered pair (M, M^*) is called an r-Jordan map of $A \times B$ if

$$\begin{cases} M(r(aM^*(x) + M^*(x)a)) = r(M(a)x + xM(a)), \\ M^*(r(M(a)x + xM(a))) = r(aM^*(x) + M^*(x)a) \end{cases} \tag{1}$$

for all $a \in A, x \in B$. Obviously, if $\phi : A \longrightarrow B$ is an r-Jordan map, that is, ϕ is a bijective map which satisfies that $\phi(r(ab+ba)) = r(\phi(a)\phi(b) + \phi(b)\phi(a))$ for all $a, b \in A$, then the pair (ϕ, ϕ^{-1}) is an r-Jordan map of $A \times B$.

It is an interesting problem to study the interrelation between the multiplicative and the additive structure of a ring. It is Martindale who first established a condition on a ring R_1 such that every multiplicative bijective map on R_1 is additive [8, Theorem]. Recently, the question of whether a Jordan map is additive is studied by many mathematicians [1-7]. In particular, in [6], Lu showed that every r-Jordan map on a standard operator algebra is additive. In this paper, we will extend this result to these mild r-Jordan maps.

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Throughout, X is a Banach space of dimension > 1. Denote by $B(X)$ the algebra of all linear bounded operators on X. A subalgebra of $B(X)$ is called a standard operator algebra if it contains all finite rank operators in $B(X)$. Our result in this paper is the following.

Theorem *Let X be a Banach space, $\dim X > 1$, and let $A \subset B(X)$ be a standard operator algebra. Let B be algebra over Q and $r \in Q$ be non-zero. Suppose (M, M^*) is an arbitrary r-Jordan map of $A \times B$, and both M and M^* are surjective. Then both M and M^* are additive.*

The proof will be organized in a series of lemmas. We begin with the following trivial one.

Lemma 1 *If (M, M^*) is an arbitrary r-Jordan map of $A \times B$, then $M(0) = 0$ and $M^*(0) = 0$.*

Proof Since (M, M^*) is an arbitrary r-Jordan map of $A \times B$, we have that $M(0) = M(r(0M^*(0)+ M^*(0)0)) = r(M(0)0 + 0M(0)) = 0$. Similarly, $M^*(0) = M^*(r(0M(0) + M(0)0)) = r(M^*(0)0 + 0M^*(0)) = 0$. $\square$

In the following, let $e_1 \in A$ be a fixed non-trivial idempotent operator and let $e_2 = 1 - e_1$, where 1 is the identity operator on X. Set $A_{ij} = e_i A e_j$, $i, j = 1, 2$. Then we can write $A = A_{11} \oplus A_{12} \oplus A_{21} \oplus A_{22}$. It should be mentioned that this idea is from Martindale [8]. In what follows, when we write a_{ij}, it indicates $a_{ij} \in A_{ij}$.

The following lemma can be found in [5].

Lemma 2 *Let $s = s_{11} + s_{12} + s_{21} + s_{22} \in A$.*

(i) For $t_{ij} \in A_{ij}$ $(1 \le i, j \le 2)$, we have that

$$t_{ij}s + st_{ij} = t_{ij}s_{j1} + t_{ij}s_{j2} + s_{1i}t_{ij} + s_{2i}t_{ij}.$$

(ii) If $t_{ij}s_{jk} = 0$ for every $t_{ij} \in A_{ij}$ $(1 \le i, j, k \le 2)$, then $s_{jk} = 0$. Dually, if $s_{ki}t_{ij} = 0$ for every $t_{ij} \in A_{ij}$ $(1 \le i, j, k \le 2)$, then $s_{ki} = 0$.

(iii) If $t_{ij}s + st_{ij} \in A_{ij}$ for every $t_{ij} \in A_{ij}$ $(1 \le i \ne j \le 2)$, then $s_{ji} = 0$.

(iv) If $s_{ii}t_{ii} + t_{ii}s_{ii} = 0$ for every $t_{ii} \in A_{ii}$ $(i = 1, 2)$, then $s_{ii} = 0$.

(v) If $t_{jj}s + st_{jj} \in A_{ij}$ for every $t_{jj} \in A_{jj}$ $(1 \le i \ne j \le 2)$, then $s_{ji} = 0$ and $s_{jj} = 0$. Dually, if $t_{jj}s + st_{jj} \in A_{ji}$ for every $t_{jj} \in A_{jj}$ $(1 \le i \ne j \le 2)$, then $s_{ij} = 0$ and $s_{jj} = 0$.

