Journal of Applied Analysis and Computation
Volume 2, Number 3, August 2012
Website:http://jaac-online.com/
pp. 325–339

ON THE LIMIT CYCLES OF A KIND OF LIÉNARD SYSTEM WITH A NILPOTENT CENTER UNDER PERTURBATIONS

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Abstract In this paper, we study the number of limit cycles of a kind of Liénard system with a nilpotent center under perturbations. Let $L(m,n)$ denote the maximal number of limit cycles of this Liénard system $\dot{x} = y - \varepsilon F(x), \dot{y} = -g(x)$ near the origin, where $m = \deg g, n = \deg F$. We obtain some results on the lower bound of $L(m,n)$ for $m = 4, 2 \le n \le 20$, $m = 5$, $2 \le n \le 10$, $m = 6, 2 \le n \le 5$ and $m = 7, 2 \le n \le 4$, where some results are new.

Keywords Nilpotent center, Liénard system, limit cycle.

MSC(2000) 35D, 35C.

1. Introduction and main results

Consider the C^∞ near-Hamiltonian system

$$\dot{x} = H_y + \varepsilon p(x,y,\varepsilon,\delta),\ \dot{y} = -H_x + \varepsilon q(x,y,\varepsilon,\delta), \tag{1.1}$$

where $H(x,y), p(x,y,\varepsilon,\delta), q(x,y,\varepsilon,\delta)$ are C^∞ functions, $\varepsilon \ge 0$ is small and $\delta \in D \subset \mathbf{R}^m$ is a vector parameter with D compact. For $\varepsilon = 0$, system (1.1) becomes

$$\dot{x} = H_y, \quad \dot{y} = -H_x. \tag{1.2}$$

Suppose that system (1.2) has a nilpotent singular point at the origin. Then the function H satisfies $H_x(0,0) = H_y(0,0) = 0$ and

$$\frac{\partial(H_y,-H_x)}{\partial(x,y)}(0,0) \ne 0, \quad \det \frac{\partial(H_y,-H_x)}{\partial(x,y)}(0,0) = 0.$$

Without loss of generality, we may suppose

$$H_{yy}(0,0) = 1, \quad H_{xy}(0,0) = H_{xx}(0,0) = 0,$$

which implies that the expansion of $H(x,y)$ at the origin can be written as

$$H(x,y) = \frac{1}{2}y^2 + \sum_{i+j\ge 3} h_{ij}x^iy^j. \tag{1.3}$$

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*The project was supported by National Natural Science Foundation of China (11101118) and Natural Science Foundation of Hebei Province(A2012205074).

By the implicit function theorem there exists a unique C^ω function $\varphi(x) = O(x^2)$ such that $H_y(x, \varphi(x)) = 0$ for $|x|$ small. Then

$$H(x,y) = H(x,\varphi(x)) + \frac{1}{2}\frac{\partial^2 H}{\partial y^2}(x,\varphi(x))(y-\varphi(x))^2 + O(|y-\varphi(x)|^3).$$

Introduce

$$H_0^*(x) = H(x,\varphi(x)) = \sum_{j\geq 3} h_j x^j.$$

It is obvious that $H(x,y)$ is definitely positive if there exists an even number $k = 2m \geq 4$ such that

$$h_k > 0, \quad h_j = 0, \ j < k. \tag{1.4}$$

In this case, according to Han, Shu, Yang & Chian [2], the origin is called a nilpotent center of order $m-1$ of the system (1.2).

The equations $H(x,y) = h$ define a family of clockwise periodic orbits L_h near the origin for $0 < h \ll 1$ under (1.4). We have the following first order Melnikov function

$$M(h,\delta) = \oint_{L_h} q dx - p dy$$

which plays an important role on the study of limit cycles.

On the limit cycle bifurcation of system (1.1) many works have been done if the unperturbed system (1.2) has a nilpotent singular point. See Llibre & Zhang [3], Li, Li, Llibre & Zhang [4], Han, Jiang & Zhu [1] and Jiang & Han [5] for different cases.

For the cubic Hamiltonian system (1.2) with

$$H(x,y) = \frac{1}{2}y^2 + \sum_{3\leq i+j\leq 4} h_{ij}x^i y^j,$$

Jiang, Han & Zhu [1] gave a sufficient and necessary condition that the origin is a nilpotent center. Then Han, Shu, Yang & Chian [2] provided a complete classification for the nilpotent singular point.

To study the limit cycles bifurcated from the nilpotent center for general near-Hamiltonian system, Han, Jiang & Zhu [1] obtained the following lemma.

Lemma 1.1. *Let* (1.4) *hold with* $k = 2m$ *(i.e., the origin is a nilpotent center of order* $m-1$ *). Then there exists a* C^∞ *function* $N(v,\delta)$ *such that*

$$M(h,\delta) = h^{\frac{1+m}{2m}} N(h^{\frac{1}{m}}, \delta).$$

Moreover, if (1.1) *is analytic then so is* $N(v,\delta)$*. Thus, if*

$$N(v,\delta) = \sum_{l\geq 0} b_l(\delta) v^l$$

formally for $|v|$ *small then*

$$M(h,\delta) = h^{\frac{1+m}{2m}} \sum_{l\geq 0} b_l(\delta) h^{\frac{l}{m}} \tag{1.5}$$

formally for $0 < h \ll 1$.

Consider a general Liénard system of the form

$$\begin{cases} \dot{x} = y - \varepsilon F(x, a), \\ \dot{y} = -g(x, c), \end{cases} \tag{1.6}$$

where $a \in R^{n_1}$, $c \in R^{n_2}$, F and g are C^ω functions satisfying

$$F(0, a) = 0, \ g(x, c) = x^{2m-1}(g_0 + O(x))$$

with $m \geq 1$ and $g_0 > 0$. For (1.6) we have

$$H(x, y) = \frac{1}{2}y^2 + G(x, c), \quad G(x, c) = \int_0^x g(x, c)dx,$$

and for $h > 0$ small

$$M(h, a, c) = \oint_{L_h} F(x, a)dy = h^{\frac{1+m}{2m}} \sum_{l \geq 0} b_l(a, c)h^{\frac{l}{m}}. \tag{1.7}$$

On the coefficients b_l in (1.7) Su, Yang & Han [6] gave the following lemma.

Lemma 1.2. *Consider* (1.6) *with* $m \geq 1$. *Let* $\alpha(x, c) = -x + O(|x|^2)$ *satisfy* $G(\alpha(x, c), c) \equiv G(x, c)$ *and*

$$F(\alpha(x, c), a) - F(x, a) = \sum_{i \geq 1} B_i(a, c)x^i. \tag{1.8}$$

Then there exist constants $N_l > 0$, $l = 0, 1, \cdots$, *such that the coefficients* b_l *in* (1.7) *satisfy*

$$b_0 = B_1 N_1,$$
$$b_l = B_{2l+1}N_{2l+1} + O(|B_1, B_3, B_5, \cdots, B_{2l-1}|), \ l \geq 1.$$

Then one can use the coefficients $B_{2l+1}, l \geq 0$ to study the limit cycles near the origin. Further, [6] gave the following lemma.

Lemma 1.3. *Consider the system* (1.6), *where the function* F *is linear in* $a \in R^{n_1}$. *Let there exist integers* $k > 0$ *and* $n > 0$, $a_0 = (a_{10}, \cdots, a_{n_1,0}) \in R^{n_1}$ *and* $c_0 \in R^{n_2}$ *such that*

$$B_{2j+1}(a_0, c_0) = 0, j = 0, \cdots, k-1, \ \det \frac{\partial(B_1, B_3, \cdots, B_{2k-1})}{\partial(a_1, \cdots, a_k)}(a_0, c_0) \neq 0 \tag{1.9}$$

and

$$B_{2(k+j)+1}\Big|_{(a_1, \cdots, a_k) = \varphi(a_{k+1}, \cdots, a_{n_1}, c)} = L_j(a_{k+1}, \cdots, a_{n_1}) \triangle_j(c), j = 0, \cdots, n \tag{1.10}$$

with

$$\begin{aligned} &L_j(a_{k+1,0}, \cdots, a_{n_1,0}) \neq 0, \ j = 0, \cdots, n, \\ &\triangle_j(c_0) = 0, j = 0, \cdots, n-1, \ \triangle_n(c_0) \neq 0, \end{aligned} \tag{1.11}$$

and

$$\det \frac{\partial(\triangle_0, \cdots, \triangle_{n-1})}{\partial(c_1, \cdots, c_n)}(c_0) \neq 0, \tag{1.12}$$

where $\varphi(a_{k+1}, \cdots, a_{n_1}, c)$ *is the unique solution of the linear equations* $B_{2j+1} = 0$, $j = 0, \cdots, k-1$ *in* $(a_1, a_2, \cdots, a_k)$ *for* c *near* c_0. *Then for some* (ε, a, c) *near* $(0, a_0, c_0)$, *the system* (1.6) *has at least* $k + n$ *limit cycles near the origin.*

As an application of Lemma 1.3, consider the following polynomial Liénard system

$$\begin{cases} \dot{x} = y - \varepsilon F(x,a), \\ \dot{y} = -g(x,c), \end{cases} \tag{1.13}$$

where

$$F(x,a) = \sum_{j=1}^{n} a_j x^j, \quad g(x,c) = \sum_{j=3}^{m} c_j x^j, \tag{1.14}$$

where $a = (a_1, a_2, \cdots, a_n), c = (c_3, c_4, \cdots, c_m)$ and $c_3 = 1$.

Let $L(m,n)$ denote the maximal number of limit cycles of the system (1.13) near the origin. It was obtained in [6]

$$L(m,n) \geq 2n-5, \quad for \quad m = n = 3, 4, 5.$$

In this paper we will consider the limit cycles of system (1.13) under the cases $m = 4, 2 \leq n \leq 20$, $m = 5, 2 \leq n \leq 10$, $m = 6, 2 \leq n \leq 5$ and $m = 7, 2 \leq n \leq 4$. Our main result is as follows.

