

TRAVELING WAVE SOLUTIONS IN COUPLED CHUA'S CIRCUITS, PART I: PERIODIC SOLUTIONS*

Shui-Nee Chow¹, Ming Jiang² and Xiaobiao Lin^{3,†}

Abstract We study a singularly perturbed system of partial differential equations that models a one-dimensional array of coupled Chua's circuits. The PDE system is a natural generalization to the FitzHugh-Nagumo equation. In part I of the paper, we show that similar to the FitzHugh-Nagumo equation, the system has periodic traveling wave solutions formed alternatively by fast and slow flows. First, asymptotic method is used on the singular limit of the fast/slow systems to construct a formal periodic solution. Then, dynamical systems method is used to obtain an exact solution near the formal periodic solution. In part II, we show that the system can have more complicated periodic and chaotic traveling wave solutions that do not exist in the FitzHugh-Nagumo equation.

Keywords Chua's circuits, singular perturbations, traveling waves, periodic orbits, exponential dichotomy, Melnikov's method.

MSC(2000) 34C25, 34C37, 34E15, 35B25, 47N70.

1. Introduction and Preliminary

Coupled oscillators are important models for many engineering and biological systems such as neural networks [3] and cardiovascular systems [15, 16].

Among all the networks, electronic networks are easy to simulate analogously. The equations can be written down precisely and the parameters of the systems can be measured easily. In this paper we consider an array of infinitely many Chua's circuits as in [3, 23]. Each identical unit of the circuits consists of a parallel oscillator L and C_2 connected to a semiconductor N_R through R and C_1 . The circuits are interconnected by resistors R_1 . See Figure 1.

Using k as the index for the k th circuit, the system of equations can be written as:

$$\begin{aligned}C_1 \frac{dV_{C_1}^k}{dt} &= (V_{C_2}^k - V_{C_1}^k)/R - G(V_{C_1}^k) + (V_{C_1}^{k-1} - 2V_{C_1}^k + V_{C_1}^{k+1})/R_1, \\C_2 \frac{dV_{C_2}^k}{dt} &= (V_{C_1}^k - V_{C_2}^k)/R + i_L^k, \\L \frac{di_L^k}{dt} &= -V_{C_2}^k, \quad k \in \mathbb{Z},\end{aligned}$$

[†]the corresponding author. Email address: xblin@ncsu.edu (X. Lin)

¹School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA

²Department of Mathematics, Augustana College, Rock Island, Illinois, 61201-2296, USA

³Department of Math. NC State University, Raleigh, NC, 27695-8205, USA

*Research partially supported by NSF grant DMS-1211070.

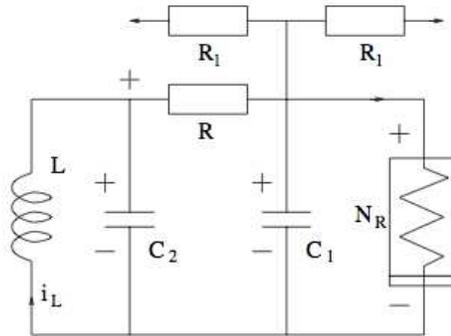


Figure 1. A one dimensional coupled Chua's Circuits

where G is the conductance of Chua's diode. The function G has negative slope in the middle of its domain, providing energy to the network to compensate the loss of energy due to the resistances of R and R_1 . If we combine $V_{C_1}^k + RG(V_{C_1}^k)$ into $h(u)$, the above system can be transformed into the dimensionless form:

$$\begin{aligned} \dot{u}_k &= \alpha(y_k - h(u_k)) + \bar{D}(u_{k-1} - 2u_k + u_{k+1}), \\ \dot{y}_k &= u_k - y_k + z_k, \\ \dot{z}_k &= -\beta y_k, \quad k \in \mathbb{Z}. \end{aligned} \tag{1.1}$$

The function h considered in [23] was a piecewise linear function. In this paper we assume that h is an N -shaped function with odd symmetry. That is:

- (1) $h \in C^\infty$ with odd symmetry. It has three roots $h(-c) = h(0) = h(c)$ for some $c > 0$;
- (2) h has a local minimum at $u = u_m$, a local maximum at $u = -u_m$, where $0 < u_m < c$;
- (3) $h(u)$ is strictly monotone decreasing for $|u| < u_m$ and strictly monotone increasing for $|u| > u_m$. $h(u) \rightarrow \pm\infty$ as $u \rightarrow \pm\infty$.

A typical example is $h(u) = mu(u + c)(u - c)$, $c > 0$, $m > 0$. For numerical works in this paper, our test function will be $H(u) = (1/30)u(u^2 - 9)$.

Let $\epsilon = 1/\alpha$, $\Delta x = \sqrt{\epsilon}$, $u_k(t) = u(t, k\Delta x)$. If Δx is small, (1.1) can be approximated by the following PDE:

$$\begin{aligned} \epsilon u_t &= (y - h(u)) + \epsilon^2 D u_{xx}, \\ y_t &= u - y + z, \\ z_t &= -\beta y. \end{aligned} \tag{1.2}$$

Our system is one of the simplest generalizations of the diffusive FitzHugh-Nagumo equation, which is a second order bistable PDE coupled with a linear first order ODE. In FitzHugh-Nagumo's system the reduced 1D slow equation has an equilibrium with only one real eigenvalue [13, 14]. However, in our system the reduced 2D slow system of ODEs has an equilibrium with two complex eigenvalues

if $\beta > 0$; or one positive, one negative eigenvalues if $\beta < 0$. The case $\beta < 0$ was studied by Acosta in his 1998 thesis [1], while in this paper we only consider the physically meaningful case $\beta > 0$.

Denote the traveling wave solution of (1.2) by $w = w(x - st) := w(\xi)$, where $w = (u, v, y, z)$. We switch the independent variable ξ to t so the traveling wave becomes $w(t)$ and $\dot{w} = dw/dt$. The system obtained is called the slow system where t is the slow time scale:

$$\begin{aligned}\epsilon \dot{u} &= v/D, \\ \epsilon \dot{v} &= h(u) - sv/D - y, \\ s \dot{y} &= y - u - z, \\ s \dot{z} &= \beta y.\end{aligned}\tag{1.3}$$

Let $\tau = t/\epsilon$ and $w' = dw/d\tau$, then we have the fast system where τ is the fast time scale:

$$\begin{aligned}u' &= v/D, \\ v' &= h(u) - sv/D - y, \\ sy' &= \epsilon(y - u - z), \\ sz' &= \epsilon\beta y.\end{aligned}\tag{1.4}$$

To find asymptotic solutions, we consider the limiting problem by setting $\epsilon = 0$ in (1.3) and (1.4):

$$\begin{aligned}0 &= v, \\ 0 &= h(u) - y, \\ s \dot{y} &= y - u - z, \\ s \dot{z} &= \beta y.\end{aligned}\tag{1.5}$$

$$\begin{aligned}u' &= v/D, \\ v' &= h(u) - sv/D - y, \\ y' &= 0, \\ z' &= 0.\end{aligned}\tag{1.6}$$

Let $y_m := \sup\{h(u) : -c \leq u \leq c\}$. Then $\pm y_m := h(\mp u_m)$ where $\mp u_m$ is the local critical points defined before. For each $-y_m < y < y_m$, equation $h(u) = y$ has three inverse functions denoted by

$$u^-(y) := h_-^{-1}(y) < u^0(y) := h_0^{-1}(y) < u^+(y) := h_+^{-1}(y).$$

For $y < -y_m$ or $y > y_m$, only the solution $u^-(y)$ or $u^+(y)$ exists. Observe that $Dh(u^\pm(y)) > 0$ and $Dh(u^0(y)) < 0$.

