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A Refinement of Hilbert's Double Series Theorem*

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The object of this note is to prove the following

Theorem Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $0<\sum a_n^2<+\infty$ and $0<\sum b_n^2<+\infty$. Then we have the inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\left\{\sum_{n=1}^{\infty}\left(\pi-\frac{\theta}{\sqrt{n}}\right)a_n^2\right\}^{\frac{1}{2}}\left\{\sum_{n=1}^{\infty}\left(\pi-\frac{\theta}{\sqrt{n}}\right)b_n^2\right\}^{\frac{1}{2}} \qquad (1)$$

where $\theta=3/\sqrt{2}-1\doteq 1.1213\,20343$.

Clearly (1) offers a refined form of Hilbert's inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\pi\left\{\sum_{n=1}^{\infty}a_n^2\right\}^{\frac{1}{2}}\left\{\sum_{n=1}^{\infty}b_n^2\right\}^{\frac{1}{2}}. \qquad (2)$$

Also, as an immediate consequence of (1) we have the inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n}<\sum_{n=1}^{\infty}\left(\pi-\frac{1}{\sqrt{n}}\right)a_n^2. \qquad (3)$$

Proof of the theorem Making use of G. H. Hardy's idea for the proof of (2), one may apply Cauchy's inequality to estimate the left side of (1) as follows

$$\begin{aligned}\sum_{m,n}\frac{a_m b_n}{m+n}&=\sum_{m,n}\left(\frac{m}{n}\right)^{\frac{1}{4}}\frac{a_m}{\sqrt{m+n}}\left(\frac{n}{m}\right)^{\frac{1}{4}}\frac{a_n}{\sqrt{m+n}}\\&\le\left\{\sum_{m,n}\left(\frac{m}{n}\right)^{\frac{1}{2}}\frac{a_m^2}{m+n}\right\}^{\frac{1}{2}}\left\{\sum_{m,n}\left(\frac{n}{m}\right)^{\frac{1}{2}}\frac{b_n^2}{m+n}\right\}^{\frac{1}{2}}\\&=\left\{\sum_m a_m^2\left(\sum_n\frac{1}{m+n}\left(\frac{m}{n}\right)^{\frac{1}{2}}\right)\cdot\sum_n b_n^2\left(\sum_m\frac{1}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{2}}\right)\right\}^{\frac{1}{2}}\\&=\left\{\sum_n a_n^2\theta_n\right\}^{\frac{1}{2}}\left\{\sum_n b_n^2\theta_n\right\}^{\frac{1}{2}},\end{aligned}$$

where θ_n is defined by

$$\theta_n:=\sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{2}}, \qquad (4)$$

Thus it suffices to verify that the following inequality

$$\theta_n<\pi-\theta/\sqrt{n} \qquad (5)$$

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holds for all positive integers n, where $\theta = 3/\sqrt{2} - 1$.

Evidently we have

$$\theta_n = \frac{\sqrt{n}}{n+1} + \frac{\sqrt{n/2}}{n+2} + \sum_{m=3}^{\infty} \frac{1}{(1+m/n)\sqrt{m/n}} (\frac{1}{n})$$

$$< \frac{\sqrt{n}}{n+1} + \frac{\sqrt{n/2}}{n+2} + \int_{2/n}^{\infty} \frac{dx}{(1+x)\sqrt{x}}$$

$$= \pi - (2\operatorname{arctg}\sqrt{2/n} - \sqrt{n}/(n+1) - \sqrt{n/2}/(n+2)) = \pi - \delta_n. \qquad (6)$$

Consequently we need only to show that the δ_n as defined by (6) is greater than $\theta/\sqrt{n}$ for all $n \geq 1$. In the first place direct computation shows that $\delta_n > \theta/\sqrt{n}$ holds for all $n \leq 6$. Indeed we have $\delta_1 \doteq 1.17509$, $\delta_2 \doteq 0.84937$, $\delta_3 \doteq 0.69164$, $\delta_4 \doteq 0.59510$, $\delta_5 \doteq 0.52874$, $\delta_6 \doteq 0.48066$, while $\theta/\sqrt{n}$ gives smaller numerical values 1.12132, 0.79289, 0.64739, 0.56066, 0.50147, 0.45778, for $n = 1, 2, 3, 4, 5, 6$, respectively.

In what follows we may assume $n \geq 7$. Plainly we have

$$2\operatorname{arctg}\sqrt{\frac{2}{n}} > 2\{(\frac{2}{n})^{\frac{1}{2}} - \frac{1}{3}(\frac{2}{n})^{\frac{3}{2}}\} = (2\sqrt{2} - \frac{4\sqrt{2}}{3n})(\frac{1}{n})^{\frac{1}{2}}$$

and we may write

$$\frac{\sqrt{n}}{n+1} = (1 - \frac{1}{n+1})(\frac{1}{n})^{\frac{1}{2}}, \quad \frac{\sqrt{n/2}}{n+2} = (1 - \frac{2}{n+2})\frac{\sqrt{2}}{2}(\frac{1}{n})^{\frac{1}{2}}$$

Thus it follows that

$$\delta_n = 2\operatorname{arctg}\sqrt{2/n} - \sqrt{n}/(n+1) - \sqrt{n/2}/(n+2)$$

$$> (2\sqrt{2} - 1 - \frac{\sqrt{2}}{2} + \frac{1}{n+1} + \frac{\sqrt{2}}{n+2} - \frac{4\sqrt{2}}{3n})(\frac{1}{n})^{\frac{1}{2}}$$

$$> (\frac{3}{2}\sqrt{2} - 1)/\sqrt{n} = \theta/\sqrt{n}, \quad \text{when } n \geq 7.$$

Consequently (5) is verified by means of (6) for all $n \geq 1$. This completes the proof of the theorem.

Remarks It may be of interest to ask the question of how to determine the largest possible value of θ that keeps (1) valid. For various classical results concerning Hilbert's inequality and its extensions, refer to Hardy-Littlewood-Polya's "Inequalities", chap.9, and D. S. Mitrinovic's "Analytic Inequalities", § 3.9.36, (Springer Verlag, 1970).