Lemma 3 *Both M and M^* are bijective.*

Proof It suffices to prove that M and M^* are injective. First we show that M is injective. Let $x, y \in A$ and suppose $M(x) = M(y)$. Note that A is dense in $B(X)$ under the strong operator topology. We can take a net $\{t_\alpha\} \subset A$ such that sot-$\lim_\alpha t_\alpha = 1$. For every t_α, by surjectivity of M^* there is $b_\alpha \in B$ such that $M^*(b_\alpha) = t_\alpha$ and we have by (1)

$$\begin{aligned} r(t_\alpha x + xt_\alpha) &= rM^*(b_\alpha)x + xM^*(b_\alpha) = M^*(r(b_\alpha M(x) + M(x)b_\alpha) \\ &= M^*(r(b_\alpha M(y) + M(y)b_\alpha) = r(t_\alpha y + yt_\alpha). \end{aligned}$$

Taking the limit in $r(t_\alpha x + xt_\alpha) = r(t_\alpha y + yt_\alpha)$, we get $2x = 2y$. That is, $x = y$.

Now we turn to proving the injectivity of M^*. Let $x, y \in B$ such that $M^*(x) = M^*(y)$. Since M^*M is also surjective, we can choose $s_\alpha \in A$ such that $M^*M(s_\alpha) = t_\alpha$ for all α. Then we have by (1)

$$\begin{aligned} &r(t_\alpha M^{-1}(x) + M^{-1}(x)t_\alpha) = r(M^*M(s_\alpha)M^{-1}(x) + M^{-1}(x)M^*M(s_\alpha)) \\ &\quad = M^*(r(M(s_\alpha)MM^{-1}(x) + MM^{-1}(x)M(s_\alpha))) = M^*(r(M(s_\alpha)x + xM(s_\alpha))) \\ &\quad = r(s_\alpha M^*(x) + M^*(x)s_\alpha) = r(s_\alpha M^*(y) + M^*(y)s_\alpha) \\ &\quad = M^*(r(M(s_\alpha)y + yM(s_\alpha))) = M^*(r(M(s_\alpha)MM^{-1}(y) + MM^{-1}(y)M(s_\alpha))) \\ &\quad = r(M^*M(s_\alpha)M^{-1}(y) + M^{-1}(y)M^*M(s_\alpha)) = r(t_\alpha M^{-1}(y) + M^{-1}(y)t_\alpha). \end{aligned}$$

Taking the limit in $t_\alpha M^{-1}(x)+M^{-1}(x)t_\alpha = t_\alpha M^{-1}(y)+M^{-1}(y)t_\alpha$, we get $2M^{-1}(x) = 2M^{-1}(y)$ and so $x = y$. □

Lemma 4 *The pair (M^{*-1}, M^{-1}) is an r-Jordan map of $A\times B$, that is, the maps $M^{*-1} : A \longrightarrow B$ and $M^{-1} : B \longrightarrow A$ satisfy*

$$\begin{cases} M^{*-1}(r(aM^{-1}(x) + M^{-1}(x)a)) = r(M^{*-1}(a)x + xM^{*-1}(a)), \\ M^{-1}(r(M^{*-1}(a)x + xM^{*-1}(a))) = r(aM^{-1}(x) + M^{-1}(x)a) \end{cases} \tag{2}$$

for all $a \in A, x \in B$.

Proof The first equality can follow from

$$\begin{aligned} &M^*(r(M^{*-1}(a)x + xM^{*-1}(a))) = M^*(r(M^{*-1}(a)MM^{-1}(x) + MM^{-1}(x)M^{*-1}(a))) \\ &\quad = r(M^*(M^{*-1}(a))M^{-1}(x) + M^{-1}(x)M^*(M^{*-1}(a))) \\ &\quad = r(aM^{-1}(x) + M^{-1}(x)a) = M^*(r(M^{*-1}(aM^{-1}(x) + M^{-1}(x)a))) \end{aligned}$$

and the second equality follows in a similar way. □

Lemma 5 *If $s, a, b \in A$ such that $M(s) = M(a) + M(b)$, then for all $t \in A$*

(i) $M(r(ts + st)) = M(r(ta + at)) + M(r(tb + bt))$;

(ii) $M^{*-1}(r(ts + st)) = M^{*-1}(r(ta + at)) + M^{*-1}(r(tb + bt))$.