Augmented by $v = 0, z \in \mathbb{R}$, the solutions of $y = h(u)$ form the slow manifold $S := \{(y, z, u, v) : y = h(u), v = 0\}$, which splits into three branches as follows:

$$S^\pm := \{(y, z, u, v) : u = u^\pm(y), v = 0\}, \quad S^0 := \{(y, z, u, v) : u = u^0(y), v = 0\}.$$

The slow manifold S consists of equilibrium points of (1.6). We are interested in the most left and right submanifolds S^- and S^+ . Given any $0 < y_0 < y_m$, we show that for $y = -y_0$, there exists a unique wave speed $s = s_0 > 0$ such that (1.6) has an increasing heteroclinic orbit w_1^0 connecting S^- to S^+ . For the same wave speed s_0 but $y = y_0$, (1.6) has a decreasing heteroclinic orbits w_3^0 connecting S^+ to S^- . To

form a closed loop, the layer solutions (w_1^0, w_3^0) must be connected by two orbits of (1.5) on the slow manifolds S^- and S^+ . The orbit w_2^0 on S^+ connects $u = u^+(-y_0)$ to $u = u^+(y_0)$ while the orbit w_4^0 on S^- connects $u = u^-(y_0)$ to $u = u^-(-y_0)$.

For $\epsilon > 0$ and small, under some generic conditions, the existence of an exact solution near the singular limit approximation has been proved in singular perturbation theory. In some papers, the proof is based on the graph transformation theory called the Exchange Lemma, similar to the Inclination Lemma or Lambda Lemma in dynamical systems [7–9, 11, 25]. Such geometric method works better if the boundary conditions at the two ends are isolated, such as the Dirichlet or Norman boundary conditions, but should also work with the periodic boundary conditions by an iteration scheme as in [4].

In this paper, we use an analytical method to show the existence of the exact solution. We look for error correction terms to the singular orbits to form the exact solution for $\epsilon > 0$. Rather than tracking the nonlinear manifolds, the analytic method tracks the linear manifolds under the flow, called the invariant stable and unstable subspaces of exponential dichotomies [5, 22]. We also use an iteration method, but the iteration is on a system of linear equations. To solve the system of linear equations, we use a combination of methods from the classic singular perturbation theory and the method of heteroclinic bifurcations as in [10, 19, 20]. If the linear system for the error corrections has been solved, the nonlinear system can be solved by the contraction mapping principle.

We present notations used in this paper and an outline of our approach in §2. In §3, we construct the 0th order approximation for the singular periodic traveling wave. The singular orbits in internal layers and outer layers are constructed in §3.1 and §3.2 respectively. In §3.3, we combine the approximations in regular and singular layers to form the 0th approximation with small residual and jump errors. By checking some generic conditions as in [19], we conclude in §4 that there is an exact periodic traveling wave for $\epsilon > 0$ and small, near the singular one for $\epsilon = 0$. The outline of the proof is given in §5. After deriving the linear variational systems for the correction terms in singular and regular layer, in §5.1 we derive the jump conditions between the regular and singular layers. In §5.2 we solve the boundary value problems that determine the correction terms $(u_i, v_i), i = 1, 2, 3, 4$. In §5.3 we obtain the solutions $(y_i, z_i), i = 2, 4$ on regular layers. Finally, in §5.4 we show how to use the correction terms to show the existence of the exact periodic traveling wave for $\epsilon > 0$ and small.

This research constitutes part of the doctoral dissertation of Ming Jiang who is grateful to Professor X.-B. Lin and faculty of the Mathematics Department of NCSU for their advise and support.

2. Notation of this paper and an outline of the approach

Consider a linear ODE in \mathbb{R}^2

$$x'(\tau) - A(\tau)x(\tau) = h(\tau), \quad \tau \in I, \quad (2.1)$$

where I is a bounded or unbounded interval and $A(\tau)$ is continuous and uniformly bounded on I . Let $T(\tau, \sigma)$ be the solution matrix for the system.

Definition 2.1. We say (2.1) has an exponential dichotomy in I if there exist projections $P_s(\tau) + P_u(\tau) = I$, constant $K \geq 1$ and exponent $\alpha > 0$ such that

$$\begin{aligned} T(\tau, \sigma)P_s(\sigma) &= P_s(\tau)T(\tau, \sigma), \quad \sigma, \tau \in I, \\ |T(\tau, \sigma)P_s(\sigma)| &\leq Ke^{-\alpha(\tau-\sigma)}, \quad \tau \geq \sigma \text{ in } I, \\ |T(\sigma, \tau)P_u(\tau)| &\leq Ke^{-\alpha(\tau-\sigma)}, \quad \tau \geq \sigma \text{ in } I. \end{aligned}$$

For example, if $A(\tau) = A_0$ is a constant matrix with eigenvalues $\lambda_1 < 0 < \lambda_2$, then $x' = A_0x$ has an exponential dichotomy, and the projections are spectral projections to stable and unstable eigenspaces. If $A(\tau)$ depends on τ , then the existence of exponential dichotomies can be proved by the perturbation theory (or the roughness) of exponential dichotomies, [5, 22]. Two such cases is given in the next lemma.

Lemma 2.1. (1) If $A(\tau) \rightarrow A_0$ as $\tau \rightarrow \infty$ (or $\tau \rightarrow -\infty$), and if $\dot{x} = A_0x$ has exponential dichotomies, then (2.1) has an exponential dichotomy on \mathbb{R}^+ (or \mathbb{R}^-).

(2) Assume that for each fixed $\tau \in I$, the matrix $A(\tau)$ has eigenvalues $\lambda_1(\tau) < 0 < \lambda_2(\tau)$ where $|\lambda_i(\tau)| \geq c > 0$ uniformly for $t \in I$. If the system is slowly varying in the sense that $|A'(\tau)| \ll c$, then the time dependent system (2.1) has an exponential dichotomy in I .

Let $\{x_i(t), t \in [a_i, b_i]\}_{i=-\infty}^{\infty}$ be a sequence of continuous functions with $a_0 < 0 < b_0$. If $x_i(b_i) = x_{i+1}(a_{i+1})$, then we define a new continuous function $x(t) = \bigvee_{i=-\infty}^{\infty} \{x_i\}, t \in \mathbb{R}$, by gluing the local functions together, where \bigvee is called the concatenation operator defined as follows. Assume $x(0) = x_0(0)$ to fix the phase. If $a_0 \leq t \leq b_0$, then $x(t) = x_0(t)$; if $t > b_0$, then there exists some $j \geq 1$ such that

$$x(t) = x_j(t - b_0 - \sum_{i=1}^{j-1} (b_i - a_i) + a_j), \text{ if } \sum_{i=1}^{j-1} (b_i - a_i) \leq t - b_0 \leq \sum_{i=1}^j (b_i - a_i);$$

and if $t < a_0$ then there exists some $j < 0$ such that

$$x(t) = x_j(t - a_0 + \sum_{i=-1}^j (b_i - a_i) + a_j), \text{ if } -\sum_{i=-1}^j (b_i - a_i) \leq t - a_0 \leq -\sum_{i=-1}^{j+1} (b_i - a_i).$$

If $j = -1$ then the summation $\sum_{i=-1}^{j+1}$ is ignored. If $j < -1$, the indices in the summations must count down from $i = -1$ to j or $j + 1$ by step size -1 . We do not require $b_i = a_{i+1}$ so the coordinate on I_i is local. This gives us the flexibility of changing the end points a_i, b_i of I_i locally without changing the definitions of x_i on other intervals.

