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THE STABILITY OF ADDITIVE (α, β)-FUNCTIONAL EQUATIONS*

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Abstract In this paper, we investigate the following (α, β)-functional equations

$$2f(x) + 2f(z) = f(x-y) + \alpha^{-1}f(\alpha(x+z)) + \beta^{-1}f(\beta(y+z)), \quad (0.1)$$
$$2f(x) + 2f(y) = f(x+y) + \alpha^{-1}f(\alpha(x+z)) + \beta^{-1}f(\beta(y-z)), \quad (0.2)$$

where α, β are fixed nonzero real numbers with $\alpha^{-1} + \beta^{-1} \neq 3$. Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the (α, β)-functional equations (0.1) and (0.2) in non-Archimedean Banach spaces.

Keywords Hyers-Ulam stability, additive (α, β)-functional equation, fixed point method, direct method,non-Archimedean Banach space.

MSC(2010) 39B52, 39B62, 47H10.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [16] in 1940, concerning the stability of group homomorphisms. Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(., .)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, Under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [8] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \to E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

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for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. In 1978, Rassias [15] proved the following theorem.

Theorem 1.1. *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for all $x, y \neq 0$, and (1.2) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}$-linear.

In 1991, Gajda [7] answered the question for the case $p > 1$, which was raised by Rassias. More generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings can be found in [5,6,10,11].

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.2 ([1,3]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = +\infty$$

for all nonnegative integers n or there exists a integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < +\infty, \forall n \geq n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0}x, y) < +\infty\}$;

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, Isac and Rassias [9] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors (see [2,4,13,14]).

Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let α, β be fixed nonzero real numbers with $\alpha^{-1} + \beta^{-1} \neq 3$.

Definition 1.1. Let X be a vector space over a non-Archimedean scalar field $\Bbbk$ with a valuation $|\cdot|$. *A* function $\|\cdot\| : X \to [0, \infty)$ is a non-Archimedean norm if it satisfies, for all $r \in \Bbbk, x, y \in X$,

(1) $\|x\| \geq 0$ if and only if $x = 0$,

(2) $\|rx\| = |r|\|x\|$,

(3) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$(the strong triangle inequality).

Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

Definition 1.2. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X.

(1) $\{x_n\}$ *converges to* $x \in X$ if, for any $\varepsilon\, 0$ there exists an integer N such that $\|x_n - x\| \leq \varepsilon$ for all $n \geq N$. Then the point x is called the *limit* of the sequence $\{x_n\}$, which is denoted by $\lim_{n\to\infty} x_n = x$.

(2) $\{x_n\}$ is a *Cauchy sequence* if the sequence $\{x_{n+1} - x_n\}$ converges to zero.

(3) X is called a *non-Archimedean Banach space* if every Cauchy sequence in X is convergent.

This paper is organized as follows. In Sections 2 and 3, we prove the Hyers-Ulam stability of the additive (α, β)-functional equation (0.1) in non-Archimedean Banach spaces by using the fixed point method and the direct method. In Sections 4 and 5, we prove the Hyers-Ulam stability of the additive (α, β)-functional equation (0.2) in non-Archimedean Banach spaces by using the fixed point method and the direct method.

2. Stability of the (α, β)-function equation (0.1): A fixed point approach

We solve the (α, β)-function equation (0.1) in non-Archimedean Banach spaces.

Lemma 2.1. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$2f(x) + 2f(z) = f(x - y) + \alpha^{-1} f(\alpha(x + z)) + \beta^{-1} f(\beta(y + z)) \tag{2.1}$$

for all $x, y, z \in X$, then $f : X \to Y$ is an additive mapping.

Proof. Assume the mapping $f : X \to Y$ satisfies (2.1). Letting $x = y = z = 0$, we get

$$3f(0) = \alpha^{-1} f(0) + \beta^{-1} f(0).$$

So $f(0) = 0$. Letting $y = z = 0$ in (2.1), we get

$$f(x) = \alpha^{-1} f(\alpha x)$$

and so

$$f(\alpha x) = \alpha f(x)$$

for all $x \in X$.