Proof Let $t \in A$. Then by (1)

$$\begin{aligned} &M(r(ts + st)) = M(r(M^*M^{*-1}(t)s + sM^*M^{*-1}(t))) = r(M^{*-1}(t)M(s) + M(s)M^{*-1}(t)) \\ &\quad = r(M^{*-1}(t)(M(a) + M(b)) + (M(a) + M(b))M^{*-1}(t)) \\ &\quad = r(M^{*-1}(t)M(a) + M(a)M^{*-1}(t)) + r(M^{*-1}(t)M(b) + M(b)M^{*-1}(t)) \\ &\quad = M(r(M^*M^{*-1}(t)a + aM^*M^{*-1}(t))) + M(r(M^*M^{*-1}(t)b + bM^*M^{*-1}(t))) \\ &\quad = M(r(ta + at)) + M(r(tb + bt)). \end{aligned}$$

This proves (i).

Similarly to the above, it follows from the first equality of (2) that (ii) holds, completing the proof. □

Lemma 6 *For any $a_{ij} \in A_{ij}$ $(1 \le i, j \le 2)$, we have the following equalities:*

(i) $M(a_{11} + a_{ij}) = M(a_{11}) + M(a_{ij})$, $1 \le i \ne j \le 2$;

(ii) $M(a_{22} + a_{ij}) = M(a_{22}) + M(a_{ij})$, $1 \le i \ne j \le 2$;

(iii) $M^{*-1}(a_{11}+a_{ij}) = M^{*-1}(a_{11}) + M^{*-1}(a_{ij})$, $1 \le i \ne j \le 2$;
(iv) $M^{*-1}(a_{22}+a_{ij}) = M^{*-1}(a_{22}) + M^{*-1}(a_{ij})$, $1 \le i \ne j \le 2$.

Proof By Lemma 4, we only prove (i) and (ii).

Suppose that $i=1$ and $j=2$. Since M is surjective, we can find an element $s = s_{11}+s_{12}+s_{21}+s_{22} \in A$ such that

$$M(s) = M(a_{11}) + M(a_{12}).$$

For $t_{22} \in A_{22}$, we see that from Lemma 5(i)

$$\begin{aligned} M(r(t_{22}s+st_{22})) &= M(r(t_{22}a_{11}+a_{11}t_{22})) + M(r(t_{22}a_{12}+a_{12}t_{22})) \\ &= M(0) + M(r(a_{12}t_{22})) = M(r(a_{12}t_{22})). \end{aligned}$$

It follows that $t_{22}s+st_{22} = a_{12}t_{22}$ for every $t_{22} \in A_{22}$. Hence $t_{22}s+st_{22} \in A_{12}$ for every $t_{22} \in A_{22}$. Thus by Lemma 2(v), we get $s_{21}=0$ and $s_{22}=0$. Hence we have $s_{12}t_{22} = a_{12}t_{22}$. By Lemma 2(i), we get $s_{12}=a_{12}$. Thus $s = s_{11}+a_{12}$.

For $t_{12} \in A_{12}$, we see that from Lemma 5(i)

$$\begin{aligned} M(r(t_{12}s+st_{12})) &= M(r(t_{12}a_{11}+a_{11}t_{12})) + M(r(t_{12}a_{12}+a_{12}t_{12})) \\ &= M(0) + M(r(a_{11}t_{12})) = M(r(a_{11}t_{12})). \end{aligned}$$

It follows that $t_{12}s+st_{12} = a_{11}t_{12}$ for every $t_{12} \in A_{12}$. Since $s = s_{11}+a_{12}$, we have $s_{11}t_{12} = a_{11}t_{12}$ for every $t_{12} \in A_{12}$. Thus by Lemma 2(i), we have $s_{11} = a_{11}$. Consequently, $s = a_{11}+a_{12}$. This proves the first equality. The second can be proved similarly. □

Lemma 7 *For any $a_{ij}, b_{ij} \in A_{ij}$ $(1 \le i,j \le 2)$, we have the following equalities:*
(i) $M(r(a_{12}+b_{12}a_{22})) = M(ra_{12}) + M(rb_{12}a_{22})$;
(ii) $M^{*-1}(r(a_{12}+b_{12}a_{22})) = M^{*-1}(ra_{12}) + M^{*-1}(rb_{12}a_{22}))$.