If $x_{i+1}(a_{i+1}) - x_i(b_i) \neq 0$ is denoted by $\mathcal{J}^{i,i+1}$, we can still define the concatenated function $x(t)$ which has a jump of size $\mathcal{J}^{i,i+1}$ between two consecutive intervals.

The exact periodic solution of (1.3), $\epsilon \neq 0$, will be denoted by $w^{\epsilon x}$. Using local time coordinates, we look for $w_i^{\epsilon x}(t), \epsilon > 0$ on intervals $I_i = [a_i, b_i], i \in \mathbb{Z}$ with $w_i^{\epsilon x}(b_i) = w_{i+1}^{\epsilon x}(a_{i+1})$, so that

$$w^{\epsilon x} = \bigvee_{i=-\infty}^{\infty} \{w_i^{\epsilon x}(t)\}.$$

For even i , w_i^{ex} satisfies (1.3). For odd i , w_i^{ex} satisfies (1.4) and we will change the variable $t \rightarrow \tau$ so the function is $w_i^{ex}(\tau)$ defined on $[a_i/\epsilon, b_i/\epsilon]$.

The exact solution will be obtained as the limit of a sequence of approximations denoted by $w^j, j = 0, \dots, \infty$, so that in some suitable topology (not in $L^\infty(\mathbb{R})$, since the period of each iteration will be different):

$$w^{ex} = \lim_{j \rightarrow \infty} w^j.$$

The first term w^0 in the sequence is the singular periodic orbit, obtained by solutions of (1.5), (1.6) where $\epsilon = 0$. The approximation w^{j+1} is constructed from w^j by an iteration process to reduce the error of approximation by a factor $0 < c < 1$.

Similar local time coordinates also apply to the j th approximation w^j which is the union of fast and slow layers defined on the intervals $I_i^j, i = 1, 2, 3, 4$. For $i = 2, 4$, we use the slow time t for the solutions so that $w_i^j = w_i^j(t)$. For $i = 1, 3$, we change t to the fast time τ so that $w_i^j = w_i^j(\tau)$ where $\tau = t/\epsilon$.

To calculate the approximations by iteration, let $w_i^{cor,j}$ be the corrections so that

$$w_i^{j+1} = w_i^j + w_i^{cor,j}.$$

To construct $w_i^{cor,j}$ from the the approximation w_i^j , we will use a linear variational system around the 0th order approximation, not the linearized system around w_i^j .

The notations on the super and sub scripts of w also apply to its components $w = (u, v, y, z)$ throughout the paper. Finally by the periodicity, $w_i^j = w_{i+4}^j$, so we can limit our attention to $i = 1, 2, 3, 4$.

3. Constructing the 0th approximation for periodic orbits

In this section, we construct formal periodic solutions for the singular system $\epsilon = 0$. Then using the formal solutions we formulate an approximation of the periodic traveling wave for $\epsilon > 0$ and small.

Let $U = (u, v)$, $Y = (y, z)$ and $w = (U, Y)$. The singular periodic solution $w^0 = (U^0, Y^0)$ is the concatenation of 4 pieces of orbits $w_i^0, 1 \leq i \leq 4$. The fast orbits $w_i^0, i = 1, 3$ satisfy (1.5) and are defined on singular layers where the solutions are written as $w_i^0(\tau) = (U_i^0(\tau), Y_i^0(\tau)), \tau \in (-\infty, \infty)$. The slow orbits $w_i^0, i = 2, 4$ satisfy (1.6) and are defined on regular layers where the solutions are written as $w_i^0(t) = (U_i^0(t), Y_i^0(t)), t \in [a_i^0, b_i^0]$. We extend the definition of orbits to $i \in \mathbb{Z}$ by $w_{i+4}^0 = w_i^0$ for convenience.

The following asymptotic matching conditions must be satisfied:

$$\lim_{\tau \rightarrow \infty} w_i^0(\tau) = w^{i+1}(a_{i+1}^0), \quad \lim_{\tau \rightarrow -\infty} w_i^0(\tau) = w_{i-1}^0(b_{i-1}^0), \quad i = 1, 3.$$

The matching conditions are satisfied if

- (i) U_1^0 and U_3^0 are a pair of heteroclinic solutions connecting $S^- \rightarrow S^+$ and $S^+ \rightarrow S^-$ with the same wave speed s_0 and constant parameters y_1^0 for U_1^0 and y_3^0 for U_3^0 .

- (ii) If (Y_2^0, Y_4^0) are a pair of slow orbits on (S^+, S^-) and form a closed loop when projected to the (y, z) plane. Y_2^0 connects $y = y_1^0$ to $y = y_3^0$ and Y_4^0 connects $y = y_3^0$ back to $y = y_1^0$.

Although (U_i, Y_i) are 2D vector-valued functions and S^\pm are 3D manifolds in the 4D space. To avoid complicated notation, the projections of S^\pm to (u, v) components or (y, z) components will be denoted by the same notation S^\pm if no confusion should arise.

In Subsection 3.1 we show how to construct (U_1^0, U_3^0) that satisfies (i). In Subsection 3.2 we show how to construct (Y_2^0, Y_4^0) that satisfies (ii).

3.1. Singular periodic solutions in the internal layers

With (y, s) as parameters, we look for $U_i^0, i = 1, 3$ that satisfies the first two equations of (1.6):

$$\begin{aligned} u' &= v/D, \\ v' &= -sv/D - y + h(u), \end{aligned} \quad (3.1)$$

which can be written as the second order ODE:

$$Du'' + su' + y - h(u) = 0. \quad (3.2)$$

The equilibrium points of (3.1) satisfy $(h(u) = y, v = 0)$ and form a smooth manifold. For convenience, we shall denote the manifold S, S^\pm, S^0 and its projection to the (u, y) coordinates by the same notation. By this convention, we say the equilibrium point of (3.1) or the graph $y = h(u)$ is in S and belong to the branch S^-, S^+ or S^0 if $u \leq -u_m, u \geq u_m$ or $|u| < u_m$. Using the characteristic equation $Dr^2 + sr - h'(u) = 0$, we conclude that the equilibrium points on S^- and S^+ are hyperbolic saddle points, while the equilibrium points on S^0 are stable (or unstable) if $s > 0$ (or $s < 0$).

A solution u_0 of (3.2) with the parameters (s_0, y_0) shall be denoted by the triplet (s_0, y_0, u_0) . Multiplying (3.2) by u_0' and integrating from $-\infty$ to ∞ , we have:

$$s_0 \left[\int_{-\infty}^{\infty} (u_0')^2 d\tau \right] = \int_{u_0(-\infty)}^{u_0(\infty)} (h(u_0) - y_0) du_0. \quad (3.3)$$

We call u_0 an increasing (or decreasing) solution if $u_0'(tau) > 0$ (or $u_0'(\tau) < 0$). For an increasing solution $u_0(\tau)$, if $y_0 > 0$ (or < 0), then the integral in the right hand side of (3.3) is negative (or positive), so $s_0 < 0$ (or > 0). We have the following results.

Lemma 3.1. (1) For an increasing solution $u_0(\tau)$, if $y_0 > 0$, then $s_0 < 0$; while if $y_0 < 0$, then $s_0 > 0$.

(2) For a decreasing solution $u_0(\tau)$, if $y_0 > 0$, then $s_0 > 0$; while if $y_0 < 0$ then $s_0 < 0$.