Theorem 1.1. *Let $L(m,n)$ denote the maximal number of limit cycles of the system* (1.13) *near the origin for all possible $F(x,a)$ and $g(x,c)$ satisfying* (1.14). *We have*

$$L(4,n) \geq n - 1 - [\tfrac{n}{5}], \ 2 \leq n \leq 20,$$

$$L(5,n) \geq \begin{cases} n-1, & 2 \leq n \leq 4, \\ 5, & n = 5,6,7, \\ n-2, & n = 8,9, \\ 9, & n = 10. \end{cases}$$

$$L(6,n) \geq \begin{cases} 1, & n = 2, \\ 3, & n = 3, \\ 5, & n = 4,5. \end{cases}$$

$$L(7,n) \geq \begin{cases} 1, & n = 2, \\ n+1, & n = 3,4. \end{cases}$$

2. Proof of the main result

(1) $m = 4$. In this case we know that

$$g(x,c) = x^3 + c_4 x^4 (c_4 \neq 0), \quad G(x,c) = \frac{1}{4}x^4 + \frac{1}{5}c_4 x^5.$$

Then by $G(\alpha(x,c),c) \equiv G(x,c)$ in Lemma 1.2, we have

$$\begin{aligned} \alpha(x,c) \;=\; & -x - \tfrac{2}{5}\, c_4 x^2 - \tfrac{4}{25}\, {c_4}^2 x^3 - \tfrac{28}{125}\, {c_4}^3 x^4 - \tfrac{136}{625}\, {c_4}^4 x^5 - \tfrac{904}{3125}\, {c_4}^5 x^6 \\ & - \tfrac{224}{625}\, {c_4}^6 x^7 - \tfrac{38144}{78125}\, {c_4}^7 x^8 - \tfrac{258976}{390625}\, {c_4}^8 x^9 - \tfrac{1821408}{1953125}\, {c_4}^9 x^{10} \\ & - \tfrac{12927552}{9765625}\, {c_4}^{10} x^{11} - \tfrac{93331136}{48828125}\, {c_4}^{11} x^{12} - \tfrac{136035584}{48828125}\, {c_4}^{12} x^{13} \\ & - \tfrac{5008582144}{1220703125}\, {c_4}^{13} x^{14} - \tfrac{37165985792}{6103515625}\, {c_4}^{14} x^{15} - \tfrac{277796923392}{30517578125}\, {c_4}^{15} x^{16} \\ & - \tfrac{2089009203712}{152587890625}\, {c_4}^{16} x^{17} - \tfrac{15795692600832}{762939453125}\, {c_4}^{17} x^{18} \\ & - \tfrac{120012697848832}{3814697265625}\, {c_4}^{18} x^{19} - \tfrac{183160567243776}{3814697265625}\, {c_4}^{19} x^{20} \\ & - \tfrac{1403128771414016}{19073486328125}\, {c_4}^{20} x^{21} - \tfrac{431476464871424}{3814697265625}\, {c_4}^{21} x^{22} \end{aligned}$$

$$
\begin{aligned}
&-\tfrac{665548892643328}{3814697265625}\,{c_4}^{22}x^{23}-\tfrac{643511472801038336}{2384185791015625}\,{c_4}^{23}x^{24}\\
&-\tfrac{2495513367525251481 6}{59604644775390625}\,{c_4}^{24}x^{25}-\tfrac{19402958951873226342 4}{298023223876953125}\,{c_4}^{25}x^{26}\\
&-\tfrac{302411310900137361408}{298023223876953125}\,{c_4}^{26}x^{27}-\tfrac{118082882602971777597 44}{7450580596923828125}\,{c_4}^{27}x^{28}\\
&-\tfrac{92397299064558060568576}{37252902984619140625}\,{c_4}^{28}x^{29}-\tfrac{28972527208234266918912}{7450580596923828125}\,{c_4}^{29}x^{30}\\
&-\tfrac{1137536840287038946148352}{186264514923095703125}\,{c_4}^{30}x^{31}-\tfrac{4473411347609820969251635 2}{465661287307739257812 5}\,{c_4}^{31}x^{32}\\
&-\tfrac{352365371585075328032833536}{23283064365386962890625}\,{c_4}^{32}x^{33}+O(x^{34}).
\end{aligned}
\tag{2.1}
$$

In the following we give the proof of $n=20$ and $n=5$, other cases can be proved similarly.

First, take $n=20$. By (1.8), (1.14) and (2.1) we have

$$
\begin{aligned}
B_1 &= -2\,a_1,\\
B_3 &= -\tfrac{4}{25}\,a_1{c_4}^2+\tfrac{4}{5}\,a_2c_4-2\,a_3,\\
B_5 &= -\tfrac{136}{625}\,a_1{c_4}^4+\tfrac{72}{125}\,a_2{c_4}^3-\tfrac{24}{25}\,a_3{c_4}^2+\tfrac{8}{5}\,a_4c_4-2\,a_5,\\
B_7 &= -\tfrac{224}{625}\,a_1{c_4}^6+\tfrac{2576}{3125}\,a_2{c_4}^5-\tfrac{168}{125}\,a_3{c_4}^4+\tfrac{48}{25}\,a_4{c_4}^3-\tfrac{12}{5}\,a_5{c_4}^2\\
&\quad+\tfrac{12}{5}\,a_6c_4-2\,a_7,\\
B_9 &= -\tfrac{258976}{390625}\,a_1{c_4}^8+\tfrac{113536}{78125}\,a_2{c_4}^7-\tfrac{36304}{15625}\,a_3{c_4}^6+\tfrac{2016}{625}\,a_4{c_4}^5-\tfrac{504}{125}\,a_5{c_4}^4\\
&\quad+\tfrac{568}{125}\,a_6{c_4}^3-\tfrac{112}{25}\,a_7{c_4}^2+\tfrac{16}{5}\,a_8c_4-2\,a_9,\\
B_{11} &= -\tfrac{12927552}{9765625}\,a_1{c_4}^{10}+\tfrac{1108672}{390625}\,a_2{c_4}^9-\tfrac{349728}{78125}\,a_3{c_4}^8+\tfrac{768}{125}\,a_4{c_4}^7-\tfrac{192}{25}\,a_5{c_4}^6\\
&\quad+\tfrac{27696}{3125}\,a_6{c_4}^5-\tfrac{1176}{125}\,a_7{c_4}^4+\tfrac{224}{25}\,a_8{c_4}^3-\tfrac{36}{5}\,a_9{c_4}^2+4\,a_{10}c_4-2\,a_{11},\\
B_{13} &= -\tfrac{136035584}{48828125}\,a_1{c_4}^{12}+\tfrac{287946368}{48828125}\,a_2{c_4}^{11}-\tfrac{17999616}{1953125}\,a_3{c_4}^{10}+\tfrac{4910208}{390625}\,a_4{c_4}^9\\
&\quad-\tfrac{1227552}{78125}\,a_5{c_4}^8+\tfrac{1431552}{78125}\,a_6{c_4}^7-\tfrac{312816}{15625}\,a_7{c_4}^6+\tfrac{12736}{625}\,a_8{c_4}^5-\tfrac{2376}{125}\,a_9{c_4}^4\\
&\quad+\tfrac{392}{25}\,a_{10}{c_4}^3-\tfrac{264}{25}\,a_{11}{c_4}^2+\tfrac{24}{5}\,a_{12}c_4-2\,a_{13},\\
B_{15} &= -\tfrac{37165985792}{6103515625}\,a_1{c_4}^{14}+\tfrac{15599332352}{1220703125}\,a_2{c_4}^{13}-\tfrac{4844126464}{244140625}\,a_3{c_4}^{12}\\
&\quad+\tfrac{1316211456}{48828125}\,a_4{c_4}^{11}-\tfrac{329052864}{9765625}\,a_5{c_4}^{10}+\tfrac{15441984}{390625}\,a_6{c_4}^9-\tfrac{17132192}{390625}\,a_7{c_4}^8\\
&\quad+\tfrac{3596288}{78125}\,a_8{c_4}^7-\tfrac{710304}{15625}\,a_9{c_4}^6+\tfrac{130384}{3125}\,a_{10}{c_4}^5-\tfrac{21736}{625}\,a_{11}{c_4}^4\\
&\quad+\tfrac{3152}{125}\,a_{12}{c_4}^3-\tfrac{364}{25}\,a_{13}{c_4}^2+\tfrac{28}{5}\,a_{14}c_4-2\,a_{15},\\
B_{17} &= -\tfrac{2089009203712}{152587890625}\,a_1{c_4}^{16}+\tfrac{174266404864}{6103515625}\,a_2{c_4}^{15}-\tfrac{53859873792}{1220703125}\,a_3{c_4}^{14}\\
&\quad+\tfrac{2918762496}{48828125}\,a_4{c_4}^{13}-\tfrac{729690624}{9765625}\,a_5{c_4}^{12}+\tfrac{4297190784}{48828125}\,a_6{c_4}^{11}-\tfrac{192524416}{1953125}\,a_7{c_4}^{10}\\
&\quad+\tfrac{41138432}{390625}\,a_8{c_4}^9-\tfrac{8375328}{78125}\,a_9{c_4}^8+\tfrac{1616512}{15625}\,a_{10}{c_4}^7-\tfrac{1464848}{15625}\,a_{11}{c_4}^6\\
&\quad+\tfrac{49056}{625}\,a_{12}{c_4}^5-\tfrac{7384}{125}\,a_{13}{c_4}^4+\tfrac{952}{25}\,a_{14}{c_4}^3-\tfrac{96}{5}\,a_{15}{c_4}^2+\tfrac{32}{5}\,a_{16}c_4-2\,a_{17},\\
B_{19} &= -\tfrac{120012697848832}{3814697265625}\,a_1{c_4}^{18}+\tfrac{9963623271424}{152587890625}\,a_2{c_4}^{17}-\tfrac{3068349694464}{30517578125}\,a_3{c_4}^{16}\\
&\quad+\tfrac{829687652352}{6103515625}\,a_4{c_4}^{15}-\tfrac{207421913088}{1220703125}\,a_5{c_4}^{14}+\tfrac{244977312768}{1220703125}\,a_6{c_4}^{13}\\
&\quad-\tfrac{55234629632}{244140625}\,a_7{c_4}^{12}+\tfrac{11940408832}{48828125}\,a_8{c_4}^{11}-\tfrac{2477162304}{9765625}\,a_9{c_4}^{10}\\
&\quad+\tfrac{19692224}{78125}\,a_{10}{c_4}^9-\tfrac{93307808}{390625}\,a_{11}{c_4}^8+\tfrac{16737024}{78125}\,a_{12}{c_4}^7-\tfrac{2807168}{15625}\,a_{13}{c_4}^6\\
&\quad+\tfrac{431984}{3125}\,a_{14}{c_4}^5-\tfrac{11832}{125}\,a_{15}{c_4}^4+\tfrac{6848}{125}\,a_{16}{c_4}^3-\tfrac{612}{25}\,a_{17}{c_4}^2+\tfrac{36}{5}\,a_{18}c_4-2\,a_{19},\\
B_{21} &= -\tfrac{1403128771414016}{19073486328125}\,a_1{c_4}^{20}+\tfrac{580236532103168}{3814697265625}\,a_2{c_4}^{19}-\tfrac{178182652168192}{762939453125}\,a_3{c_4}^{18}\\
&\quad+\tfrac{48104537622528}{152587890625}\,a_4{c_4}^{17}-\tfrac{12026134405632}{30517578125}\,a_5{c_4}^{16}+\tfrac{2846523756544}{6103515625}\,a_6{c_4}^{15}\\
&\quad-\tfrac{644824505344}{1220703125}\,a_7{c_4}^{14}+\tfrac{28108500992}{48828125}\,a_8{c_4}^{13}-\tfrac{29534159616}{48828125}\,a_9{c_4}^{12}\\
&\quad+\tfrac{5983173248}{9765625}\,a_{10}{c_4}^{11}-\tfrac{1166122496}{1953125}\,a_{11}{c_4}^{10}+\tfrac{217789824}{390625}\,a_{12}{c_4}^9-\tfrac{38733344}{78125}\,a_{13}{c_4}^8\\
&\quad+\tfrac{1299712}{3125}\,a_{14}{c_4}^7-\tfrac{1014384}{3125}\,a_{15}{c_4}^6+\tfrac{144256}{625}\,a_{16}{c_4}^5-\tfrac{18088}{125}\,a_{17}{c_4}^4\\
&\quad+\tfrac{1896}{25}\,a_{18}{c_4}^3-\tfrac{152}{5}\,a_{19}{c_4}^2+8\,a_{20}c_4,
\end{aligned}
$$