Proof (i) Compute

$$a_{12}+b_{12}a_{22} = (e_1+b_{12})(a_{12}+a_{22}) = (e_1+b_{12})(a_{12}+a_{22}) + (a_{12}+a_{22})(e_1+b_{12}).$$

Then using (1) and Lemma 6, we have that

$$\begin{aligned} M(r(a_{12}+b_{12}a_{22})) &= M(r((e_1+b_{12})(a_{12}+a_{22}) + (a_{12}+a_{22})(e_1+b_{12}))) \\ &= M(r((e_1+b_{12})M^*(M^{*-1}(a_{12}+a_{22})) + M^*(M^{*-1}(a_{12}+a_{22}))(e_1+b_{12}))) \\ &= r(M(e_1+b_{12})M^{*-1}(a_{12}+a_{22}) + M^{*-1}(a_{12}+a_{22})M(e_1+b_{12})) \\ &= r((M(e_1)+M(b_{12}))(M^{*-1}(a_{12})+M^{*-1}(a_{22})) + (M^{*-1}(a_{12})+M^{*-1}(a_{22}))(M(e_1)+(b_{12}))) \\ &= r((M(e_1)M^{*-1}(a_{12}) + M^{*-1}(a_{12})M(e_1)) + r(M(e_1)M^{*-1}(a_{22}) + M^{*-1}(a_{22})M(e_1)) + \\ &\quad r(M(b_{12})M^{*-1}(a_{12}) + M^{*-1}(a_{12})M(b_{12})) + r(M(b_{12})M^{*-1}(a_{22}) + M^{*-1}(a_{22})M(b_{12})) \\ &= M(r(e_1M^*M^{*-1}(a_{12}) + M^*M^{*-1}(a_{12})e_1)) + M(r(e_1M^*M^{*-1}(a_{22}) + M^*M^{*-1}(a_{22})e_1)) + \\ &\quad M(r(b_{12}M^*M^{*-1}(a_{12}) + M^*M^{*-1}(a_{12})b_{12})) + M(r(b_{12}M^*M^{*-1}(a_{22}) + M^*M^{*-1}(a_{22})b_{12})) \\ &= M(r(e_1a_{12}+a_{12}e_1)) + M(r(e_1a_{22}+a_{22}e_1) + M(r(b_{12}a_{12}+a_{12}b_{12})) + M(r(b_{12}a_{22}+a_{22}b_{12})) \\ &= M(ra_{12}) + 2M(0) + M(rb_{12}a_{22}) = M(ra_{12}) + M(rb_{12}a_{22}). \end{aligned}$$

(ii) Lemma 4 tells us that the pair (M^{*-1}, M^{-1}) is also an r-Jordan map of $A\times B$. Therefore (ii) holds. $\Box$

Lemma 8 *For any $a_{ij} \in A_{ij}$ $(1 \le i, j \le 2)$, we have the following equalities:*

(i) $M(r(a_{21} + a_{22}b_{21})) = M(ra_{21}) + M(ra_{22}b_{21})$;

(ii) $M^{*-1}(r(a_{12} + a_{22}b_{21})) = M^{*-1}(ra_{12}) + M^{*-1}(ra_{22}b_{21}))$.

Proof (i) Compute

$$a_{12} + a_{22}b_{21} = (a_{12} + a_{22})(e_1 + b_{21}) = (e_1 + b_{21})(a_{12} + a_{22}) + (a_{12} + a_{22})(e_1 + b_{21}).$$

Then we can complete the proof using a computation similar to that in the proof of Lemma 7.$\Box$

Lemma 9 *M and M^{*-1} are additive on A_{ij} $(1 \le i \ne j \le 2)$.*

Proof By Lemma 4, we only prove that M is additive on A_{ij} $(1 \le i \ne j \le 2)$.

Let $a_{12}, b_{12} \in A_{12}$ and choose $s = s_{11} + s_{12} + s_{21} + s_{22} \in A$ such that

$$M(s) = M(a_{12}) + M(b_{12}).$$

For $t_{22} \in A_{22}$, we see that from Lemma 5(i) and Lemma 7

$$\begin{aligned} M(r(t_{22}s + st_{22})) &= M(r(t_{22}a_{12} + a_{12}t_{22})) + M(r(t_{22}b_{12} + b_{12}t_{22})) \\ &= M(ra_{12}t_{22}) + M(rb_{12}t_{22}) = M(r(a_{12}t_{22} + b_{12}t_{22})). \end{aligned}$$

Hence $t_{22}s + st_{22} = (a_{12} + b_{12})t_{22}$ for every $t_{22} \in A_{22}$. It follows from Lemma 2(v) and (i) that $s_{22} = s_{21} = 0$ and $s_{12} = a_{12} + b_{12}$.