The function $y = h(u)$ has odd symmetry. From one heteroclinic orbit we can obtain three more orbits as follows. If $(s, y, u) = (s_0, -y_0, u_0(\tau))$ is an increasing solution of (3.2) corresponding to $-y_0 < 0, s_0 > 0$, denoted by (1) in Figure 2, then

$$(s, y, u) = (-s_0, -y_0, u_0(-\tau)), (s_0, y_0, -u_0(\tau)), (-s_0, y_0, -u_0(-\tau))$$

are also solutions of (3.2), denoted by (2), (3), (4) in Figure 2. To form a closed loop with the same s_0 , we have to connect the orbits (1) and (3) by slow flows on S^+ , or the orbits (2) and (4) by slow flows on S^- . For definiteness, only the first case will be considered in this paper.

In the rest of the paper, we assume that u_1^0 is a increasing solution with parameters $s_0 > 0, -y_0 < 0$, and $u_3^0(t) = -u^1(t)$ is an decreasing solution with parameters $s_0 > 0, y_0 > 0$.

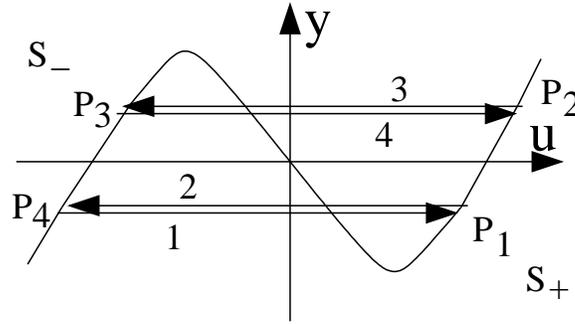


Figure 2. Orbit (1) depicts the solution $(s_0, -y_0, u_0(\tau))$. Orbits (2), (3) and (4) depict solutions $(-s_0, -y_0, u_0(-\tau))$, $(s_0, y_0, -u_0(\tau))$ and $(-s_0, y_0, -u_0(-\tau))$ respectively.

To further study the relation between y_0 and the wave speed s_0 , we introduce a linear version of Melnikov’s method.

~ **Melnikov’s method for bounded solutions on the infinite domain.**

Let $(u(\tau), v(\tau))^T$ be a heteroclinic orbit of (3.1) connecting two saddle points from $S^- \rightarrow S^+$ or $S^+ \rightarrow S^-$, and let

$$A(\tau) = \begin{pmatrix} 0 & 1/D \\ h'(u(\tau)) & -s/D \end{pmatrix}. \tag{3.4}$$

Since $(u(\tau), v(\tau))^T \rightarrow S^\pm$ as $\tau \rightarrow -\infty$, or ∞ , due to the fact that $A(\pm\infty)$ has one positive and one negative eigenvalues. from Lemma 2.1, $U' = A(\tau)U$ has exponential dichotomy on \mathbb{R}^- or \mathbb{R}^+ respectively. But the linear system has no exponential dichotomy on \mathbb{R} due to the non-transversal intersection of the ranges $RP_u(0-)$ and $RP_s(0+)$ at $\tau = 0$.

It is clear that, up to a constant factor, the derivative of the heteroclinic solution $(u', v')^T$ is the unique bounded solution to the 2x2 system $U' = A(\tau)U$. Therefore, up to a constant factor, $\Psi = e^{s\tau/D}(-v', u')^T$ is the unique bounded solution to the adjoint equation $\Psi' + A(\tau)^T\Psi = 0$.

Let $C^k(-\infty, \infty)$ be the space of C^k and uniformly bounded functions on R . We consider the following nonhomogeneous equation

$$U'(\tau) - A(\tau)U(\tau) = H(\tau). \tag{3.5}$$

Lemma 3.2. *For any $H \in C(-\infty, \infty)$, the linear system (3.5) has a unique solution $U \in C^1(-\infty, \infty)$ with $U(0) \perp (u'(0), v'(0))^T$ if and only if*

$$\int_{-\infty}^{\infty} \langle \Psi(\tau), H(\tau) \rangle d\tau = 0. \tag{3.6}$$

If (3.6) does not hold then let the left hand side be g . There exists a unique piecewise C^1 solution $U \in C^1(-\infty, 0] \cap C^1[0, \infty)$ with $U(0\pm) \perp (u'(0), v'(0))^T$ such that

$$U(0+) - U(0-) = g\Psi(0). \quad (\text{Assume } |\Psi(0)| = 1).$$

Moreover, we have the following estimates for (g, U) :

$$|g| + |U| \leq C|H|. \tag{3.7}$$

~ **Existence of internal layers and the relation between parameters.**

We go back to consider (3.1). For the cubic nonlinearity h stated before, it is known that given $y = y_0 \in (-y_m, y_m)$, there exists a unique s_0 such that (3.5) has a heteroclinic orbit connecting $S^- \rightarrow S^+$ or $S^+ \rightarrow S^-$. In fact, for the "typical system" $u'' + su' + u(u - a)(1 - u) = 0, 0 < a < 1$, the exact solution corresponding to the unique wave speed has been obtained by Huxley [24]. In the following lemma we show the relation between s_0 and y_0 for the existence of an increasing heteroclinic orbit connecting saddle points $u^-(y_0)$ to $u^+(y_0)$. Similar results on the decreasing heteroclinic orbit, from $u^+(y_0)$ to $u^-(y_0)$, can be obtained by symmetry.

Before stating the lemma, consider the limiting case $y_0 = -y_m$ where $u^-(y_0)$ is still a saddle point on S^- , while $u^0(y_0) = u^+(y_0)$ becomes the local minimum point (turning point) of S where S^0 meets S^+ . It is known that there exists a minimum wave speed s_m such that for any $s \geq s_m$, the heteroclinic orbit connecting the saddle point $(u^-(y_0), y_0)$ on S^- to the turning point $(u, y) = (u_m, -y_m)$ on S exists, see [2].

Lemma 3.3. (1) For each $y_0 \in (-y_m, y_m)$, there exists a unique $s = s^*(y_0)$ such that system (3.1) has a unique increasing heteroclinic solution $(u_0(\tau), v_0(\tau))$ connecting $(u^-(y_0), 0)$ to $(u^+(y_0), 0)$. Moreover, $\frac{ds^*(y_0)}{dy_0} < 0$.

(2) Let $s_m > 0$ be the minimum wave speed of the connection from the saddle point $(u^-(y_m), -y_m)$ on S^- to the turning point $(u_m, -y_m)$ on S . Then $s_m = \sup\{s^*(y_0) : |y_0| < y_m\}$. As $y_0 \rightarrow \mp y_m$, the wave speed $s^*(y_0) \rightarrow \pm s_m$.

Proof. Part (1): Assume that (u_0, v_0) is a heteroclinic solution of (3.1) corresponding to parameters (s_0, y_0) . Let $(u_0 + \Delta u, v_0 + \Delta v)$ be a nearby solution with the parameters $(s_0 + \Delta s, y_0 + \Delta y)$. The linear variational system for $\Delta U = (\Delta u, \Delta v)^T$ is

$$\Delta U'(\tau) - A(\tau)\Delta U(\tau) = (0, -\Delta y - v_0(\tau)\Delta s/D)^T. \tag{3.8}$$

The linear homogeneous part of the above has a unique bounded solution $(u'_0, v'_0)^T$, so the adjoint equation $\Psi' + A(\tau)^T\Psi = 0$ has a unique bounded solution $\Psi = (\psi_1, \psi_2)^T$ where $\psi_2 = e^{s\tau/D}u'_0(\tau)$.