Letting $x = y = 0$ in (2.1), we get

$$2f(z) = \alpha^{-1} f(\alpha z) + \beta^{-1} f(\beta z)$$

and so

$$f(\beta z) = \beta f(z)$$

for all $z \in X$. Thus

$$\begin{aligned} &2f(x) + 2f(z) \\ =&f(x-y) + \alpha^{-1} f(\alpha(x+z)) + \beta^{-1} f(\beta(y+z)) \\ =&f(x-y) + f(x+z) + f(y+z). \end{aligned} \tag{2.2}$$

for all $x, y, z \in X$. Letting $y = 0$ in (2.2), we get

$$f(x+z) = f(x) + f(z)$$

for all $x, z \in X$. Thus $f : X \to Y$ is additive. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive (α, β)-functional equation (0.1) in non-Archimedean spaces.

Theorem 2.1. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < |2|$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \le \frac{L}{|2|} \varphi(x, y, z) \tag{2.3}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &\|2f(x) + 2f(z) - f(x-y) - \alpha^{-1} f(\alpha(x+z)) - \beta^{-1} f(\beta(y+z))\| \\ \le&\varphi(x, y, z) \end{aligned} \tag{2.4}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\begin{aligned} &\|f(x) - A(x)\| \\ \le&\frac{L}{|2|(1-L)} \max\left\{\varphi(x,0,x), \varphi(2x,0,0), \varphi(x,0,0), \varphi(0,0,x)\right\} \end{aligned} \tag{2.5}$$

for all $x \in X$.

Proof. Letting $y = z = 0$ in (2.4), we get

$$\|f(x) - \alpha^{-1} f(\alpha x)\| \le \varphi(x, 0, 0) \tag{2.6}$$

for all $x \in X$.

Letting $x = y = 0$ in (2.4), we get

$$\|2f(z) - \alpha^{-1} f(\alpha z) - \beta^{-1} f(\beta z)\| \le \varphi(0, 0, z) \tag{2.7}$$

for all $z \in X$. It follows from (2.6) and (2.7) that

$$\begin{aligned} &\|f(z) - \beta^{-1} f(\beta z)\| \\ =\ &\left\|2f(z) - \alpha^{-1} f(\alpha z) - \beta^{-1} f(\beta z) + \alpha^{-1} f(\alpha z) - f(z)\right\| \\ \le\ &\max\left\{\left\|2f(z) - \alpha^{-1} f(\alpha x) - \beta^{-1} f(\beta x)\right\|, \left\|f(z) - \alpha^{-1} f(\alpha z)\right\|\right\} \\ =\ &\max\left\{\varphi(0,0,z), \varphi(z,0,0)\right\} \end{aligned}$$

for all $z \in X$. Thus

$$
\begin{aligned}
&\|2f(x)+2f(z)-f(x-y)-f(x+z)-f(y+z)\| \\
&\le \big\|2f(x)+2f(z)-f(x-y)-\alpha^{-1}f(\alpha(x+z))-\beta^{-1}f(\beta(y+z)) \\
&\quad +\alpha^{-1}f(\alpha(x+z))+\beta^{-1}f(\beta(y+z))-f(x+z)-f(y+z)\big\| \\
&\le \max\big\{\|2f(x)+2f(z)-f(x-y)-\alpha^{-1}f(\alpha(x+z))-\beta^{-1}f(\beta(y+z))\|, \\
&\quad \|\alpha^{-1}f(\alpha(x+z)-f(x+z)\|, \|\beta^{-1}f(\beta(y+z))-f(y+z)\|\big\} \\
&\le \max\{\varphi(x,y,z),\varphi(x+z,0,0),\varphi(y+z,0,0),\varphi(0,0,y+z)\}
\end{aligned} \tag{2.8}
$$

for all $x, y, z \in X$.

Replacing x, z by $\frac{x}{2}, \frac{x}{2}$ and letting $y = 0$ in (2.8), we get

$$
\begin{aligned}
\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| &\le \max\left\{\varphi\left(\frac{x}{2},0,\frac{x}{2}\right),\varphi\left(x,0,0\right),\varphi\left(\frac{x}{2},0,0\right),\varphi\left(0,0,\frac{x}{2}\right)\right\} \\
&\le \frac{L}{|2|}\max\left\{\varphi\left(x,0,x\right),\varphi\left(2x,0,0\right),\varphi\left(x,0,0\right),\varphi\left(0,0,x\right)\right\}
\end{aligned} \tag{2.9}
$$

for all $x \in X$.