$$
\begin{aligned}
B_{23} \;=\;& -\frac{6655488892643328}{3814697265625}\,a_1{c_4}^{22}+\frac{274374209736704}{762939453125}\,a_2{c_4}^{21}-\frac{10508132318201856}{19073486328125}\,a_3{c_4}^{20}\\
&+\frac{2833337563222016}{3814697265625}\,a_4{c_4}^{19}-\frac{708334390805504}{762939453125}\,a_5{c_4}^{18}+\frac{167923824288768}{152587890625}\,a_6{c_4}^{17}\\
&-\frac{38175735374336}{30517578125}\,a_7{c_4}^{16}+\frac{8371410911232}{6103515625}\,a_8{c_4}^{15}-\frac{1775695454208}{1220703125}\,a_9{c_4}^{14}\\
&+\frac{364634545152}{244140625}\,a_{10}{c_4}^{13}-\frac{362188876544}{244140625}\,a_{11}{c_4}^{12}+\frac{69463980288}{48828125}\,a_{12}{c_4}^{11}\\
&-\frac{12816549824}{9765625}\,a_{13}{c_4}^{10}+\frac{452606784}{390625}\,a_{14}{c_4}^{9}-\frac{75909408}{78125}\,a_{15}{c_4}^{8}+\frac{59824128}{78125}\,a_{16}{c_4}^{7}\\
&-\frac{8727392}{15625}\,a_{17}{c_4}^{6}+\frac{1152144}{3125}\,a_{18}{c_4}^{5}-\frac{1064}{5}\,a_{19}{c_4}^{4}+\frac{2544}{25}\,a_{20}{c_4}^{3},\\
B_{25} \;=\;& -\frac{2495513367525251481 6}{59604644775390625}\,a_1{c_4}^{24}+\frac{10261434188811075584}{11920928955078125}\,a_2{c_4}^{23}\\
&-\frac{3138106666361327616}{2384185791015625}\,a_3{c_4}^{22}+\frac{845268085861801984}{476837158203125}\,a_4{c_4}^{21}\\
&-\frac{211317021465450496}{95367431640625}\,a_5{c_4}^{20}+\frac{50159541565876224}{19073486328125}\,a_6{c_4}^{19}\\
&-\frac{11435196777324544}{3814697265625}\,a_7{c_4}^{18}+\frac{2519409019670528}{762939453125}\,a_8{c_4}^{17}\\
&-\frac{107640508137984}{30517578125}\,a_9{c_4}^{16}+\frac{111638206615552}{30517578125}\,a_{10}{c_4}^{15}-\frac{22488072037376}{6103515625}\,a_{11}{c_4}^{14}\\
&+\frac{4394805270528}{1220703125}\,a_{12}{c_4}^{13}-\frac{831570102272}{244140625}\,a_{13}{c_4}^{12}+\frac{151869033344}{48828125}\,a_{14}{c_4}^{11}\\
&-\frac{26651213184}{9765625}\,a_{15}{c_4}^{10}+\frac{4466604544}{1953125}\,a_{16}{c_4}^{9}-\frac{141784352}{78125}\,a_{17}{c_4}^{8}\\
&+\frac{105323904}{78125}\,a_{18}{c_4}^{7}-\frac{14409296}{15625}\,a_{19}{c_4}^{6}+\frac{1772768}{3125}\,a_{20}{c_4}^{5},\\
B_{27} \;=\;& -\frac{302411310900137361408}{298023223876953125}\,a_1{c_4}^{26}+\frac{620406084640747094016}{298023223876953125}\,a_2{c_4}^{25}\\
&-\frac{7577320656023683072}{2384185791015625}\,a_3{c_4}^{24}+\frac{16314066686312448}{3814697265625}\,a_4{c_4}^{23}\\
&-\frac{4078516671578112}{762939453125}\,a_5{c_4}^{22}+\frac{24227316616687616}{3814697265625}\,a_6{c_4}^{21}\\
&-\frac{138393056841209856}{19073486328125}\,a_7{c_4}^{20}+\frac{30605263385370624}{3814697265625}\,a_8{c_4}^{19}\\
&-\frac{6574451908471808}{762939453125}\,a_9{c_4}^{18}+\frac{274879398530048}{30517578125}\,a_{10}{c_4}^{17}-\frac{279792781869568}{30517578125}\,a_{11}{c_4}^{16}\\
&+\frac{55450729193472}{6103515625}\,a_{12}{c_4}^{15}-\frac{10686665213952}{1220703125}\,a_{13}{c_4}^{14}+\frac{1998979003392}{244140625}\,a_{14}{c_4}^{13}\\
&-\frac{361902168576}{48828125}\,a_{15}{c_4}^{12}+\frac{315862744064}{48828125}\,a_{16}{c_4}^{11}-\frac{52889021504}{9765625}\,a_{17}{c_4}^{10}\\
&+\frac{1687445696}{390625}\,a_{18}{c_4}^{9}-\frac{254131232}{78125}\,a_{19}{c_4}^{8}+\frac{35684096}{15625}\,a_{20}{c_4}^{7},\\
B_{29} \;=\;& -\frac{92397299064558060568576}{37252902984619140625}\,a_1{c_4}^{28}+\frac{37841114300012374654976}{7450580596923828125}\,a_2{c_4}^{27}\\
&-\frac{115389452480303978250 24}{1490116119384765625}\,a_3{c_4}^{26}+\frac{620641043054888615936}{59604644775390625}\,a_4{c_4}^{25}\\
&-\frac{155160260763722153984}{11920928955078125}\,a_5{c_4}^{24}+\frac{184493985171834372096}{11920928955078125}\,a_6{c_4}^{23}\\
&-\frac{42233469435369074688}{2384185791015625}\,a_7{c_4}^{22}+\frac{9368374886882787328}{476837158203125}\,a_8{c_4}^{21}\\
&-\frac{2021508664003086336}{95367431640625}\,a_9{c_4}^{20}+\frac{85046180314273792}{3814697265625}\,a_{10}{c_4}^{19}\\
&-\frac{87287551142516736}{3814697265625}\,a_{11}{c_4}^{18}+\frac{17487682730194944}{762939453125}\,a_{12}{c_4}^{17}\\
&-\frac{683544835138048}{30517578125}\,a_{13}{c_4}^{16}+\frac{650822544576512}{30517578125}\,a_{14}{c_4}^{15}-\frac{24107733629952}{1220703125}\,a_{15}{c_4}^{14}\\
&+\frac{21659103485952}{1220703125}\,a_{16}{c_4}^{13}-\frac{3763218324736}{244140625}\,a_{17}{c_4}^{12}+\frac{629501678208}{48828125}\,a_{18}{c_4}^{11}\\
&-\frac{100809177344}{9765625}\,a_{19}{c_4}^{10}+\frac{3068264064}{390625}\,a_{20}{c_4}^{9},\\
B_{31} \;=\;& -\frac{1137536840287038946148352}{1862645149230957031 25}\,a_1{c_4}^{30}+\frac{465131421991786886201344}{37252902984619140625}\,a_2{c_4}^{29}\\
&-\frac{1416711561698231892049 92}{7450580596923828125}\,a_3{c_4}^{28}+\frac{761533428126338331443 2}{298023223876953125}\,a_4{c_4}^{27}\\
&-\frac{19038335703158458286 08}{59604644775390625}\,a_5{c_4}^{26}+\frac{11326916086330132463616}{298023223876953125}\,a_6{c_4}^{25}\\
&-\frac{1038773311161130352 64}{2384185791015625}\,a_7{c_4}^{24}+\frac{184807427854434304}{3814697265625}\,a_8{c_4}^{23}\\
&-\frac{40025292119138304}{762939453125}\,a_9{c_4}^{22}+\frac{8462181818667008}{152587890625}\,a_{10}{c_4}^{21}\\
&-\frac{1092940030235682816}{19073486328125}\,a_{11}{c_4}^{20}+\frac{220866561851240448}{3814697265625}\,a_{12}{c_4}^{19}\\
&-\frac{43642453830636544}{762939453125}\,a_{13}{c_4}^{18}+\frac{1685303941694464}{30517578125}\,a_{14}{c_4}^{17}\\
&-\frac{317593943407104}{6103515625}\,a_{15}{c_4}^{16}+\frac{291581894623232}{6103515625}\,a_{16}{c_4}^{15}-\frac{52050414178304}{1220703125}\,a_{17}{c_4}^{14}\\
&+\frac{9006676540416}{244140625}\,a_{18}{c_4}^{13}-\frac{301016845568}{9765625}\,a_{19}{c_4}^{12}+\frac{241739635456}{9765625}\,a_{20}{c_4}^{11},
\end{aligned}
$$

$$
\begin{aligned}
B_{33} \;=\; & -\tfrac{352365371585075328032833536}{23283064365386962890625}\, a_1 {c_4}^{32} + \tfrac{143878924525133549354876928}{4656612873077392578125}\, a_2 {c_4}^{31} \\
& -\tfrac{43779577882150409432727552}{931322574615478515625}\, a_3 {c_4}^{30} + \tfrac{2352059979231898488537088}{37252902984619140625}\, a_4 {c_4}^{29} \\
& -\tfrac{588014994807974622134272}{7450580596923828125}\, a_5 {c_4}^{28} + \tfrac{700111130483531584765952}{7450580596923828125}\, a_6 {c_4}^{27} \\
& -\tfrac{160727139091616048480256}{1490116119384765625}\, a_7 {c_4}^{26} + \tfrac{7164025192513357479936}{59604644775390625}\, a_8 {c_4}^{25} \\
& -\tfrac{311275990338594373632}{2384185791015625}\, a_9 {c_4}^{24} + \tfrac{330438401314401091584}{2384185791015625}\, a_{10} {c_4}^{23} \\
& -\tfrac{343311710976556978176}{2384185791015625}\, a_{11} {c_4}^{22} + \tfrac{69869012750607654912}{476837158203125}\, a_{12} {c_4}^{21} \\
& -\tfrac{13928929670192183296}{95367431640625}\, a_{13} {c_4}^{20} + \tfrac{543856299898333184}{3814697265625}\, a_{14} {c_4}^{19} \\
& -\tfrac{103898027137191936}{762939453125}\, a_{15} {c_4}^{18} + \tfrac{97004709023490048}{762939453125}\, a_{16} {c_4}^{17} \\
& -\tfrac{3535500487475712}{30517578125}\, a_{17} {c_4}^{16} + \tfrac{3137449148977152}{30517578125}\, a_{18} {c_4}^{15} \\
& -\tfrac{108185148060672}{1220703125}\, a_{19} {c_4}^{14} + \tfrac{18060313430016}{244140625}\, a_{20} {c_4}^{13}.
\end{aligned}
\tag{2.2}
$$