From Lemma 3.2, (3.8) has a piecewise C^1 solution $\Delta U \in C^1(-\infty, 0] \cap C^1[0, \infty)$ with $\Delta U(0\pm) \perp (u'(0), v'(0))^T$ such that at $\tau = 0$, the solution admits a jump

$$\Delta U(0+) - \Delta U(0-) = g\Psi(0). \quad (\text{Assume } |\Psi(0)| = 1),$$

where

$$g(\Delta s, \Delta y) := \int_{-\infty}^{\infty} \psi_2(\tau)(-\Delta s u'_0/D - \Delta y) d\tau.$$

To have a C^1 solution near (u_0, v_0) we must have $g(\Delta s, \Delta y) = 0$. Observe that

$$\int \psi_2(\tau)u'_0(\tau)d\tau = \int [u'_0(\tau)]^2 e^{(-s\tau)/D} d\tau > 0,$$

and

$$\int \psi_2(\tau)d\tau = \int e^{(-s\tau)/D}u'_0(\tau)d\tau > 0 \text{ if } u'_0(\tau) > 0,$$

$$< 0 \text{ if } u'_0(\tau) < 0.$$

Therefore equation $g(\Delta s, \Delta y) = 0$ can be solved for $\Delta s = s^*(\Delta y)$ with

$$\frac{ds^*(\Delta y)}{d\Delta y} < 0, \text{ if } u'_0 > 0,$$

$$> 0, \text{ if } u'_0 < 0.$$

Part (2): When $y_0 = -y_m$, Billingham and Needham [2] showed that there is a minimum wave speed s_m such that a heteroclinic connection from the saddle point $(u^-(y_0), 0)$ to the turning point $(u^0(y_0), 0)$ exists iff $s \geq s_m$. When $s = s_m$, the connection is from the unstable manifold $W^u(u^-(y_0))$ of the saddle point to the stable manifold $W^s(u^0(y_0))$ of the turning point, while for $s > s_m$ the connection is tangent to the weakly stable part of the center manifold of the turning point.

Increasing y_0 from $-y_m$, the function $u = h^{-1}(y_0)$ at $(u_m, -y_m)$ splits into two branches – one is $u^0(y_0) < u_m$, and the other one is $u^+(y_0) > u_m$. If we follow the branch formed by $(u^+(y_0), 0)$ on S^+ , then the connection of $W^u(u^-(y_0))$ to $W^s(u^+(y_0))$ persists and the wave speed $s^*(y_0)$ is a continuous decreasing function of y_0 . This shows that $s_m = \sup\{s^*(y_0) : |y_0| < y_m\}$. As $y_0 \rightarrow \mp y_m$, the wave speed $s^*(y_0)$ monotonically approaches $\pm s_m$.

To make the picture more complete, if we follow the other branch where the turning point becomes a stable node $(u^0(y_0), 0)$ on S^0 . The saddle to node connection exists if the wave speed $s > s^*(y_0)$, which is a well-known fact in ODEs. □

3.2. Singular periodic solution in the slow layers.

Recall that $(s_0, -y_0)$ are the parameters for U_1^0 and (s_0, y_0) are the parameters for U_3^0 . We look for w_2^0 on S^+ and w_4^0 on S^- so that Y_2^0 connects $y = -y_0$ to $y = y_0$, and Y_4^0 connects $y = y_0$ back to $y = -y_0$.

Observe that the slow system and its limit, (1.3) and (1.5), obey the odd symmetry about the origin, so we will assume that the solutions w_2^0 and w_4^0 are symmetric about the origin. Plug $u = h_{\pm}^{-1}(y)$ into the last two equations of (1.5), we have:

$$s\dot{z} = \beta y, \quad s\dot{y} = y - h_{\pm}^{-1}(y) - z, \quad \text{where } s > 0. \tag{3.9}$$

Several orbits of (3.9) are plotted in Figure 3 with the z -axis pointing to the right and the y -axis pointing upward. The left half of each smooth arc represents Y_2^0 on S^+ plotted with the nonlinear function $h_+^{-1}(y)$. The right half of each smooth arc represents Y_4^0 on S^- plotted with the nonlinear function $h_-^{-1}(y)$. The orbit Y_2^0 (or Y_4^0) rotates around the equilibrium points $(z, y) = (-c, 0)$ (or $(z, y) = (c, 0)$) in clock wise direction. They are symmetric to each other and form a continuous closed loop, but the derivatives are discontinuous at the junction points.

On S^+ , the z -nullcline is defined by $y = 0$, and y -nullcline is defined by $z = y - h_+^{-1}(y)$. The equilibrium point $(z, y) = (-c, 0)$ separates the z -nullcline into the left and right half lines, and separates the y nullcline into the upper lower half lines.

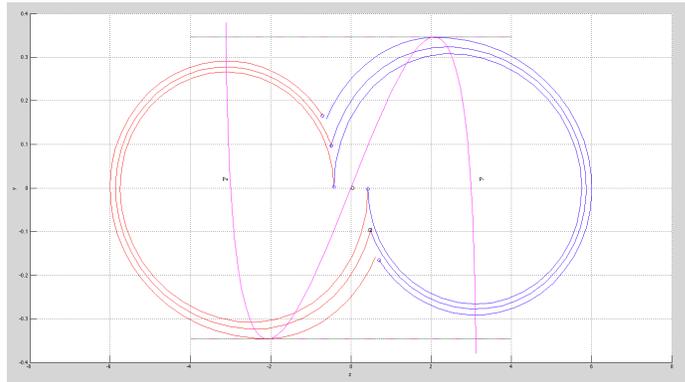


Figure 3. For the nonlinearity $H(u)$ and $\beta = 100$, several closed loops that consist of Y_A on S^+ and Y_B on S^- are plotted. The largest loop touches the lines $y = \pm y_m$. The smallest loop has junction points on $y_0 = 0$. To save space, the z -axis is horizontal and the y axis is vertical.

Remark 3.1. We comment that to avoid computing $h_{\pm}^{-1}(y)$ numerically, we first compute orbits on S using the (z, u) coordinates:

$$s\dot{z} = \beta h(u), \quad s\dot{u} = \frac{1}{h'(u)}(h(u) - u - z)$$

then map the orbits to the (y, z) plane. The y -nullclines, $z = y - h_{\pm}^{-1}(y)$ on S^{\pm} , are also obtained by computing $z = h(u) - u$ first then mapped to the (z, y) plane.

In the rest of this subsection, we study the existence of Y_2^0, Y_4^0 analytically. The following hypothesis ensures that the divergence of the vector field of (3.9) is negative in the region of interest:

(H1): For $|y| \leq y_m$, we assume that $(d/dy)h_{\pm}^{-1}(y) > 1$.

For the cubic polynomial $h(u) = mu(u^2 - c^2)$, condition (H1) is satisfied if $m < 1/(3c^2)$. In particular, it is satisfied by the test function $H(u)$.

Definition 3.1. Let $z = M_z, y = M_y$ be the coordinates of a point $M = (y, z)$ on the yz -plane. For any $0 > \bar{y} \geq -y_m$, let $A \in S^+$ be a point on the y -nullcline with $A_y = \bar{y}$ and let $Y_A(t) = (z_A(t), y_A(t))$ be the solution of (3.9) with $Y_A(0) = A$. Similarly, let $B \in S^-$ be a point on the y -nullcline with $B_y = -\bar{y}$, and let $Y_B(t)$ be the solution of (3.9) with $Y_B(0) = B$.

It is easy to see Y_A and Y_B obey the odd symmetry. In Fig. 4, (A, Y_A) and (B, Y_B) are plotted with $0 > \bar{y} \geq -y_m$. The y coordinate $y_A(t)$ is locally decreasing (or increasing) to the right (or left) of the y -nullcline. So $Y_A(t)$ will hit the z -nullcline in backward time at some point C and hit the z -nullcline, y -nullcline and z -nullcline consecutively in forward time at (K, A', D) . Similarly $Y_B(t)$ will hit the z -nullcline and y -nullcline consecutively at (E, B, L, B', F) .