Now there remains to prove that $s_{11} = 0$. For $t_{12} \in A_{12}$, applying Lemma 5(i) and Lemma 7 to $M(s) = M(a_{12}) + M(b_{12})$ again, we get that $t_{12}s + st_{12} = 0$. Since we have shown that $s_{22} = s_{21} = 0$, we have that $s_{11}t_{12} = 0$ for every $t_{12} \in A_{12}$. Hence from Lemma 2(ii) we get $s_{11} = 0$. Therefore M is additive on A_{12}.

It can be proved similarly that M is additive on A_{21}. $\Box$

Lemma 10 *M and M^{*-1} are additive on A_{ii} $(i = 1, 2)$.*

Proof By Lemma 4, we only prove that M is additive on A_{ii} $(i = 1, 2)$.

Let $a_{ii}, b_{ii} \in A_{ii}$ and choose $s = s_{11} + s_{12} + s_{21} + s_{22} \in A$ such that

$$M(s) = M(a_{11}) + M(b_{11}).$$

Let $j \ne i$. For $t_{jj} \in A_{jj}$, we see that from Lemma 5(i)

$$M(r(t_{jj}s + st_{jj})) = M(r(t_{jj}a_{ii} + a_{ii}t_{jj})) + M(r(t_{jj}b_{ii} + b_{ii}t_{jj})) = 0.$$

Hence, we have $t_{jj}s + st_{jj} = 0$ for every $t_{jj} \in A_{jj}$. It follows from Lemma 2(v) that $s_{ji} = s_{ij} = s_{jj} = 0$.

Now there remains to prove that $s_{ii} = a_{ii} + b_{ii}$. For $t_{ij} \in A_{ij}$, we see that from Lemma 5(i) and Lemma 9

$$M(r(t_{ij}s + st_{ij})) = M(r(t_{ij}a_{ii} + a_{ii}t_{ij})) + M(r(t_{ij}b_{ii} + b_{ii}t_{ij}))$$

$$= M(r(a_{ii}t_{ij})) + M(r(b_{ii}t_{ij})) = M(r(a_{ii}t_{ij} + b_{ii}t_{ij})).$$

Hence, we have $t_{ij}s + st_{ij} = a_{ii}t_{ij} + b_{ii}t_{ij}$ for every $t_{ij} \in A_{ij}$. Since $s_{ji} = s_{ij} = s_{jj} = 0$, it follows that $s_{11}t_{12} = (a_{11} + b_{11})t_{12}$ for every $t_{ij} \in A_{ij}$. Hence by Lemma 2(ii), we have that $s_{ii} = a_{ii} + b_{ii}$. □

Remark 11 We have shown that M and M^{*-1} are additive on A_{ij} for $1 \leq i, j \leq 2$. Therefore, for $a_{ij} \in A_{ij}$, we have that $M(r(a_{ij})) = rM(a_{ij})$ and $M^{*-1}(r(a_{ij})) = rM^{*-1}(a_{ij})$.

Lemma 12 *M and M^* are additive on $e_1A = A_{11} + A_{12}$.*

Proof Let $a_{11}, b_{11} \in A_{11}$ and let $a_{12}, b_{12} \in A_{12}$. Then by Lemmas 6, 9, and 10, we see that

$$\begin{aligned} &M((a_{11} + a_{12}) + (b_{11} + b_{12})) = M((a_{11} + b_{11}) + (a_{12} + b_{12})) \\ &\quad = M(a_{11} + b_{11}) + M(a_{12} + b_{12}) = M(a_{11}) + M(b_{11}) + M(a_{12}) + M(b_{12}) \\ &\quad = M(a_{11} + a_{12}) + M(b_{11} + b_{12}). \end{aligned}$$

Similarly, we can get that $M^{*-1}((a_{11} + a_{12}) + (b_{11} + b_{12})) = M^{*-1}(a_{11} + a_{12}) + M^{*-1}(b_{11} + b_{12})$. □

Lemma 13 *For any $a_{11} \in A_{11}$, $a_{22} \in A_{22}$, we get that*
(i) $M(a_{11} + a_{22}) = M(a_{11}) + M(a_{22})$;
(ii) $M^{*-1}(a_{11} + a_{22}) = M^{*-1}(a_{11}) + M^{*-1}(a_{22})$.