We find that there exists a point

$$a_0 = (a_1^*, \cdots, a_4^*, a_5, a_6^*, \cdots, a_9^*, a_{10}, a_{11}^*, \cdots, a_{14}^*, a_{15} a_{16}^*, a_{17}^*, a_{18}^*, a_{19}, a_{20})$$

with

$$
\begin{aligned}
& a_1^* = 0, a_2^* = -\tfrac{3814697265625}{2756434688}\, \tfrac{-5\, a_{20} + a_{19} c_4}{{c_4}^{18}}, a_3^* = -\tfrac{762939453125}{1378217344}\, \tfrac{-5\, a_{20} + a_{19} c_4}{{c_4}^{17}}, \\
& a_4^* = \tfrac{5}{2756434688}\, \tfrac{-457763671875\, a_{20} + 91552734375\, a_{19} c_4 + 689108672\, a_5 {c_4}^{15}}{{c_4}^{16}}, \\
& a_6^* = -\tfrac{1505126953125}{172277168}\, \tfrac{-5\, a_{20} + a_{19} c_4}{{c_4}^{14}}, a_7^* = -\tfrac{3625732421875}{344554336}\, \tfrac{-5\, a_{20} + a_{19} c_4}{{c_4}^{13}}, \\
& a_8^* = \tfrac{25}{393776384}\, \tfrac{-402021484375\, a_{20} + 80404296875\, a_{19} c_4 + 24611024\, a_{10} {c_4}^{10}}{{c_4}^{12}}, \\
& a_9^* = \tfrac{5}{344554336}\, \tfrac{172277168\, a_{10} {c_4}^{10} - 4063876953125\, a_{20} + 812775390625\, a_{19} c_4}{{c_4}^{11}}, \\
& a_{11}^* = -\tfrac{27753515625}{3076378}\, \tfrac{-5\, a_{20} + a_{19} c_4}{{c_4}^{9}}, \\
& a_{12}^* = \tfrac{125}{1378217344}\, \tfrac{-53918758125\, a_{19} c_4 + 269593790625\, a_{20} + 21534646\, a_{15} {c_4}^{5}}{{c_4}^{8}}, \\
& a_{13}^* = \tfrac{25}{49222048}\, \tfrac{-6074646875\, a_{20} + 9229134\, a_{15} {c_4}^{5} + 1214929375\, a_{19} c_4}{{c_4}^{7}}, \\
& a_{14}^* = \tfrac{5}{86138584}\, \tfrac{-89568078125\, a_{20} + 64603938\, a_{15} {c_4}^{5} + 17913615625\, a_{19} c_4}{{c_4}^{6}}, \\
& a_{16}^* = -\tfrac{125}{2756434688}\, \tfrac{-29223012550\, a_{20} + 5833835187\, a_{19} c_4}{{c_4}^{4}}, \\
& a_{17}^* = -\tfrac{25}{172277168}\, \tfrac{-3800137350\, a_{20} + 749260147\, a_{19} c_4}{{c_4}^{3}}, \\
& a_{18}^* = -\tfrac{15}{2970296}\, \tfrac{-11183750\, a_{20} + 1865463\, a_{19} c_4}{{c_4}^{2}},
\end{aligned}
$$

such that $B_1(a_0) = B_3(a_0) = \cdots = B_{29}(a_0) = 0$ and

$$
\begin{aligned}
B_{31}(a_0) &= -\tfrac{27072733952}{1437255859375}\, (-5\, a_{20} + a_{19} c_4)\, {c_4}^{11}, \\
B_{33}(a_0) &= -\tfrac{31467352767172608}{65718524169921875}\, (-5\, a_{20} + a_{19} c_4)\, {c_4}^{13}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \det \tfrac{\partial(B_1, B_3, B_5, \cdots, B_{29})}{\partial(a_1, \cdots, a_4, a_6, \cdots, a_9, a_{11}, \cdots, a_{14}, a_{16}, a_{17}, a_{18})}(a_0) \\
= \;& -\tfrac{1394436119043895772653751208385286306111661723029386247012352}{10339757656912845935892608650874535669572651386260986328125}\, {c_4}^{84}.
\end{aligned}
$$

Then by Lemma 1.3, for some (ε, a) near $(0, a_0)$, the system (1.13) has 15 limit cycles if $a_{20} \neq -\frac{c_4}{5} a_{19}$ and $c_4 \neq 0$.

Next, we consider the case of $n = 5$. In this case, (2.2) holds with $a_6 = a_7 = \cdots = a_{20} = 0$. We can find $a_0 = (a_1^*, a_2^*, a_3^*, a_4, a_5)$ with

$$a_1^* = 0,\; a_2^* = -\frac{25}{12}\, \frac{4\, a_4 c_4 - 5\, a_5}{{c_4}^{3}},\; a_3^* = -\frac{5}{6}\, \frac{4\, a_4 c_4 - 5\, a_5}{{c_4}^{2}},$$

such that $B_1(a_0) = B_3(a_0) = B_5(a_0) = 0$ and

$$B_7(a_0) = \frac{44}{75}c_4^2(a_5 - \frac{4}{5}c_4a_4), \quad B_9(a_0) = \frac{2672}{1875}c_4^4(a_5 - \frac{4}{5}c_4a_4).$$

Suppose $a_5 \neq \frac{4}{5}c_4a_4, c_4 \neq 0$. Note that

$$\det \frac{\partial(B_1, B_3, B_5)}{\partial(a_1, a_2, a_3,)}(a_0) = -\frac{96}{125}\,{c_4}^3.$$

Then by Lemma 1.3, it is easy to see that for some (ε, a) near $(0, a_0)$, the system (1.13) has 3 limit cycles near the origin.

(2) $m = 5$.