Using the divergence theorem and (H1), we can show that $0 < A'_y < |A_y|$, and $0 > B'_y > -|B_y|$. Therefore, $(Y_A(t), Y_B(t))$ stay within the region $|y| \leq y_m$ when they rotate one cycle around $(-c, 0)$ and $(c, 0)$ as in Figure 4. By the same method we can show $-c < D_z < C_z$ and $E_z < F_z < c$.

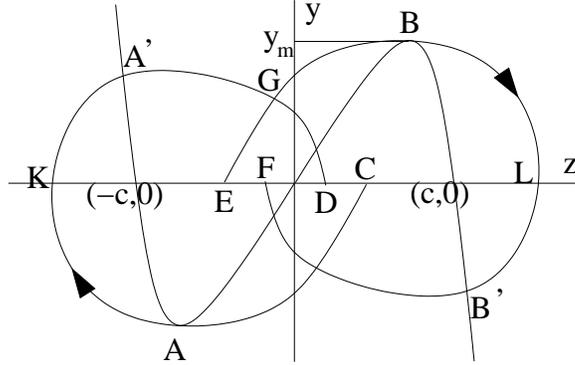


Figure 4. For the nonlinearity $H(u)$, the y nullclines on S^\pm are the most left and the most right branches of the S shaped curve. The trajectories (Y_A, Y_B) in the figure are for the extreme case $\bar{y} = -y_m$.

Lemma 3.4. *For any $0 > \bar{y} \geq -y_m$, let (A, Y_A) be defined as in Definitions 3.1. Assume the orbit of Y_A hits the z -nullcline and y -nullcline at (C, A, K, A', D) as in Fig. 4. Then there exists $\beta > 0$ such that the following properties are satisfied:*

- (P1): $C_z + D_z > 0$, (P2): $C_z \leq c$ and (P3): $D_z + A_z \leq 0$.

Proof. Using the comparison argument, one can show that there exists $\beta = \bar{\beta} > 0$ so that $C_z = c$. Using $D_z > -c$, we see (P1) is satisfied. For this $\beta = \bar{\beta}$, if (P3) is not satisfied, i.e., $D_z + A_z > 0$. Then we can reduce β so that $D_z + A_z = 0$. For such β , by $C_z > D_z = -A_z > 0$, we have $C_z + D_z > 0$ so (P1) still holds. For the smaller β , we still have $C_z \leq c$ so (P2) still holds. \square

For each pair of $(\beta, h(u))$, condition (P1)–(P3) can be verified numerically. In particular, (P1)–(P3) are satisfied for the test function $h(u) = H(u)$ and $\beta = 100$.

Theorem 3.1. *For any $0 > \bar{y} \geq -y_m$, there exists a non-unique $\beta > 0$ such that if $(Y_A(t), Y_B(t))$ are the solutions described in Definition 3.1, then (Y_A, Y_B) intersect transversely at two symmetric points (z_0, y_0) and $(-z_0, -y_0)$ with $y_0 > 0$, if they are plotted on the same zy -plane.*

Moreover, if between the intersecting points, $Y_A = (y_A, z_A), Y_B = (y_B, z_B)$ are defined on the domain $t \in [a, b]$ with $y_A(a) = -y_0, y_A(b) = y_0$, then $dy_A(a)/dt < 0, dy_A(b)/dt < 0$ and $dy_B(a)/dt > 0, dy_B(b)/dt > 0$.

Proof. In Figure 4, y -nullclines $y - h_\pm^{-1}(y) - z = 0$ are plotted as part of the mirror image of an N shaped curve. The left branch of the curve is the y -nullcline on S^+ and the right branch of the curve is the y -nullcline on S^- .

Recall that if $(z(t), y(t))$ is a solution of (3.9) on S^+ where $h^{-1} = h_+^{-1}$, then $(-z(t), -y(t))$ is a solution of (3.9) on S^- where $h^{-1} = h_-^{-1}$.

From the symmetry of the solutions Y_A and Y_B and the property (P1), $E_z = -C_z < D_z$ in Figure 4. From (P2), $E_z = -C_z \geq -c > A'_z$. From (P3), $D_z \leq -A_z = B_z$. We then have the following ordering on the z -axis:

$$A'_z < E_z < D_z \leq B_z.$$

Consider the y coordinates of the curves. We see that E is under the curve $A'D$, and D is under the curve EB . Thus $A'D$ and EB must intersect at some point G ,

of which the z coordinate is between E and D .

At the point G we have $dz_A(t)/dt = dz_B(t)/dt > 0$. By $A'_z < G_z < D_z$ we have $dy_A(t)/dt < 0$, and by $E_z < G_z < B_z$ we have $dy_B(t)/dt > 0$. The intersection of Y_A and Y_B at G is transversal.

We already proved $dy_A(b)dt < 0$, $dy_B(a) > 0$. By symmetry, $dy_A(a)/dt < 0$, $dy_B(b) > 0$. \square

Although (Y_A, Y_B) are solutions to (3.9), the phase conditions are not satisfied. Let $d > 0$ be such that $Y_A(d) = K, Y_B(d) = L$. The construction of the singular slow orbits are complete if we let $Y_2^0(t) = Y_A(t + d)$ and $Y_4^0(t) = Y_B(t + d)$ with the domains $[a_i, b_i] = [a - d, b - d], i = 2, 4$.

The construction of (Y_2^0, Y_4^0) can be viewed as the fixed points of Poincare mappings defined below. The two lines $y = \pm y_0$ are transverse to the flow of (3.9). So we can define the Poincare mapping Π_2 that maps points on the line $\Sigma(-y_0) := \{y = -y_0\}$ to the line $\Sigma(y_0) := \{y = y_0\}$, and the Poincare mapping $\Pi_4 : \Sigma(y_0) \rightarrow \Sigma(-y_0)$. The two mappings are symmetric about the origin. We may say that $Y_2^0(a_2)$ is a fix point of the the mapping $\Pi_4 \circ \Pi_2$, and $Y_4^0(a_4)$ is a fixed point of the mapping $\Pi_4 \circ \Pi_2$.

In the future we need the following assumption.

(H2): The Poincare mappings Π_2 and Π_4 are exponentially stable in a neighborhood of $Y_2^0(a_2)$ and $Y_4^0(a_4)$.

Assumption (H2) is based on the numerical simulation of our system. It seems to be related to (H1) but we do have a rigorous proof of that.

3.3. Approximation of periodic solutions for small $\epsilon > 0$.

When $\epsilon = 0$, using the time scale t , the fast solution takes no time and the slow layers are defined for $t \in [a_i^0, b_i^0], i = 2, 4$. This is not the case when $\epsilon > 0$. Let $\sqrt{\epsilon}$ be an intermediate time scale. When $\epsilon > 0$, define the slow orbits $w_i^0, i = 2, 4$ on $t \in [a_i^0, b_i^0]$ and the fast orbits (w_1^0, w_3^0) on $t \in (-\sqrt{\epsilon}, \sqrt{\epsilon})$ in the time scale t . This is a short interval in time t but long enough in time τ where the same interval becomes $(-\epsilon^{-0.5}, \epsilon^{-0.5})$.