Proof By Lemma 4, we only prove (i). Since M is surjective, we can find an element $s = s_{11} + s_{12} + s_{21} + s_{22} \in A$ such that

$$M(s) = M(a_{11}) + M(a_{22}).$$

We see that from Lemma 5(i)

$$M(r(e_1s + se_1)) = M(r(e_1a_{11} + a_{11}e_1)) + M(r(e_1a_{22} + a_{22}e_1)) = M(0) + M(2ra_{11}) = M(2ra_{11}).$$

It follows that $2s_{11} + s_{12} + _{21} = 2a_{11}$. Hence we have that $s_{12} = s_{21} = 0$ and $s_{11} = a_{11}$.

For $t_{22} \in A_{22}$, we see that from Lemma 5(i)

$$\begin{aligned} M(r(t_{22}s + st_{22})) &= M(r(t_{22}a_{11} + a_{11}t_{22})) + M(r(t_{22}a_{22} + a_{22}t_{22})) \\ &= M(r(t_{22}a_{22} + a_{22}t_{22})). \end{aligned}$$

It follows that $t_{22}s + st_{22} = t_{22}a_{22} + a_{22}t_{22}$ for every $t_{22} \in A_{22}$. Since $s_{12} = s_{21} = 0$, we have that $t_{22}s + st_{22} = t_{22}a_{22} + a_{22}t_{22}$ for every $t_{22} \in A_{22}$. Thus by Lemma 2(iv), we get that $s_{22} = a_{22}$. Consequently, $s = a_{11} + a_{22}$. □

It should be mentioned that the idea of the proof of the following lemma is from [6].

Lemma 14 *If a is a finite rank operator on X, then $M(ra) = rM(a)$.*

Proof If $\dim X < \infty$, then A must contain the identity operator 1 in $B(X)$. By Lemma 13 and Remark 11, we have that

$$M^{*-1}(r1) = M^{*-1}(re_1 + re_2) = M^{*-1}(re_1) + M^{*-1}(re_2)$$

$$= rM^{*-1}(e_1) + rM^{*-1}(e_2) = rM^{*-1}(e_1+e_2) = rM^{*-1}(1).$$

Further, for every $a \in A$, we have that

$$\begin{aligned}
M(ra) &= M(r(\frac{a}{2r}M^*M^{*-1}(r1) + M^*M^{*-1}(r1)\frac{a}{2r})) \\
&= r(M(\frac{a}{2r})M^{*-1}(r1) + M^{*-1}(r1)M(\frac{a}{2r})) \\
&= r^2(M(\frac{a}{2r})M^{*-1}(1) + M^{*-1}(1)M(\frac{a}{2r})) \\
&= r(M(r(\frac{a}{2r})M^*M^{*-1}(1) + M^*M^{*-1}(1)(\frac{a}{2r}))) = rM(a).
\end{aligned}$$

We now assume that $\dim X = \infty$.

For every non-trivial idempotent operator $q \in A$, set $e_1 = q$. By Lemma 12, M and M^{*-1} are additive on qA. Therefore, for every $a \in qA$, we have that $M(ra) = rM(a)$.

Let a be a finite rank operator of X. Suppose that the range of a is $\mathrm{sp}\{h_1, h_2, \dots, h_n\}$ ($n < \infty$), where $h_1, \dots, h_n$ are linearly independent. By the Hahn-Banach Extension Theorem, there are $f_1, \dots, f_n \in X^*$, the dual Banach space of X, such that $f_j(h_i) = \delta_{ij}$ (Kronecker delta). Let $q = h_1 \otimes f_1 + \cdots h_n \otimes f_n$. Then q is a finite rank idempotent operator in A. Clearly, $qa = a$. Thus, we have that $M(ra) = M(rqa) = rM(qa) = rM(a)$. $\square$

Lemma 15 *Let $a_{12} \in A_{12}$ and $a_{21} \in A_{21}$. Then $M(a_{12} + a_{21}) = M(a_{12}) + M(a_{21})$.*