Now we have

$$g(x, c) = x^3 + c_4x^4 + c_5x^5$$

and

$$G(x, c) = \frac{1}{4}x^4 + \frac{1}{5}c_4x^5 + \frac{1}{6}c_5x^6.$$

By Lemma 1.2, let

$$\alpha(x, c) = -x + \sum_{i\geq 2} \alpha_i x^i \tag{2.3}$$

satisfying $G(\alpha(x, c), c) = G(x, c)$. Then using Maple 13 we easily get

$$\begin{aligned}
\alpha_2 &= -\tfrac{2}{5}\,c_4,\ \alpha_3 = -\tfrac{4}{25}\,{c_4}^2,\ \alpha_4 = -\tfrac{28}{125}\,{c_4}^3 + \tfrac{2}{5}\,c_5c_4,\ \alpha_5 = -\tfrac{136}{625}\,{c_4}^4 + \tfrac{12}{25}\,c_5{c_4}^2,\\
\alpha_6 &= -\tfrac{904}{3125}\,{c_4}^5 + \tfrac{344}{375}\,{c_4}^3c_5 - \tfrac{2}{5}\,{c_5}^2c_4,\ \alpha_7 = -\tfrac{224}{625}\,{c_4}^6 + \tfrac{512}{375}\,{c_4}^4c_5 - \tfrac{24}{25}\,{c_5}^2{c_4}^2,\\
\alpha_8 &= -\tfrac{38144}{78125}\,{c_4}^7 + \tfrac{21112}{9375}\,{c_4}^5c_5 - \tfrac{184}{75}\,{c_5}^2{c_4}^3 + \tfrac{2}{5}\,{c_5}^3c_4,\\
\alpha_9 &= \tfrac{55104}{15625}\,{c_4}^6c_5 - \tfrac{376}{75}\,{c_4}^4{c_5}^2 + \tfrac{8}{5}\,{c_5}^3{c_4}^2 - \tfrac{258976}{390625}\,{c_4}^8,\\
\alpha_{10} &= \tfrac{1327744}{234375}\,{c_4}^7c_5 - \tfrac{95072}{9375}\,{c_4}^5{c_5}^2 + \tfrac{16}{3}\,{c_4}^3{c_5}^3 - \tfrac{2}{5}\,{c_5}^4c_4 - \tfrac{1821408}{1953125}\,{c_4}^9,\\
\alpha_{11} &= \tfrac{10551808}{1171875}\,{c_4}^8c_5 - \tfrac{301504}{15625}\,{c_4}^6{c_5}^2 + \tfrac{352}{25}\,{c_4}^4{c_5}^3 - \tfrac{12}{5}\,{c_5}^4{c_4}^2 - \tfrac{12927552}{9765625}\,{c_4}^{10}\\
\alpha_{12} &= \tfrac{16895008}{1171875}\,{c_4}^9c_5 - \tfrac{25414336}{703125}\,{c_4}^7{c_5}^2 + \tfrac{322112}{9375}\,{c_4}^5{c_5}^3 - \tfrac{764}{75}\,{c_5}^4{c_4}^3\\
&\quad -\tfrac{93331136}{48828125}\,{c_4}^{11} + \tfrac{2}{5}\,{c_5}^5c_4,\\
\alpha_{13} &= \tfrac{675421504}{29296875}\,{c_4}^{10}c_5 - \tfrac{5163968}{78125}\,{c_4}^8{c_5}^2 + \tfrac{435904}{5625}\,{c_4}^6{c_5}^3 - \tfrac{12544}{375}\,{c_5}^4{c_4}^4\\
&\quad +\tfrac{84}{25}\,{c_5}^5{c_4}^2 - \tfrac{136035584}{48828125}\,{c_4}^{12},\\
\alpha_{14} &= -\tfrac{2}{5}\,{c_5}^6c_4 + \tfrac{5415938944}{146484375}\,{c_4}^{11}c_5 - \tfrac{139973344}{1171875}\,{c_4}^9{c_5}^2 + \tfrac{4705664}{28125}\,{c_4}^7{c_5}^3\\
&\quad -\tfrac{908656}{9375}\,{c_5}^4{c_4}^5 + \tfrac{6664}{375}\,{c_5}^5{c_4}^3 - \tfrac{5008582144}{1220703125}\,{c_4}^{13},\\
\alpha_{15} &= -\tfrac{112}{25}\,{c_5}^6{c_4}^2 + \tfrac{14486810624}{244140625}\,{c_4}^{12}c_5 - \tfrac{749588224}{3515625}\,{c_4}^{10}{c_5}^2 + \tfrac{48894976}{140625}\,{c_4}^8{c_5}^3\\
&\quad -\tfrac{3979136}{15625}\,{c_5}^4{c_4}^6 + \tfrac{132608}{1875}\,{c_5}^5{c_4}^4 - \tfrac{37165985792}{6103515625}\,{c_4}^{14},\\
\alpha_{16} &= -\tfrac{10864}{375}\,{c_5}^6{c_4}^3 + \tfrac{349326490112}{3662109375}\,{c_4}^{13}c_5 - \tfrac{55293146624}{146484375}\,{c_4}^{11}{c_5}^2\\
&\quad +\tfrac{7419305696}{10546875}\,{c_4}^9{c_5}^3 - \tfrac{88189504}{140625}\,{c_5}^4{c_4}^7 + \tfrac{149968}{625}\,{c_5}^5{c_4}^5 - \tfrac{277796923392}{30517578125}\,{c_4}^{15}\\
&\quad +\tfrac{2}{5}\,{c_5}^7c_4,\\
\alpha_{17} &= -\tfrac{17136}{125}\,{c_4}^4{c_5}^6 + \tfrac{562161289216}{3662109375}\,{c_4}^{14}c_5 - \tfrac{97163216384}{146484375}\,{c_4}^{12}{c_5}^2\\
&\quad +\tfrac{122302292992}{87890625}\,{c_4}^{10}{c_5}^3 - \tfrac{22917664}{15625}\,{c_4}^8{c_5}^4 + \tfrac{11325888}{15625}\,{c_4}^6{c_5}^5 + \tfrac{144}{25}\,{c_5}^7{c_4}^2\\
&\quad -\tfrac{2089009203712}{152587890625}\,{c_4}^{16},\\
\alpha_{18} &= -\tfrac{1681568}{3125}\,{c_4}^5{c_5}^6 + \tfrac{22644204335104}{91552734375}\,{c_4}^{15}c_5 - \tfrac{2546279472128}{2197265625}\,{c_4}^{13}{c_5}^2\\
&\quad +\tfrac{3565343455232}{1318359375}\,{c_4}^{11}{c_5}^3 - \tfrac{104380253824}{31640625}\,{c_4}^9{c_5}^4 + \tfrac{1418485376}{703125}\,{c_4}^7{c_5}^5\\
&\quad +\tfrac{224}{5}\,{c_4}^3{c_5}^7 - \tfrac{15795692600832}{762939453125}\,{c_4}^{17} - \tfrac{2}{5}\,{c_5}^8c_4,
\end{aligned}$$

$$\begin{aligned}\alpha_{19} &= -\tfrac{86688448}{46875}\,c_5{}^6c_4{}^6+\tfrac{6208}{25}\,c_5{}^7c_4{}^4+\tfrac{182608797810688}{457763671875}\,c_4{}^{16}c_5\\ &\quad-\tfrac{7377060235264}{3662109375}\,c_4{}^{14}c_5{}^2+\tfrac{2275528493056}{439453125}\,c_4{}^{12}c_5{}^3-\tfrac{5684973868288}{791015625}\,c_4{}^{10}c_5{}^4\\ &\quad+\tfrac{6173649152}{1171875}\,c_5{}^5c_4{}^8-\tfrac{36}{5}\,c_5{}^8c_4{}^2-\tfrac{120012697848832}{3814697265625}\,c_4{}^{18}.\end{aligned}\tag{2.4}$$

In the following we give the proof of $n = 10$ and $n = 4$, other cases can be proved similarly.

First, take $n = 10$. By (1.8), (1.14) and (2.4) we can give the formulas of $B_1, B_3, \cdots, B_{19}$ using Maple 13. Here we only present the formulas of $B_1, B_3, \cdots, B_{11}$ since the formulas of $B_{13}, B_{15}, B_{17}, B_{19}$ are too long. As an appendix, we give the programs written by Maple 13 by which we can compute $B_i, 1 \le i \le n$. For the formulas of $B_1, B_3, \cdots, B_{11}$, we have

$$\begin{aligned}
B_1 &= -2a_1,\\
B_3 &= -\tfrac{4}{25}\,a_1c_4{}^2+\tfrac{4}{5}\,a_2c_4-2\,a_3,\\
B_5 &= -\tfrac{136}{625}\,a_1c_4{}^4+\tfrac{12}{25}\,a_1c_5c_4{}^2+\tfrac{72}{125}\,a_2c_4{}^3-\tfrac{4}{5}\,a_2c_5c_4-\tfrac{24}{25}\,a_3c_4{}^2+\tfrac{8}{5}\,a_4c_4-2\,a_5,\\
B_7 &= -\tfrac{224}{625}\,a_1c_4{}^6+\tfrac{512}{375}\,a_1c_4{}^4c_5-\tfrac{24}{25}\,a_1c_5{}^2c_4{}^2+\tfrac{2576}{3125}\,a_2c_4{}^5-\tfrac{176}{75}\,a_2c_4{}^3c_5\\
&\quad+\tfrac{4}{5}\,a_2c_5{}^2c_4-\tfrac{168}{125}\,a_3c_4{}^4+\tfrac{12}{5}\,a_3c_5c_4{}^2+\tfrac{48}{25}\,a_4c_4{}^3-\tfrac{8}{5}\,a_4c_5c_4-\tfrac{12}{5}\,a_5c_4{}^2\\
&\quad+\tfrac{12}{5}\,a_6c_4-2\,a_7,\\
B_9 &= \tfrac{55104}{15625}\,a_1c_4{}^6c_5-\tfrac{376}{75}\,a_1c_4{}^4c_5{}^2+\tfrac{8}{5}\,a_1c_5{}^3c_4{}^2-\tfrac{258976}{390625}\,a_1c_4{}^8+\tfrac{113536}{78125}\,a_2c_4{}^7\\
&\quad-\tfrac{58864}{9375}\,a_2c_4{}^5c_5+\tfrac{464}{75}\,a_2c_5{}^2c_4{}^3-\tfrac{4}{5}\,a_2c_5{}^3c_4-\tfrac{36304}{15625}\,a_3c_4{}^6+\tfrac{192}{25}\,a_3c_4{}^4c_5\\
&\quad-\tfrac{108}{25}\,a_3c_5{}^2c_4{}^2+\tfrac{2016}{625}\,a_4c_4{}^5-\tfrac{2816}{375}\,a_4c_4{}^3c_5+\tfrac{8}{5}\,a_4c_5{}^2c_4-\tfrac{504}{125}\,a_5c_4{}^4\\
&\quad+\tfrac{28}{5}\,a_5c_5c_4{}^2+\tfrac{568}{125}\,a_6c_4{}^3-\tfrac{12}{5}\,a_6c_5c_4-\tfrac{112}{25}\,a_7c_4{}^2+\tfrac{16}{5}\,a_8c_4-2\,a_9,\\
B_{11} &= \tfrac{768}{125}\,a_4c_4{}^7-\tfrac{192}{25}\,a_5c_4{}^6+4\,a_{10}c_4-\tfrac{36}{5}\,a_9c_4{}^2+\tfrac{27696}{3125}\,a_6c_4{}^5-\tfrac{1176}{125}\,a_7c_4{}^4\\
&\quad+\tfrac{224}{25}\,a_8c_4{}^3-\tfrac{12927552}{9765625}\,a_1c_4{}^{10}+\tfrac{1108672}{390625}\,a_2c_4{}^9-\tfrac{349728}{78125}\,a_3c_4{}^8-\tfrac{16}{5}\,a_8c_5c_4\\
&\quad-\tfrac{3854848}{234375}\,a_2c_4{}^7c_5-\tfrac{219232}{9375}\,a_4c_4{}^5c_5+\tfrac{8192}{375}\,a_5c_4{}^4c_5-\tfrac{992}{75}\,a_2c_4{}^3c_5{}^3\\
&\quad+\tfrac{7136}{375}\,a_4c_5{}^2c_4{}^3+\tfrac{10551808}{1171875}\,a_1c_4{}^8c_5-\tfrac{301504}{15625}\,a_1c_4{}^6c_5{}^2+\tfrac{352}{25}\,a_1c_4{}^4c_5{}^3\\
&\quad-\tfrac{12}{5}\,a_1c_5{}^4c_4{}^2+\tfrac{259264}{9375}\,a_2c_4{}^5c_5{}^2+\tfrac{4}{5}\,a_2c_5{}^4c_4-\tfrac{3304}{125}\,a_3c_4{}^4c_5{}^2\\
&\quad+\tfrac{335104}{15625}\,a_3c_4{}^6c_5+\tfrac{168}{25}\,a_3c_5{}^3c_4{}^2-\tfrac{8}{5}\,a_4c_5{}^3c_4-\tfrac{48}{5}\,a_5c_5{}^2c_4{}^2-\tfrac{2128}{125}\,a_6c_4{}^3c_5\\
&\quad+\tfrac{12}{5}\,a_6c_5{}^2c_4+\tfrac{252}{25}\,a_7c_5c_4{}^2.
\end{aligned}\tag{2.5}$$