Consider the set

$$S = \{h : X \to Y, h(0) = 0\}$$

and introduce the generalized metric space on S:

$$
\begin{aligned}
d(g,h) = & inf\,\{\mu \in \mathbb{R}_+ : \|g(x)-h(x)\| \\
&\le \mu\max\left\{\varphi\left(x,0,x\right),\varphi\left(2x,0,0\right),\varphi\left(x,0,0\right),\varphi\left(0,0,x\right)\right\}, \forall x \in X\}.
\end{aligned}
$$

It is easy to show that (S, d) is complete, for details, see [12].

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) = 2g\left(\frac{x}{2}\right)$$

for all $x \in X$. Let $g, h \in S$ be give such that $d(g, h) = \varepsilon$. Then

$$\|g(x)-h(x)\| \le \varepsilon\max\left\{\varphi\left(x,0,x\right),\varphi\left(2x,0,0\right),\varphi\left(x,0,0\right),\varphi\left(0,0,x\right)\right\}$$

for all $x \in X$. Hence

$$
\begin{aligned}
\|Jg(x)-Jh(x)\| = \left\|2g\left(\frac{x}{2}\right)-2h\left(\frac{x}{2}\right)\right\| &\le \left\|g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right)\right\| \\
&\le \varepsilon\max\left\{\varphi\left(\frac{x}{2},0,\frac{x}{2}\right),\varphi\left(x,0,0\right),\varphi\left(\frac{x}{2},0,0\right),\varphi\left(0,0,\frac{x}{2}\right)\right\} \\
&\le \frac{L}{|2|}\varepsilon\max\left\{\varphi\left(x,0,x\right),\varphi\left(2x,0,0\right),\varphi\left(x,0,0\right),\varphi\left(0,0,x\right)\right\}
\end{aligned}
$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le \frac{L}{|2|}\varepsilon$. This means that

$$d(Jg, Jh) \le \frac{L}{|2|}d(g,h)$$

for all $g, h \in S$. It follows from (2.9) that $d(f, Jf) \le \frac{L}{|2|}$.

By Theorem 1.2, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is

$$A(x) = 2A\left(\frac{x}{2}\right) \tag{2.10}$$

for all $x \in X$. The mapping A is an unique fixed point of J in the set

$$M = \{g \in S : d(g, h) < +\infty\}$$

This implies that A is an unique mapping satisfying (2.10) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\| \leq \mu \max\left\{\varphi\left(x, 0, x\right), \varphi\left(2x, 0, 0\right), \varphi\left(x, 0, 0\right), \varphi\left(0, 0, x\right)\right\}$$

for all $x \in X$.

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$.

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x) - A(x)\| \leq \frac{L}{|2|(1-L)} \max\left\{\varphi\left(x, 0, x\right), \varphi\left(2x, 0, 0\right), \varphi\left(x, 0, 0\right), \varphi\left(0, 0, x\right)\right\}$$

for all $x \in X$.

It follows from (2.3) and (2.4) that

$$\begin{aligned}
&\|2A(x) + 2A(z) - A(x-y) - \alpha^{-1}A(\alpha(x+z)) - \beta^{-1}A(\beta(y+z))\| \\
&= \lim_{n\to\infty} \left\| 2^n \left(2f\left(\frac{x}{2^n}\right) + 2f\left(\frac{z}{2^n}\right) - f\left(\frac{x-y}{2^n}\right) \right.\right. \\
&\quad \left.\left. -\alpha^{-1} f\left(\alpha \frac{x+z}{2^n}\right) - \beta^{-1} f\left(\beta \frac{y+z}{2^n}\right)\right)\right\| \\
&= \lim_{n\to\infty} \left\| 2f\left(\frac{x}{2^n}\right) + 2f\left(\frac{z}{2^n}\right) - f\left(\frac{x-y}{2^n}\right) - \right. \\
&\quad \left. \alpha^{-1} f\left(\alpha \frac{x+z}{2^n}\right) - \beta^{-1} f\left(\beta \frac{y+z}{2^n}\right)\right\| \\
&\leq \lim_{n\to\infty} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \leq \lim_{n\to\infty} \frac{L^n}{|2|^n} \varphi\left(x, y, z\right) = 0
\end{aligned}$$

for all $x \in X$. So

$$2A(x) + 2A(z) - A(x-y) - \alpha^{-1}A(\alpha(x+z)) - \beta^{-1}A(\beta(y+z)) = 0$$

for all $x \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive. $\square$

Theorem 2.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < |2|$ with*

$$\varphi(2x, 2y, 2z) \leq |2| L \varphi(x, y, z)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.4). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{|2|(1-L)} \max\left\{\varphi\left(x, 0, x\right), \varphi\left(2x, 0, 0\right), \varphi\left(x, 0, 0\right), \varphi\left(0, 0, x\right)\right\}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g\left(2x\right)$$

for all $x \in X$.