Proof Choose $s = s_{11} + s_{12} + s_{21} + s_{22} \in A$ such that

$$M(s) = M(a_{12}) + M(a_{21}). \tag{3}$$

For $t_{12} \in A_{12}$, we see that from Lemma 5(i)

$$M(r(t_{12}s + st_{12})) = M(r(t_{12}a_{12} + a_{12}t_{12})) + M(r(t_{12}a_{21} + a_{21}t_{12})) = M(r(t_{12}a_{21} + a_{21}t_{12})).$$

Hence, by Lemma 3, we have that $t_{12}s + st_{12} = t_{12}a_{21} + a_{21}t_{12}$ for every $t_{12} \in A_{12}$. Multiplying this equality by e_1 from the right, we have that $t_{12}s_{21} = t_{12}a_{21}$ for every $t_{12} \in A_{12}$. It follows from Lemma 2(ii) that $s_{21} = a_{21}$. Hence by Lemma 2(i), we get that $t_{12}s_{22} + s_{11}t_{12} = 0$ for every $t_{12} \in A_{12}$. An argument similar to what has led to the equality $s_{21} = a_{21}$ proves that $s_{12} = a_{12}$ also holds.

By Lemma 5(i), from (3), we get that

$$\begin{aligned}
M(r(e_1s + se_1)) &= M(r(e_1a_{12} + a_{12}e_1)) + M(r(e_1a_{21} + a_{21}e_1)) \\
&= M(ra_{12}) + M(ra_{21}).
\end{aligned}$$

Hence we deduce from Lemma 14 that

$$rM(e_1s + se_1) = rM(a_{12}) + rM(a_{21}) = rM(s).$$

By the injectivity of M, we have that $e_1s + se_1 = s$. Thus $s_{11} = s_{22} = 0$. Consequently $s = a_{12} + a_{21}$. $\square$

Lemma 16 *Let $a_{11} \in A_{11}$, $a_{12} \in A_{12}$ and $a_{21} \in A_{21}$. Then $M(a_{11} + a_{12} + a_{21}) = M(a_{11}) + M(a_{12}) + M(a_{21})$.*

Proof Choose $s = s_{11} + s_{12} + s_{21} + s_{22} \in A$ such that

$$M(s) = M(a_{11}) + M(a_{12}) + M(a_{21}).$$

Then by Lemma 6, we have that

$$M(s) = M(a_{11} + a_{12}) + M(a_{21}), \tag{4}$$

$$M(s) = M(a_{11} + a_{21}) + M(a_{12}). \tag{5}$$

For $t_{21} \in A_{21}$, we see that from Lemma 5(i)

$$\begin{aligned} M(r(t_{21}s + st_{21})) &= M(r(t_{21}(a_{11} + a_{12}) + (a_{11} + a_{12})t_{21})) + M(r(t_{21}a_{21} + a_{21}t_{21})) \\ &= M(r(t_{21}a_{11} + t_{21}a_{12} + a_{12}t_{21})). \end{aligned}$$

By Lemma 3, we have that

$$t_{21}s + st_{21} = t_{21}a_{11} + t_{21}a_{12} + a_{12}t_{21} \tag{6}$$

for every $t_{21} \in A_{21}$. Multiplying this equality by e_1 from the left, we get that $s_{12}t_{21} = a_{12}t_{21}$ for every $t_{21} \in A_{21}$. By Lemma 2(ii), it follows that $s_{12} = a_{12}$. Multiplying (6) by e_1 from the right, we get that

$$t_{21}s_{11} + s_{22}t_{21} = t_{21}a_{11} \tag{7}$$

for every $t_{21} \in A_{21}$. Similarly, for $t_{12} \in A_{12}$, by Lemma 5(i), we get $s_{21} = a_{21}$ from (5).

For $t_{22} \in A_{22}$, by Lemma 5(i) and Lemma 15, we get from (4)

$$M(r(t_{22}s + st_{22})) = M(ra_{12}t_{22}) + M(rt_{22}a_{21}) = M(r(a_{12}t_{22} + t_{22}a_{21})).$$

Therefore, $t_{22}s + st_{22} = a_{12}t_{22} + t_{22}a_{21}$ for every $t_{22} \in A_{22}$. Since $s_{12} = a_{12}$ and $s_{21} = a_{21}$, it follows that $t_{22}s_{22} + s_{22}t_{22} = 0$ for every $t_{22} \in A_{22}$. It follows from Lemma 2(iv) that $s_{22} = 0$. Hence, from (7), we have $s_{11} = a_{11}$. Consequently, $s = a_{11} + a_{12} + a_{21}$. □