We can find $a_0 = (a_1^*, \cdots, a_5^*, a_6, a_7^*, a_8, a_9^*, a_{10}^*)$ with

$$\begin{aligned}
a_1^* &= 0,\ a_2^*=\tfrac{1875000}{K}\left(6\,c_4{}^2-25\,c_5\right)a_8c_5,\ a_3^*=\tfrac{750000}{K}\left(6\,c_4{}^2-25\,c_5\right)a_8c_5c_4,\\
a_4^* &= \tfrac{3a_6}{2c_5}+\tfrac{3a_8}{2Kc_5}\left(-9000000\,c_4{}^6c_5+63500000\,c_4{}^4c_5{}^2-133750000\,c_5{}^3c_4{}^2\right.\\
&\quad\left.+78125000\,c_5{}^4\right),\\
a_5^* &= \tfrac{6a_6c_4}{5c_5}+\tfrac{6c_4a_8}{5Kc_5}\left(64400000\,c_4{}^4c_5{}^2-141250000\,c_5{}^3c_4{}^2+93750000\,c_5{}^4\right.\\
&\quad\left.-9000000\,c_4{}^6c_5\right),\\
a_7^* &= \tfrac{800c_4c_5a_8}{K}\left(13896\,c_4{}^6-99900\,c_4{}^4c_5+218750\,c_5{}^2c_4{}^2-140625\,c_5{}^3\right),\\
a_9^* &= \tfrac{8c_4a_8}{5K}\left(-608400\,c_4{}^8c_5-27250000\,c_4{}^4c_5{}^3+49218750\,c_5{}^4c_4{}^2\right.\\
&\quad\left.+6244000\,c_4{}^6c_5{}^2+26784\,c_4{}^{10}-29296875\,c_5{}^5\right),\\
a_{10}^* &= \tfrac{4a_8}{25K}\left(1962000\,c_4{}^8c_5{}^2+25550000\,c_4{}^6c_5{}^3+246093750\,c_5{}^5c_4{}^2\right.\\
&\quad\left.-136250000\,c_5{}^4c_4{}^4-1205280\,c_4{}^{10}c_5+107136\,c_4{}^{12}-146484375\,c_5{}^6\right)
\end{aligned}$$

and

$$\begin{aligned} K &= 26784\, c_4{}^{10} + 1694160\, c_4{}^8 c_5 - 3006000\, c_4{}^6 c_5{}^2 - 43075000\, c_4{}^4 c_5{}^3 \\ &\quad + 138281250\, c_5{}^4 c_4{}^2 - 99609375\, c_5{}^5, \end{aligned}$$

such that

$$B_1(a_0) = B_3(a_0) = \cdots = B_{13}(a_0) = B_{15}(a_0) = 0,$$

and

$$B_{17}(a_0) = L_0 \Delta_0, \quad B_{19}(a_0) = L_1 \Delta_1,$$

where

$$\begin{aligned} L_0 &= \tfrac{7424}{703125} a_8, \ L_1 = \tfrac{512}{10546875} a_8, \\ \Delta_0 &= \tfrac{1}{K} \left(1630368\, c_4{}^{10} - 28177200\, c_4{}^8 c_5 + 190890000\, c_4{}^6 c_5{}^2 \right. \\ &\quad \left. -630375000\, c_4{}^4 c_5{}^3 + 1007031250\, c_5{}^4 c_4{}^2 - 615234375\, c_5{}^5 \right) c_4{}^7 c_5, \\ \Delta_1 &= \tfrac{1}{K} \left(3248105184\, c_4{}^{12} - 57821020560\, c_4{}^{10} c_5 + 409264930800\, c_4{}^8 c_5{}^2 \right. \\ &\quad -1450804725000\, c_4{}^6 c_5{}^3 + 2644761843750\, c_5{}^4 c_4{}^4 - 2234995703125\, c_5{}^5 c_4{}^2 \\ &\quad \left. +607851562500\, c_5{}^6 \right) c_4{}^7 c_5. \end{aligned}$$

It is obvious that $L_0 \neq 0, L_1 \neq 0$ if $a_8 \neq 0$. Further, from the equation $\Delta_0 = 0$ in c_5, we get its real roots

$$\begin{aligned} c_5^{(1)} &= \tfrac{2}{25} \left(\tfrac{361}{63} + \tfrac{4}{63}\sqrt{715}\right) c_4{}^2, \quad c_5^{(2)} = \tfrac{2}{25} \left(\tfrac{361}{63} - \tfrac{4}{63}\sqrt{715}\right) c_4{}^2, \\ c_5^{(3)} &= \tfrac{2}{5} \left(-\tfrac{1}{15}\sqrt[3]{2443 + 2\sqrt{369031}} - \frac{11}{\sqrt[3]{2443+2\sqrt{369031}}} - \tfrac{4}{15}\right) c_4{}^2. \end{aligned}$$

Take $c_0 = (c_4, c_5^{(1)})$, we obtain

$$\Delta_1(c_0) = -\frac{273792}{875} \frac{c_4{}^{11}\left(599731061\sqrt{715} + 16008764624\right)}{76643019625 + 3138338158\sqrt{715}} \neq 0$$

if $c_4 \neq 0$. We also have

$$\begin{aligned} & \det \tfrac{\partial(B_1, B_3, \cdots, B_{15})}{\partial(a_1, a_2, a_3, a_4, a_5, a_7, a_9, a_{10})}(a_0, c_0) \\ = \ & \tfrac{31907312041984}{126782032558332681655883789062 5} c_4{}^{23} \left(361 + 4\sqrt{715}\right) \\ & \cdot \left(211591560419\sqrt{715} + 5539561626845\right), \\ & \det \tfrac{\partial(\Delta_0, \Delta_1)}{(c_4, c_5)}(a_0, c_0) \\ = \ & \left(\tfrac{16659879018706322295655373157171 2}{72378769244693625} + \tfrac{31151836430727728558598107693056}{361893846223468125}\sqrt{715}\right) c_4{}^{37}. \end{aligned}$$

Then by Lemma 1.3 there exist 9 limit cycles for some (ε, a, c) near $(0, a_0, c_0)$ for system (1.13) if $a_8 \neq 0$ and $c_4 \neq 0$.

Next, take $n = 4$. Now (2.5) holds with $a_5 = a_6 = a_7 = \cdots = a_{10} = 0$. Suppose $c_4 \neq 0$. Then we can find $a_0 = (a_1^*, a_2^*, a_3, a_4^*)$ with

$$a_1^* = 0, \quad a_2^* = \frac{5a_3}{2c_4}, \quad a_4^* = -\frac{a_3\left(6\, c_4{}^2 - 25\, c_5\right)}{20\, c_4},$$

such that $B_1(a_0) = B_3(a_0) = B_5(a_0) = 0$ and

$$\begin{aligned} B_7(a_0) &= \tfrac{44}{25} c_4^2 a_3 \left(\tfrac{2}{25} c_4^2 - \tfrac{1}{3} c_5\right), \\ B_9(a_0) &= \tfrac{16}{625}\, a_3 c_4{}^2 \left(\tfrac{2}{25} c_4^2 - \tfrac{1}{3} c_5\right) \left(167\, c_4{}^2 - 150\, c_5\right). \end{aligned}$$

We also have

$$\det \frac{\partial(B_1, B_3, B_5)}{\partial(a_1, a_2, a_4)}(a_0) = -\frac{64}{25}\, {c_4}^2.$$

By Lemma 1.3 there exist 3 limit cycles for some (ε, a) near $(0, a_0)$ for system (1.13) if $a_3 \neq 0, c_5 \neq \frac{6}{25}c_4^2$ and $c_4 \neq 0$.

(3) $m = 6$ and $m = 7$

First, take $m = 7$. We have

$$g(x, c) = x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7,$$

and

$$G(x, c) = \frac{1}{4}x^4 + \frac{1}{5}c_4x^5 + \frac{1}{6}c_5x^6 + \frac{1}{7}c_6x^7 + \frac{1}{8}c_7x^8.$$

Using Maple 13 we easily get the coefficients α_i in (2.3) as follows.

$$\begin{aligned}
\alpha_2 &= -\tfrac{2}{5}\, c_4, \alpha_3 = -\tfrac{4}{25}\, {c_4}^2, \alpha_4 = -\tfrac{28}{125}\, {c_4}^3 + \tfrac{2}{5}\, c_5c_4 - \tfrac{2}{7}\, c_6,\\
\alpha_5 &= -\tfrac{136}{625}\, {c_4}^4 + \tfrac{12}{25}\, c_5{c_4}^2 - \tfrac{12}{35}\, c_6c_4,\\
\alpha_6 &= -\tfrac{904}{3125}\, {c_4}^5 + \tfrac{344}{375}\, c_5{c_4}^3 - \tfrac{132}{175}\, c_6{c_4}^2 - \tfrac{2}{5}\, {c_5}^2c_4 + \tfrac{2}{7}\, c_5c_6 + \tfrac{2}{5}c_7c_4, ,\\
\alpha_7 &= \tfrac{16}{25}\, c_7{c_4}^2 + \tfrac{32}{35}\, c_4c_5c_6 + \tfrac{512}{375}\, c_5{c_4}^4 - \tfrac{992}{875}\, c_6{c_4}^3 - \tfrac{24}{25}\, {c_5}^2{c_4}^2 - \tfrac{8}{49}\, {c_6}^2\\
&\quad -\tfrac{224}{625}\, {c_4}^6,\\
\alpha_8 &= \tfrac{512}{175}\, {c_4}^2c_5c_6 - \tfrac{4}{5}\, c_5c_7c_4 + \tfrac{21112}{9375}\, {c_4}^5c_5 - \tfrac{1688}{875}\, {c_4}^4c_6 - \tfrac{184}{75}\, {c_4}^3{c_5}^2\\
&\quad +\tfrac{156}{125}\, {c_4}^3c_7 - \tfrac{188}{245}\, c_4{c_6}^2 + \tfrac{2}{5}\, {c_5}^3c_4 - \tfrac{2}{7}\, {c_5}^2c_6 + \tfrac{2}{7}\, c_7c_6 - \tfrac{38144}{78125}\, {c_4}^7,\\
\alpha_9 &= \tfrac{3424}{525}\, {c_4}^3c_5c_6 + \tfrac{55104}{15625}\, {c_4}^6c_5 - \tfrac{66704}{21875}\, {c_4}^5c_6 - \tfrac{376}{75}\, {c_4}^4{c_5}^2 + \tfrac{256}{125}\, {c_4}^4c_7\\
&\quad -\tfrac{96}{49}\, {c_4}^2{c_6}^2 + \tfrac{8}{5}\, {c_5}^3{c_4}^2 + \tfrac{20}{49}\, c_5{c_6}^2 - \tfrac{12}{5}\, c_5c_7{c_4}^2 - \tfrac{12}{7}\, c_4{c_5}^2c_6 + \tfrac{8}{7}\, c_4c_7c_6\\
&\quad -\tfrac{258976}{390625}\, {c_4}^8,\\
\alpha_{10} &= \tfrac{37112}{2625}\, c_5{c_4}^4c_6 - \tfrac{1304}{175}\, {c_5}^2{c_4}^2c_6 + \tfrac{704}{245}\, c_5{c_6}^2c_4 - \tfrac{2432}{375}\, c_7{c_4}^3c_5 + \tfrac{132}{35}\, c_7{c_4}^2c_6\\
&\quad +\tfrac{6}{5}\, {c_5}^2c_7c_4 - \tfrac{4}{7}\, c_5c_7c_6 + \tfrac{1327744}{234375}\, {c_4}^7c_5 - \tfrac{542208}{109375}\, {c_4}^6c_6 - \tfrac{95072}{9375}\, {c_5}^2{c_4}^5\\
&\quad -\tfrac{28656}{6125}\, {c_6}^2{c_4}^3 + \tfrac{16}{3}\, {c_5}^3{c_4}^3 + \tfrac{10856}{3125}\, c_7{c_4}^5 - \tfrac{2}{5}\, {c_5}^4c_4 + \tfrac{2}{7}\, {c_5}^3c_6\\
&\quad -\tfrac{2}{5}\, {c_7}^2c_4 - \tfrac{1821408}{1953125}\, {c_4}^9 - \tfrac{12}{49}\, {c_6}^3,\\
\alpha_{11} &= \tfrac{613696}{21875}\, {c_4}^5c_5c_6 + \tfrac{13056}{1225}\, {c_4}^2c_5{c_6}^2 - \tfrac{1792}{125}\, {c_4}^4c_5c_7 - \tfrac{3936}{175}\, {c_4}^3{c_5}^2c_6 + \tfrac{64}{7}\, {c_4}^3c_7c_6\\
&\quad +\tfrac{96}{35}\, {c_5}^3c_4c_6 + \tfrac{144}{25}\, {c_5}^2{c_4}^2c_7 - \tfrac{144}{35}\, c_5c_6c_7c_4 + \tfrac{10551808}{1171875}\, {c_4}^8c_5 - \tfrac{4337152}{546875}\, {c_4}^7c_6\\
&\quad -\tfrac{301504}{15625}\, {c_4}^6{c_5}^2 + \tfrac{3552}{625}\, {c_4}^6c_7 - \tfrac{60112}{6125}\, {c_4}^4{c_6}^2 + \tfrac{352}{25}\, {c_4}^4{c_5}^3 - \tfrac{2384}{1715}\, {c_6}^3c_4\\
&\quad -\tfrac{12}{5}\, {c_5}^4{c_4}^2 - \tfrac{36}{49}\, {c_5}^2{c_6}^2 - \tfrac{36}{25}\, {c_7}^2{c_4}^2 + \tfrac{24}{49}\, c_7{c_6}^2 - \tfrac{12927552}{9765625}\, {c_4}^{10}.
\end{aligned} \tag{2.6}$$