Definition 3.2. We denote $\epsilon^{-0.5}$ by ℓ for the convenience. We sometimes say $w_i^0, i = 1, 2, 3, 4$, are defined on $[\alpha_i, \beta_i]$ with the understanding that $\alpha_i = a_i, \beta_i = b_i, i = 2, 4$, are independent of ϵ while $\alpha_i = -\ell, \beta_i = \ell, i = 1, 3$, are unbounded as $\epsilon \rightarrow 0$.

$$w_i^0 = \begin{cases} w_i^0(t), & t \in [\alpha_i, \beta_i] = [a_i, b_i], & i = 2, 4, \\ w_i^0(\tau), & \tau \in [\alpha_i, \beta_i] = [-\ell, \ell], & i = 1, 3. \end{cases}$$

Here $w_i^0(t), i = 2, 4$ and $w_i^0(\tau), i = 1, 3$ are formal solutions to the singular limit systems.

If $\epsilon \neq 0$, (u^0, v^0, y^0, z^0) does not satisfy (1.3), or (1.4). The formal approximation has residual errors in singular and regular layers. It is straightforward to check that in the slow layers, the residual errors $R_i^0(t), i = 2, 4$ satisfy:

$$R_i^0(t) = \begin{pmatrix} \epsilon \dot{u}_i^0(t) - v_i^0(t)/D \\ \epsilon \dot{v}_i^0(t) - sv_i^0(t)/D + y_i^0(t) - h(u_i^0(t)) \\ -sy_i^0(t) - u_i^0(t) + y_i^0(t) - z_i^0(t) \\ -sz_i^0(t) + \beta y_i^0(t) \end{pmatrix} = \begin{pmatrix} O(\epsilon) \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.10)$$

In the fast layers, the residual errors $R_i^0(\tau), i = 1, 3$ satisfy:

$$R_i^0(\tau) = \begin{pmatrix} (u_i^0)'(\tau) - v_i^0(\tau)/D \\ (v_i^0)'(\tau) - sv_i^0(t)/D + y_i^0(t) - h(u_i^0(t)) \\ s(y_i^0)'(\tau) - \epsilon[y_i^0(\tau) - u_i^0(\tau) - z_i^0(\tau)] \\ s(z_i^0)'(\tau) - \epsilon\beta y_i^0(\tau) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ O(\epsilon) \\ O(\epsilon) \end{pmatrix}. \tag{3.11}$$

When $\epsilon = 0$, due to the matching condition, there is no jump error between singular and regular layers. When $\epsilon \neq 0$, the approximation has jump errors at the junction points.

We use $\mathcal{J}^{i,i+1}$ to denote the jump errors of an approximations for $\epsilon > 0$. Since the internal layers approach the limits exponentially as $\tau \rightarrow \pm\infty$, for the 0th approximation, it is easy to see that there exists $\gamma > 0$ such that

$$\mathcal{J}^{i,i+1} := w_{i+1}^0(\alpha_{i+1}) - w_i^0(\beta_i) = O(\exp(-\gamma\ell)) = o(\epsilon). \tag{3.12}$$

4. Existence of Periodic Solutions for small $\epsilon > 0$.

Let us rewrite (1.3) as

$$\begin{aligned} \epsilon\dot{U}(t) &= F(U, Y, \epsilon), \\ \dot{Y}(t) &= G(U, Y, \epsilon). \end{aligned} \tag{4.1}$$

General conditions for the existence of exact solutions of boundary value problems near the formal approximations have been obtained in [19] where the boundary conditions include isolated boundary conditions and the periodic boundary conditions. For the periodic solutions, these conditions are (based on [19]):

(C1): We assume the normal hyperbolicity at the approximation on S^\pm ,

$$|Re\sigma\{F_U(U_i^0, Y^i, 0)\}| \geq \alpha_0, \alpha_0 > 0, \text{ for all } t \in [\alpha_i, \beta_i].$$

(C2): The breaking of the heteroclinic solutions due to the change of Y is nontrivial. It is represented by the following Melnikov integral:

$$\Delta_i := \int_{-\infty}^{\infty} \Psi_i(\tau) \cdot F_Y(U_i^0(\tau), Y_i^0(\tau), 0) d\tau \neq 0, \quad i = 1, 3.$$

After a rescaling of Ψ_i , $\Delta_i = (1, 0)$ for our system. It reflects the fact that the change of the gap between $U_i^0(0+)$ and $U_i^0(0-)$ depends on the parameter y , not z .

(C3): The vector fields at the beginning and ending of the regular layers $t = a_i, b_i, i = 2, 4$, are transverse to the Poincare section $\Sigma(\pm y_0)$:

$$\begin{aligned} \Delta_i \cdot G(U_i^0(a_i), Y_i^0(a_i), 0) &\neq 0, \\ \Delta_i \cdot G(U_i^0(b_i), Y_i^0(b_i), 0) &\neq 0. \end{aligned}$$

(C4): The Poincare mappings Π_2 and Π_4 defined in §3.2 are exponentially stable.

It is straightforward to check that Conditions (C1)-(C3) are satisfied. Condition (C4) is from the hypothesis (H2) which based on the numerical results. Therefore we obtain the existence of a real periodic solution near the singular periodic solution from Theorem 7.1 in [19].

However, the paper [19] is not dedicated to periodic solutions. The focus of that paper was on singular perturbation problems with isolated boundary conditions and asymptotic expansion of the solutions to any order of ϵ , which are irrelevant to this paper. Thus we will give an outline of the proof for the existence of a real periodic solution even the result is not new.

5. Outline of the proofs

We shall construct a sequence of approximations $\{w^j\}_{j=0}^\infty$, using the singular periodic orbit as the initial approximation w^0 . We present an iteration method to obtain the $j+1$ th approximation from the j th approximation in this section. Each approximation should reduce the errors by a factor $0 < c < 1$, so the sequence $\{w^j\}_{j=0}^\infty$ converges to the exact periodic solution.

For $\epsilon > 0$, the process of obtaining w^{j+1} from w^j can be viewed as an operator that takes any approximation w^{ap} to a better approximation \bar{w}^{ap} . Since our goal is a better approximation, we have the freedom of dropping higher order terms which can make the iteration much easier to carry out. Let w^{ex} be the exact periodic solution near an arbitrarily approximation w^{ap} . The correction term $w = w^{ex} - w^{ap}$ in I_i will be denoted by w_i^{cor} . We will drop the superscript *cor* if no confusion should arise. This goes to all the components (u, v, y, z) of w .

We use fast time τ for corrections in fast layers and use slow time t for corrections in slow layers, so:

$$\begin{aligned} w_i(\tau) &= (U_i(\tau), Y_i(\tau)), & \tau \in [\alpha_i, \beta_i] &= [-\ell, \ell], & i &= 1, 3, \\ w_i(t) &= (U_i(t), Y_i(t)), & t \in [\bar{\alpha}_i, \bar{\beta}_i] &= [a_i + \delta a_i, b_i + \delta b_i], & i &= 2, 4, \end{aligned}$$

where $\delta a_i, \delta b_i$ are correction terms to boundaries of the domain I_i , $i = 2, 4$.

Let the (U, Y) components of the residual errors of the approximation w^{ap} in I_i be denoted by $(p^i(t), q^i(t))$, $i = 2, 4$ and $(p^i(\tau), q^i(\tau))$, $i = 1, 3$. For the initial approximatoin w_i^0

$$\begin{pmatrix} p^i(t) \\ q^i(t) \end{pmatrix} = R_i^0(t), \quad i = 2, 4, \quad \begin{pmatrix} p^i(\tau) \\ q^i(\tau) \end{pmatrix} = R_i^0(\tau), \quad i = 1, 3,$$

which are of $O(\epsilon)$ as from (3.10) and (3.11).