Lemma 17 *If $a_{ij} \in A_{ij}$ $(1 \le i, j \le 2)$, then $M(a_{11} + a_{12} + a_{21} + a_{22}) = M(a_{11}) + M(a_{12}) + M(a_{21}) + M(a_{22})$.*

Proof Choose $s = s_{11} + s_{12} + s_{21} + s_{22} \in A$ such that

$$M(s) = M(a_{11}) + M(a_{12}) + M(a_{21}) + M(a_{22}).$$

Then, by Lemma 5(i) and Lemma 16, we have that

$$M(r(e_1s + se_1)) = M(2ra_{11}) + M(ra_{12}) + M(ra_{21}) = M(r(2a_{11} + a_{21}) + a_{12})).$$

By Lemma 3, it follows that $e_1s + se_1 = 2a_{11} + a_{21} + a_{12}$. By a simple computation, we get that $s_{11} = a_{11}$, $s_{12} = a_{12}$ and $s_{21} = a_{21}$. For $t_{12} \in A_{12}$, we see that from Lemma 5(i)

$$M(r(t_{12}s + st_{12})) = M(ra_{11}t_{12}) + M(r(t_{12}a_{21} + a_{21}t_{12})) + M(rt_{12}a_{22}).$$

Making a use of Lemma 5(i) and Lemma 12 to the above equality, we have that

$$\begin{aligned} M(r^2(e_1t_{12}s + e_1st_{12} + t_{12}se_1)) &= M(r^2a_{11}t_{12}) + M(2r^2t_{12}a_{21}) + M(r^2t_{12}a_{22}) \\ &= M(r^2(a_{11}t_{12} + 2t_{12}a_{21} + t_{12}a_{22})). \end{aligned}$$

Hence we have that

$$t_{12}s_{21} + t_{12}s_{22} + s_{11}t_{12} + t_{12}s_{21} = a_{11}t_{12} + 2t_{12}a_{21} + t_{12}a_{22}$$

for every $t_{12} \in A_{12}$. Since we have shown that $s_{11} = a_{11}$, $s_{12} = a_{12}$ and $s_{21} = a_{21}$, it follows that $t_{12}s_{22} = t_{12}a_{22}$ for every $t_{12} \in A_{12}$ and hence $s_{22} = a_{22}$. Consequently, $s = a_{11}+a_{12}+a_{21}+a_{22}$. □

Proof of Theorem Let $a = a_{11} + a_{12} + a_{21} + a_{22}$, $b = b_{11} + b_{12} + b_{21} + b_{22} \in A$. Then Lemmas 17, 9, and 10 are all used in seeing the equalities

$$\begin{aligned} M(a+b) &= M((a_{11}+b_{11}) + (a_{12}+b_{12}) + (a_{21}+b_{21}) + (a_{22}+b_{22})) \\ &= M(a_{11}+b_{11}) + M(a_{12}+b_{12}) + M(a_{21}+b_{21}) + M(a_{22}+b_{22}) \\ &= M(a_{11}) + M(b_{11}) + M(a_{12}) + M(b_{12}) + M(a_{21}) + M(b_{21}) + M(a_{22}) + M(b_{22}) \\ &= M(a_{11}+a_{12}+a_{21}+a_{22}) + M(b_{11}+b_{12}+b_{21}+b_{22}) = M(a) + M(b) \end{aligned}$$

hold true. That is, M is additive on A.

Now let us show that M^* is additive on B. Let $x, y \in B$. For every $t \in A$, by using the additivity of M, we have

$$\begin{aligned} &M(r(t(M^*(x) + M^*(y)) + (M^*(x) + M^*(y))t)) \\ &= M(r(tM^*(x) + M^*(x)t)) + M(r(tM^*(y) + M^*(y)t)) \\ &= r(M(t)x + xM(t)) + r(M(t)y + yM(t)) = r(M(t)(x+y) + (x+y)M(t)) \\ &= M(r(tM^*(x+y) + M^*(x+y)t)). \end{aligned}$$

Since M is injective, it follows that

$$t(M^*(x) + M^*(y)) + (M^*(x) + M^*(y))t = tM^*(x+y) + M^*(x+y)t.$$

Since A is dense in $B(X)$ under the strong operator topology, we have that $2(M^*(x)+M^*(y)) = 2M^*(x+y)$. Therefore $M^*(x+y) = M^*(x) + M^*(y)$. This completes the proof. □

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