It follows from (2.9) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{1}{|2|} \max\left\{\varphi(x, 0, x), \varphi\left(2x, 0, 0\right), \varphi\left(x, 0, 0\right), \varphi\left(0, 0, x\right)\right\}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

3. Stability of the (α, β)-function equation (0.1): A direct method

In this section, using the direct method, we prove the Hyers-Ulam stability of the (α, β)-functional equation (2.1) in non-Archimedean spaces.

Theorem 3.1. *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\lim_{j \to \infty} |2|^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) = 0 \tag{3.1}$$

for all $x, y, z \in X$. Suppose that, for each $x \in X$, the limit

$$\begin{aligned} \psi(x) := \lim_{n \to \infty} \max_{0 \leq j \leq n} &\left\{ |2|^j \max\left\{ \varphi\left(\frac{x}{2^{j+1}}, 0, \frac{x}{2^{j+1}}\right), \varphi\left(\frac{x}{2^j}, 0, 0\right), \right.\right. \\ &\left.\left. \varphi\left(\frac{x}{2^{j+1}}, 0, 0\right), \varphi\left(0, 0, \frac{x}{2^{j+1}}\right) \right\} \right\} \end{aligned}$$

exists and

$$\begin{aligned} &\|2f(x) + 2f(z) - f(x-y) - \alpha^{-1} f(\alpha(x+z)) - \beta^{-1} f(\beta(y+z))\| \\ \leq &\varphi(x, y, z) \end{aligned} \tag{3.2}$$

for all $x, y, z \in X$. Then there exists an unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \psi(x) \tag{3.3}$$

for all $x \in X$.

Proof. It follows from (2.9) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \max\left\{\varphi\left(\frac{x}{2}, 0, \frac{x}{2}\right), \varphi(x,0,0), \varphi\left(\frac{x}{2}, 0, 0\right), \varphi\left(0, 0, \frac{x}{2}\right)\right\}$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \\ &\leq \max_{l \leq j < m}\left\{\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|\right\} \\ &\leq \max_{l \leq j < m}\left\{|2|^j \max\left\{\varphi\left(\frac{x}{2^{j+1}}, 0, \frac{x}{2^{j+1}}\right), \varphi\left(\frac{x}{2^j}, 0, 0\right),\right.\right. \\ &\left.\left.\varphi\left(\frac{x}{2^{j+1}}, 0, 0\right), \varphi\left(0, 0, \frac{x}{2^{j+1}}\right)\right\}\right\} \end{aligned} \tag{3.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.1) that the sequence $\{|2|^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{|2|^k f(\frac{x}{2^k})\}$ converges. So one can define the mappings $A : X \to Y$ by

$$A(x) = \lim_{k\to\infty} |2|^k f(\frac{x}{2^k})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.4), we get (3.3). By (3.1) and (3.2), we get

$$\begin{aligned} &\|2A(x) + 2A(z) - A(x-y) - \alpha^{-1}A(\alpha(x+z)) - \beta^{-1}A(\beta(y+z))\| \\ &= \lim_{n\to\infty} |2|^n \left\|2f\left(\frac{x}{2^n}\right) + 2f\left(\frac{z}{2^n}\right) - f\left(\frac{x-y}{2^n}\right) - \right. \\ &\left. \alpha^{-1} f\left(\frac{\alpha(x+z)}{2^n}\right) - \beta^{-1} f\left(\frac{\beta(y+z)}{2^n}\right)\right\| \\ &\leq \lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \end{aligned}$$

for all $x, y, z \in X$. Therefore, the mapping $A : X \to Y$ satisfies (2.1). So $A : X \to Y$ is additive.