In the following we only give the proof of $n = 4$ since other cases can be proved similarly.

By (1.8), (1.14) and (2.6) we can give the formulas of $B_1, B_3, \cdots, B_{11}$ using

Maple 13.

$$
\begin{aligned}
B_1 &= -2a_1,\\
B_3 &= -\tfrac{4}{25}\,a_1{c_4}^2+\tfrac{4}{5}\,a_2c_4-2\,a_3,\\
B_5 &= -\tfrac{136}{625}\,a_1{c_4}^4+\tfrac{12}{25}\,a_1c_5{c_4}^2-\tfrac{12}{35}\,a_1c_6c_4+\tfrac{72}{125}\,a_2{c_4}^3-\tfrac{4}{5}\,a_2c_5c_4+\tfrac{4}{7}\,a_2c_6\\
&\quad -\tfrac{24}{25}\,a_3{c_4}^2+\tfrac{8}{5}\,a_4c_4,\\
B_7 &= \tfrac{16}{25}\,a_1c_7{c_4}^2+\tfrac{32}{35}\,a_1c_4c_5c_6+\tfrac{512}{375}\,a_1c_5{c_4}^4-\tfrac{992}{875}\,a_1c_6{c_4}^3-\tfrac{24}{25}\,a_1{c_5}^2{c_4}^2\\
&\quad -\tfrac{8}{49}\,a_1{c_6}^2-\tfrac{224}{625}\,a_1{c_4}^6+\tfrac{2576}{3125}\,a_2{c_4}^5-\tfrac{176}{75}\,a_2c_5{c_4}^3+\tfrac{328}{175}\,a_2c_6{c_4}^2\\
&\quad +\tfrac{4}{5}\,a_2{c_5}^2c_4-\tfrac{4}{7}\,a_2c_5c_6-\tfrac{4}{5}\,a_2c_7c_4-\tfrac{168}{125}\,a_3{c_4}^4+\tfrac{12}{5}\,a_3c_5{c_4}^2-\tfrac{12}{7}\,a_3c_6c_4\\
&\quad +\tfrac{48}{25}\,a_4{c_4}^3-\tfrac{8}{5}\,a_4c_5c_4+\tfrac{8}{7}\,a_4c_6,\\
B_9 &= \tfrac{3424}{525}\,a_1{c_4}^3c_5c_6-\tfrac{12}{5}\,a_1c_5c_7{c_4}^2-\tfrac{12}{7}\,a_1c_4{c_5}^2c_6+\tfrac{8}{7}\,a_1c_4c_7c_6-\tfrac{1264}{175}\,a_2{c_4}^2c_5c_6\\
&\quad +\tfrac{8}{5}\,a_2c_5c_7c_4+\tfrac{144}{35}\,a_3c_4c_5c_6-\tfrac{258976}{390625}\,a_1{c_4}^8+\tfrac{113536}{78125}\,a_2{c_4}^7+\tfrac{55104}{15625}\,a_1{c_4}^6c_5\\
&\quad -\tfrac{66704}{21875}\,a_1{c_4}^5c_6-\tfrac{376}{75}\,a_1{c_4}^4{c_5}^2+\tfrac{256}{125}\,a_1{c_4}^4c_7-\tfrac{96}{49}\,a_1{c_4}^2{c_6}^2+\tfrac{8}{5}\,a_1{c_5}^3{c_4}^2\\
&\quad +\tfrac{20}{49}\,a_1c_5{c_6}^2-\tfrac{58864}{9375}\,a_2{c_4}^5c_5+\tfrac{4624}{875}\,a_2{c_4}^4c_6+\tfrac{464}{75}\,a_2{c_4}^3{c_5}^2-\tfrac{392}{125}\,a_2{c_4}^3c_7\\
&\quad +\tfrac{456}{245}\,a_2c_4{c_6}^2-\tfrac{4}{5}\,a_2{c_5}^3c_4+\tfrac{4}{7}\,a_2{c_5}^2c_6-\tfrac{4}{7}\,a_2c_7c_6+\tfrac{192}{25}\,a_3c_5{c_4}^4\\
&\quad +\tfrac{72}{25}\,a_3c_7{c_4}^2-\tfrac{5424}{875}\,a_3c_6{c_4}^3-\tfrac{108}{25}\,a_3{c_5}^2{c_4}^2-\tfrac{2816}{375}\,a_4c_5{c_4}^3+\tfrac{144}{25}\,a_4c_6{c_4}^2\\
&\quad +\tfrac{8}{5}\,a_4{c_5}^2c_4-\tfrac{8}{7}\,a_4c_5c_6-\tfrac{8}{5}\,a_4c_7c_4-\tfrac{36}{49}\,a_3{c_6}^2-\tfrac{36304}{15625}\,a_3{c_4}^6+\tfrac{2016}{625}\,a_4{c_4}^5,\\
B_{11} &= -\tfrac{144}{35}\,a_1c_5c_6c_7c_4+\tfrac{13056}{1225}\,a_1{c_4}^2c_5{c_6}^2-\tfrac{1792}{125}\,a_1{c_4}^4c_5c_7+\tfrac{613696}{21875}\,a_1{c_4}^5c_5c_6\\
&\quad +\tfrac{96}{35}\,a_1{c_5}^3c_4c_6+\tfrac{144}{25}\,a_1{c_5}^2{c_4}^2c_7-\tfrac{3936}{175}\,a_1{c_4}^3{c_5}^2c_6+\tfrac{64}{7}\,a_1{c_4}^3c_7c_6\\
&\quad -\tfrac{36}{5}\,a_3c_4{c_5}^2c_6+\tfrac{4192}{125}\,a_3{c_4}^3c_5c_6-\tfrac{252}{25}\,a_3c_5c_7{c_4}^2+\tfrac{3184}{175}\,a_2{c_5}^2{c_4}^2c_6\\
&\quad -\tfrac{1696}{245}\,a_2c_5{c_6}^2c_4-\tfrac{19952}{525}\,a_2c_5{c_4}^4c_6-\tfrac{12}{5}\,a_2{c_5}^2c_7c_4+\tfrac{8}{7}\,a_2c_5c_7c_6\\
&\quad +\tfrac{6016}{375}\,a_2c_7{c_4}^3c_5-\tfrac{1608}{175}\,a_2c_7{c_4}^2c_6+\tfrac{24}{5}\,a_3c_4c_7c_6-\tfrac{3776}{175}\,a_4{c_4}^2c_5c_6\\
&\quad +\tfrac{16}{5}\,a_4c_5c_7c_4+\tfrac{1108672}{390625}\,a_2{c_4}^9+\tfrac{200}{343}\,a_2{c_6}^3-\tfrac{12927552}{9765625}\,a_1{c_4}^{10}-\tfrac{349728}{78125}\,a_3{c_4}^8\\
&\quad +\tfrac{768}{125}\,a_4{c_4}^7-\tfrac{219232}{9375}\,a_4{c_4}^5c_5+\tfrac{10551808}{1171875}\,a_1{c_4}^8c_5-\tfrac{4337152}{546875}\,a_1{c_4}^7c_6\\
&\quad -\tfrac{301504}{15625}\,a_1{c_4}^6{c_5}^2+\tfrac{3552}{625}\,a_1{c_4}^6c_7-\tfrac{60112}{6125}\,a_1{c_4}^4{c_6}^2+\tfrac{352}{25}\,a_1{c_4}^4{c_5}^3\\
&\quad -\tfrac{2384}{1715}\,a_1{c_6}^3c_4-\tfrac{12}{5}\,a_1{c_5}^4{c_4}^2-\tfrac{36}{49}\,a_1{c_5}^2{c_6}^2-\tfrac{36}{25}\,a_1{c_7}^2{c_4}^2+\tfrac{24}{49}\,a_1c_7{c_6}^2\\
&\quad +\tfrac{1554304}{109375}\,a_2{c_4}^6c_6-\tfrac{3854848}{234375}\,a_2{c_4}^7c_5+\tfrac{259264}{9375}\,a_2{c_5}^2{c_4}^5+\tfrac{608}{49}\,a_2{c_6}^2{c_4}^3\\
&\quad -\tfrac{992}{75}\,a_2{c_5}^3{c_4}^3-\tfrac{5904}{625}\,a_2c_7{c_4}^5+\tfrac{4}{5}\,a_2{c_5}^4c_4-\tfrac{4}{7}\,a_2{c_5}^3c_6+\tfrac{4}{5}\,a_2{c_7}^2c_4\\
&\quad +\tfrac{335104}{15625}\,a_3{c_4}^6c_5-\tfrac{56496}{3125}\,a_3{c_4}^5c_6-\tfrac{3304}{125}\,a_3{c_4}^4{c_5}^2+\tfrac{1344}{125}\,a_3{c_4}^4c_7\\
&\quad -\tfrac{1728}{175}\,a_3{c_4}^2{c_6}^2+\tfrac{168}{25}\,a_3{c_5}^3{c_4}^2+\tfrac{12}{7}\,a_3c_5{c_6}^2+\tfrac{16736}{875}\,a_4{c_4}^4c_6\\
&\quad +\tfrac{7136}{375}\,a_4{c_4}^3{c_5}^2-\tfrac{48}{5}\,a_4{c_4}^3c_7+\tfrac{1328}{245}\,a_4c_4{c_6}^2-\tfrac{8}{5}\,a_4{c_5}^3c_4+\tfrac{8}{7}\,a_4{c_5}^2c_6\\
&\quad -\tfrac{8}{7}\,a_4c_7c_6.
\end{aligned}
\tag{2.7}
$$

Suppose $c_4 \neq 0$. We can obtain $a_0 = (a_1^*, a_2, a_3^*, a_4^*)$ with

$$
a_1^* = 0, \quad a_3^* = \frac{2}{5}\,a_2c_4, \quad a_4^* = -\frac{1}{350}\,\frac{a_2\left(42\,{c_4}^3-175\,c_5c_4+125\,c_6\right)}{c_4}
\tag{2.8}
$$

such that $B_1(a_0) = B_3(a_0) = B_5(a_0)$ and

$$
B_7(a_0) = \frac{4}{459375c_4}a_2\Delta_0, B_9(a_0) = \frac{4}{11484375c_4}a_2\Delta_1, B_{11}(a_0) = \frac{4}{401953125c_4}a_2\Delta_2,
$$

where

$$\begin{aligned}
\Delta_0 &= 6468\,c_4{}^6 - 26950\,c_5c_4{}^4 + 42000\,c_6c_4{}^3 + 65625\,c_4c_5c_6 \\
&\quad -91875\,c_7c_4{}^2 - 46875\,c_6{}^2, \\
\Delta_1 &= -1640625\,c_4c_5{}^2c_6 + 350000\,c_4{}^3c_5c_6 + 2296875\,c_5c_7c_4{}^2 \\
&\quad -1989400\,c_4{}^6c_5 + 2761500\,c_4{}^5c_6 + 1470000\,c_4{}^4c_5{}^2 \\
&\quad -5145000\,c_4{}^4c_7 - 1406250\,c_4{}^2c_6{}^2 + 392784\,c_4{}^8 + 1171875\,c_5c_6{}^2, \\
\Delta_2 &= -410497500\,c_4{}^5c_5c_6 + 420000000\,c_4{}^2c_5c_6{}^2 + 686000000\,c_4{}^4c_5c_7 \\
&\quad -80390625\,c_5{}^2c_4{}^2c_7 - 241937500\,c_4{}^3c_5{}^2c_6 - 372093750\,c_4{}^3c_7c_6 \\
&\quad +57421875\,c_5{}^3c_4c_6 + 80390625\,c_7{}^2c_4{}^2 + 98175000\,c_4{}^4c_6{}^2 \\
&\quad +312130000\,c_4{}^6c_5{}^2 - 401310000\,c_4{}^6c_7 - 135937500\,c_6{}^3c_4 \\
&\quad -83606250\,c_4{}^4c_5{}^3 + 250194000\,c_4{}^7c_6 - 200023880\,c_4{}^8c_5 \\
&\quad +31182816\,c_4{}^{10} - 41015625\,c_5{}^2c_6{}^2 + 41015625\,c_7c_6{}^2 \\
&\quad -57421875\,c_5c_6c_7c_4.
\end{aligned}$$

Next, let $\Delta_0 = 0$. We can solve

$$c_5 = \frac{3}{175}\,\frac{2156\,c_4{}^6 - 15625\,c_6{}^2 + 14000\,c_6c_4{}^3 - 30625\,c_7c_4{}^2}{c_4\left(-375\,c_6 + 154\,c_4{}^3\right)}$$

if $c_6 \neq \frac{154}{375}c_4^3$. Substituting c_5 into Δ_1, we obtain

$$\Delta_1 = -\,\frac{341250c_4^4}{(-375\,c_6 + 154\,c_4{}^3)^2}\widetilde{\Delta}_1,$$

where

$$\begin{aligned}
\widetilde{\Delta}_1 &= 656250\,c_6{}^2c_7 + 232750\,c_4{}^3c_7c_6 - 237500\,c_4c_6{}^3 - 15400\,c_4{}^4c_6{}^2 \\
&\quad +15092\,c_4{}^6c_7 - 643125\,c_4{}^2c_7{}^2.
\end{aligned}$$

Further, let $\Delta_1 = 0$, i.e. $\widetilde{\Delta}_1 = 0$. We get

$$c_7^{(1)} = \frac{50c_6^2}{49c_4^2}, \quad c_7^{(2)} = \frac{44}{1875}\,c_4{}^4 + \frac{38}{105}\,c_4c_6.$$

Now, we have $c_5 = \frac{1}{175}\,\frac{28\,c_4{}^3 + 125\,c_6}{c_4} \equiv c_5^*$. Take $c_0 = (c_5^*, c_7^{(2)})$. Then, it is easy to see that $B_1(a_0,c_0) = B_3(a_0,c_0) = B_5(a_0,c_0) = B_7(a_0,c_0) = B_9(a_0,c_0) = 0$. We also can obtain

$$\begin{aligned}
B_{11}(a_0,c_0) &= -\tfrac{1088}{41015625}\,\left(7\,c_4{}^3 + 125\,c_6\right)a_2c_4{}^6, \\
B_{13}(a_0,c_0) &= -\tfrac{1088}{287109375}\,\left(7\,c_4{}^3 + 125\,c_6\right)\left(28\,c_4{}^3 - 5\,c_6\right)a_2c_4{}^5
\end{aligned}$$

and

$$\det \tfrac{\partial(B_1,B_3,B_5)}{\partial(a_1,a_3,a_4)}(a_0,c_0) = \tfrac{32}{5}\,c_4,$$

$$\tfrac{\partial(\Delta_0,\Delta_1)}{\partial(c_5,c_7)}(a_0,c_0) = -836062500c_4^5(7\,c_4{}^3 + 125\,c_6).$$

Then, by Lemma 1.3 there exist 5 limit cycles of system (1.13) for some (ε, a, c) near $(0, a_0, c_0)$ if $c_6 \neq \frac{154}{375}c_4^3$, $c_6 \neq -\frac{7}{125}c_4^3$, $a_2c_4 \neq 0$.

When $m = 6, n = 4$, (2.6) and (2.7) hold with $c_7 = 0$. By the similar way, we can find $a_0 = (a_1^*, a_3^*, a_4^*)$ such that $B_1(a_0) = B_3(a_0) = B_5(a_0) = 0$ and

$$B_7(a_0) = \frac{4}{459375c_4} a_2\Delta_0, \quad B_9(a_0) = \frac{4}{11484375c_4} a_2\Delta_1,$$

where a_1^*, a_3^*, a_4^* satisfying (2.8) and

$$\begin{aligned} \Delta_0 &= 6468\,c_4{}^6 - 26950\,c_5c_4{}^4 + 42000\,c_6c_4{}^3 + 65625\,c_4c_5c_6 - 46875\,c_6{}^2, \\ \Delta_1 &= 1470000\,c_4{}^4c_5{}^2 - 1640625\,c_4c_5{}^2c_6 + 350000\,c_4{}^3c_5c_6 + 1171875\,c_5c_6{}^2 \\ &\quad +392784\,c_4{}^8 + 2761500\,c_4{}^5c_6 - 1989400\,c_4{}^6c_5 - 1406250\,c_4{}^2c_6{}^2. \end{aligned}$$

Further, suppose $c_6 \neq \frac{154}{375}c_4^3$, then we can find $c_0 = (c_5^*, c_6^*)$ with

$$c_5^* = \frac{54}{475}\,c_4{}^2, \quad c_6^* = -\frac{154}{2375}\,c_4{}^3$$

such that $B_7(a_0, c_0) = B_9(a_0, c_0) = 0$ and

$$B_{11}(a_0, c_0) = \frac{1088}{37109375}\,a_2c_4{}^9.$$

Note that

$$\det \frac{\partial(B_1, B_3, B_5)}{\partial(a_1, a_3, a_4)}(a_0, c_0) = \frac{32}{5}\,c_4, \quad \frac{\partial(\Delta_0, \Delta_1)}{\partial(c_5, c_6)}(a_0, c_0) = -334425000\,c_4{}^9.$$

Then, by Lemma 1.3 there exist 5 limit cycles of system (1.13) for some (ε, a, c) near $(0, a_0, c_0)$ if $c_6 \neq \frac{154}{375}c_4^3, a_2c_4 \neq 0$.

This ends the proof.

Acknowledgments

The author would like to thank the referees very much for their valuable comments and suggestions.

Appendix

the program to compute $\alpha_i, (1 \leq i \leq n)$ in (2.3):

```
restart:with(LinearAlgebra):n:=19:Alph:=-x:
for i from 2 to n do
 Alph:=Alph+alpha[i]*x^i:
od:
G:=1/4*x^4+1/5*c[4]*x^5+1/6*c[5]*x^6;
GA:=subs(x=Alph,G):
temp1:=GA-G:
for i from 5 to n+3 do
 tem[i]:=coeff(temp1,x,i):
 alpha[i-3]:=solve(tem[i],alpha[i-3]):
 print(i-3,alpha[i-3]):
od:save alpha,"alpha.m":
```

the program to compute $B_i, (1 \le i \le n)$ in (1.8):

```
restart:with(LinearAlgebra):n:=19:read "alpha.m":
for i from 11 to n do
a[i]:=0:
od:
F:=0:
 for i from 1 to n do
 F:=F+a[i]*x^i:
od:
Alph:=-x:
 for i from 2 to n do
 Alph:=Alph+alpha[i]*x^i:
od:
temp:=subs(x=Alph,F)-F:
for i from 1 to n do
 B[i]:=simplify(coeff(temp,x,i)):
 print(i,B[i]):
od:
```

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