To prove the uniqueness property of A, let $T : X \to Y$ be another mapping satisfying (3.3). Then we have

$$\begin{aligned} &\|A(x) - T(x)\| = |2|^j \left\|A\left(\frac{x}{2^j}\right) - T\left(\frac{x}{2^j}\right)\right\| \\ &\leq |2|^j \max\left\{\left\|f\left(\frac{x}{2^j}\right) - T\left(\frac{x}{2^j}\right)\right\|, \left\|f\left(\frac{x}{2^j}\right) - A\left(\frac{x}{2^j}\right)\right\|\right\} \\ &\leq |2|^j \psi\left(\frac{x}{2^j}\right), \end{aligned}$$

which tends to zero as $j \to \infty$, we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. $\square$

Theorem 3.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that*

$$\lim_{j\to\infty} \frac{1}{|2|^j} \varphi(2^j x, 2^j y, 2^j z) = 0$$

for all $x, y, z \in X$. Suppose that, for each $x \in X$, the limit

$$\psi(x) := \lim_{n\to\infty} \max_{0\le j\le n}\{\frac{1}{|2|^{j+1}}\max\{\varphi\left(2^j x, 0, 2^j x\right), \varphi\left(2^{j+1}x, 0, 0\right), \\ \varphi\left(2^j x, 0, 0\right), \varphi\left(0, 0, 2^j x\right)\}\}$$

exists. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.2). Then there exists an unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \psi(x)$$

for all $x \in X$.

Proof. It follows from (2.9) that

$$\left\|f(x) - \frac{1}{2}f\left(2x\right)\right\| \le \frac{1}{|2|}\max\left\{\varphi\left(x, 0, x\right), \varphi\left(2x, 0, 0\right), \varphi\left(x, 0, 0\right), \varphi\left(0, 0, x\right)\right\}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.1. □

4. Stability of the (α,β)-function equation (0.2): A fixed point approach

We solve the (α,β)-function equation (0.2) in non-Archimedean Banach spaces.

Lemma 4.1. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$2f(x) + 2f(y) = f(x+y) + \alpha^{-1}f(\alpha(x+z)) + \beta^{-1}f(\beta(y-z)) \tag{4.1}$$

for all $x, y, z \in X$, then $f : X \to Y$ is an additive mapping.

Proof. Assume a mapping $f : X \to Y$ satisfies (4.1). Letting $x = y = z = 0$, we get

$$3f(0) = \alpha^{-1}f(0) + \beta^{-1}f(0)$$

So $f(0) = 0$. Letting $y = z = 0$ in (4.1), we get

$$f(x) = \alpha^{-1}f(\alpha x).$$

and so

$$f(\alpha x) = \alpha f(x)$$

for all $x \in X$.

Letting $x = z = 0$ in (4.1), we get

$$f(y) = \beta^{-1}f(\beta y)$$

and so

$$f(\beta y) = \beta f(y)$$

for all $y \in X$. Thus

$$\begin{aligned} 2f(x) + 2f(y) &= f(x+y) + \alpha^{-1}f(\alpha(x+z)) + \beta^{-1}f(\beta(y-z)) \\ &= f(x+y) + f(x+z) + f(y-z). \end{aligned} \tag{4.2}$$

for all $x, y, z \in X$. Letting $z = 0$ in (4.2), we get

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. Thus $f : X \to Y$ is additive. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive (α, β)-functional equation (4.1) in non-Archimedean spaces.

Theorem 4.1. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < |2|$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{|2|}\varphi(x, y, z)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\|2f(x) + 2f(y) - f(x+y) - \alpha^{-1}f(\alpha(x+z)) - \beta^{-1}f(\beta(y-z))\| \leq \varphi(x, y, z) \quad (4.3)$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{L}{|2|(1-L)} \max\left\{\varphi\left(x, x, 0\right), \varphi\left(x, 0, 0\right), \varphi\left(0, x, 0\right)\right\}$$

for all $x \in X$.

Proof. Letting $x = z = 0$ in (4.3), we get

$$\|f(y) - \beta^{-1}f(\beta y)\| \leq \varphi(0, y, 0) \quad (4.4)$$

for all $y \in X$.

Letting $y = z = 0$ in (4.3), we get

$$\|f(x) - \alpha^{-1}f(\alpha x)\| \leq \varphi(x, 0, 0) \quad (4.5)$$

for all $x \in X$.

It follows from (4.4), (4.5) and (4.3) that

$$\begin{aligned}
&\|2f(x) + 2f(y) - f(x+y) - f(x+z) - f(y-z)\| \qquad (4.6)\\
&= \|2f(x) + 2f(y) - f(x+y) - \alpha^{-1}f(\alpha(x+z)) - \beta^{-1}f(\beta(y-z))\\
&\quad +\alpha^{-1}f(\alpha(x+z)) - f(x+z) + \beta^{-1}f(\beta(y-z)) - f(y-z)\|\\
&\leq \max\{\|2f(x) + 2f(y) - f(x+y) - \alpha^{-2}f(\alpha(x+z)) - \beta^{-1}f(\beta(y-z))\|,\\
&\quad \|\alpha^{-1}f(\alpha(x+z)) - f(x+z)\|, \|\beta^{-1}f(\beta(y-z)) - f(y-z)\|\}\\
&\leq \max\{\varphi(x, y, z), \varphi(x+z, 0, 0), \varphi(0, y-z, 0)\}
\end{aligned}$$

for all $x, y, z \in X$.

Letting $z = 0$ and replacing x, y by $\frac{x}{2}, \frac{x}{2}$ in (4.6), we get

$$\begin{aligned}
\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| &\leq \max\left\{\varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right), \varphi\left(\frac{x}{2}, 0, 0\right), \varphi\left(0, \frac{x}{2}, 0\right)\right\} \qquad (4.7)\\
&\leq \frac{L}{|2|}\max\{\varphi\left(x, x, 0\right), \varphi\left(x, 0, 0\right), \varphi\left(0, x, 0\right)\}
\end{aligned}$$

for all $x \in X$.

Consider the set

$$S = \{h : X \to Y, h(0) = 0\}$$

and introduce the generalized metric space on S:

$$d(g, h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu \max\{\varphi(x, x, 0), \varphi(x, 0, 0), \varphi(0, x, 0), \forall x \in X\}.$$

It is easy to show that (S, d) is complete (see [12]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) = 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 4.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < |2|$ with*

$$\varphi(2x, 2y, 2z) \leq |2| L\varphi(x, y, z)$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (4.3). Then there exists an unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{|2|(1 - L)} \max\{\varphi(x, x, 0), \varphi(x, 0, 0), \varphi(0, x, 0)\}.$$

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 4.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g(2x)$$

for all $x \in X$.

It follows from (4.7) that

$$\left\|f(x) - \frac{1}{2} f(2x)\right\| \leq \frac{1}{|2|} \max\{\varphi(x, x, 0), \varphi(x, 0, 0, \varphi(0, x, 0)\}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 4.1. □

5. Stability of the (α, β)-function equation (0.2): A direct method

In this section, using the direct method, we prove the Hyers-Ulam stability of the (α, β)-functional equation (0.2) in non-Archimedean spaces.

Theorem 5.1. *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\lim_{j\to\infty} |2|^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) = 0$$

for all $x, y, z \in X$. Suppose that, for each $x \in X$, the limit

$$\psi(x) := \lim_{n\to\infty} \max_{0\leq j\leq n} \left\{|2|^j \max\left\{\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right), \varphi\left(\frac{x}{2^{j+1}}, 0, 0\right), \varphi\left(0, \frac{x}{2^{j+1}}, 0\right)\right\}\right\}$$

exists and

$$\|2f(x)+2f(y)-f(x+y)-\alpha^{-1}f(\alpha(x+z))-\beta^{-1}f(\beta(y-z))\| \leq \varphi(x,y,z) \quad (5.1)$$

for all $x, y, z \in X$. Then there exists an unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \psi(x)$$

for all $x \in X$.

Proof. It follows from (4.7) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \max\left\{\varphi\left(\frac{x}{2},\frac{x}{2},0\right), \varphi\left(\frac{x}{2},0,0\right), \varphi\left(0,\frac{x}{2},0\right)\right\} \quad (5.2)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 3.1 and 4.1. □

Theorem 5.2. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that*

$$\lim_{j\to\infty} \frac{1}{|2|^j}\varphi(2^j x, 2^j y, 2^j z) = 0$$

for all $x, y, z \in X$. Suppose that, for each $x \in X$, the limit

$$\psi(x) := \lim_{n\to\infty} \max_{0\leq j\leq n}\left\{\frac{1}{|2|^{j+1}}\max\left\{\varphi\left(2^j x, 2^j x, 0\right), \varphi\left(2^j x, 0, 0\right), \varphi\left(0, 2^j x, 0\right)\right\}\right\}$$

exists. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (5.1). Then there exists an unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \psi(x)$$

for all $x \in X$.

Proof. It follows from (5.2) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{1}{|2|}\max\left\{\varphi(x,x,0), \varphi(x,0,0), \varphi(0,x,0)\right\}$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 3.2 and 4.2. □

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