The approximation and the residual error satisfy the following equations:

$$\begin{aligned} \epsilon \dot{U}_i^{ap}(t) &= F(U_i^{ap}, Y_i^{ap}, 0) + p^i, & \dot{Y}_i^{ap}(t) &= G(F(U_i^{ap}, Y_i^{ap}, 0) + q^i, & i &= 2, 4. \\ (U_i^{ap})'(\tau) &= F(U_i^{ap}, Y_i^{ap}, 0) + p^i, & (Y_i^{ap})'(\tau) &= \epsilon F(U_i^{ap}, Y_i^{ap}, 0) + q^i, & i &= 1, 3. \end{aligned}$$

Thus the **exact** correction terms satisfy:

$$\begin{aligned} \epsilon \dot{U}_i(t) &= F(U_i^{ex}, Y_i^{ex}, s) - [F(U_i^{ap}, Y_i^{ap}, s) + p^i(t)] \\ \dot{Y}_i(t) &= G(U_i^{ex}, Y_i^{ex}, s) - [G(U_i^{ap}, Y_i^{ap}, s) + q^i(t)], & i &= 2, 4, \\ U_i'(\tau) &= F(U_i^{ex}, Y_i^{ex}, s) - [F(U_i^{ap}, Y_i^{ap}, s) + p^i(\tau)] \\ Y_i'(\tau) &= \epsilon (G(U_i^{ex}, Y_i^{ex}, s) - [G(U_i^{ap}, Y_i^{ap}, s) + q^i(t)]), & i &= 1, 3. \end{aligned}$$

By higher order term we mean a function of the form $o(\epsilon + |w|)$ and its Lipschitz number with respect to the unknown function w_i is $o(1)$. Let $(p_i, q_i) = -(p^i, q^i)$.

The linear variational system for the **approximate** correction terms can be written as:

$$\begin{aligned} \epsilon \dot{U}_i(t) &= F_U U_i + F_Y Y_i + p_i + h.o.t. & i = 2, 4, \\ \dot{Y}_i(t) &= G_U U_i + G_Y Y_i + q_i + h.o.t. & i = 2, 4, \\ U'_i(\tau) &= F_U U_i + F_Y Y_i + p_i + h.o.t. & i = 1, 3, \\ Y'_i(\tau) &= \epsilon(G_U U_i + G_Y Y_i) + \epsilon q_i + h.o.t. & i = 1, 3. \end{aligned} \quad (5.1)$$

The coefficients are calculated by linearizing around (U_i^0, Y_i^0) . We use the slow time t for $i = 2, 4$ and the fast time τ for $i = 1, 3$ so that

$$\begin{aligned} F_U &= F_U(t) = F_U(U_i^0(t), Y_i^0(t), 0), & i = 2, 4, \\ F_U &= F_u(\tau) = F_U(U_i^0(\tau), Y_i^0(\tau), 0), & i = 1, 3. \end{aligned}$$

Similar rules apply to F_Y, G_U and G_Y .

We will drop the *h.o.t.* in (5.1) since we only look for approximate correction terms. Then we use the smallness of ϵ to simplify the system to a triangular form which can easily be solved.

(i) For $i = 1, 3$: Since $|Y_i|_{L^\infty} \leq C|Y'_i|_{L^1}$, and the $L^1(-\ell, \ell)$ norm of the two terms, $\epsilon(G_u U_i + G_y Y_i)$, are of $O(\sqrt{\epsilon}(|U_i| + |Y_i|))$ on the interval $(-\ell, \ell)$, thus $\epsilon(G_u U_i + G_y Y_i)$ can be considered as higher order terms. We cannot drop ϵq_i even it is small. The simplified fast system is:

$$U'_i(\tau) = F_U U_i + F_Y Y_i + p_i + h.o.t. \quad Y'_i(\tau) = \epsilon q_i + h.o.t. \quad i = 1, 3. \quad (5.2)$$

(ii) For $i = 2, 4$: From (5.1), the vectors near the slow manifold S decompose into two components. The first is tangent to the slow manifold, and the second is parallel to the U -axis, each of them is invariant under the singular homogeneous flow of (5.1):

$$(U_i, Y_i) = (-F_U^{-1} F_Y Y_i, Y_i) + (U_i + F_U^{-1} F_Y Y_i, 0).$$

We describe the decomposition in the following figure.

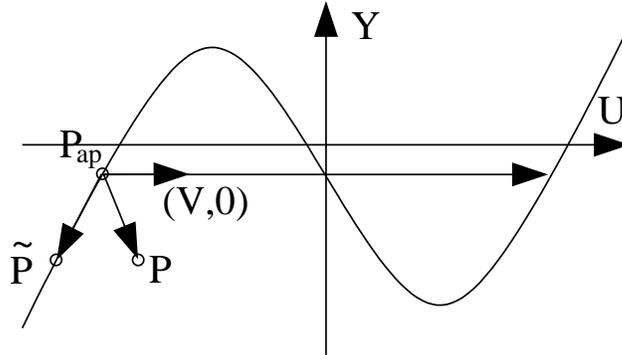


Figure 5. $P_{ap} = (U_{ap}, Y_{ap})$ is a point on the slow manifold S and $P = (U_{ap} + U, Y_{ap} + Y)$ is a point near S . The vector $P_{ap} \tilde{P} = (-F_U^{-1} F_Y Y, Y)$ is tangent to S . Let $V = U + F_U^{-1} F_Y Y$. Then $(V, 0)$ is on the U -plane where system $U' = F_U U$ has an exponential dichotomy for $\tau \geq 0$ or $\tau \leq 0$.

Let $V_i = U_i + F_U^{-1}(t)F_Y(t)Y_i$. After the change of variables $(U_i, Y_i) \rightarrow (V_i, Y_i)$, using $\epsilon\dot{Y}_i = \epsilon q_i + h.o.t.$, we have

$$\begin{aligned}\epsilon\dot{V}_i(t) &= F_U(t)V_i(t) + \epsilon\frac{d}{dt}[F_U^{-1}F_Y Y_i] + p_i + h.o.t. \\ &= F_U(t)V_i(t) + p_i + \epsilon F_U^{-1}F_Y q_i + h.o.t. \quad i = 2, 4, \\ \dot{Y}_i(t) &= (G_Y - G_U F_U^{-1}F_Y)Y_i + G_U V_i + q_i + h.o.t. \quad i = 2, 4.\end{aligned}\quad (5.3)$$

5.1. Jump conditions for the intervals with varying end points

Recall that $I_i = [\alpha_i, \beta_i]$ for all i . For $i = 1, 3$, we use the fast time τ , then $\alpha_i = -\ell, \beta_i = \ell$. For $i = 2, 4$, we use the slow time t , then $\alpha_i = a_i, \beta_i = b_i$. Define jump between the adjacent intervals of the approximations by:

$$\mathcal{J}_U^{i,i+1} := U_{i+1}^{ap}(\alpha_{i+1}) - U_i^{ap}(\beta_i), \quad \mathcal{J}_Y^{i,i+1} := Y_{i+1}^{ap}(\alpha_{i+1}) - Y_i^{ap}(\beta_i). \quad (5.4)$$

To correct the jump errors, we allow the boundaries of I_2, I_4 to change:

$$[a_i, b_i] \rightarrow [a_i + \delta a_i, b_i + \delta b_i], \quad i = 2, 4.$$

The total change of boundary values on $I_i, i = 2, 4$, partially comes from the correction terms (Y_i, U_i) and partially comes from the change of boundary points. Thus, for $i = 2, 4$, if $c = a_i$ or b_i and δc is the variation of that point, then

$$\begin{aligned}Y_i^{ex}(c + \delta c) - Y_i^{ap}(c) \\ = Y_i(c) + \dot{Y}_i^{ap}(c)\delta c + h.o.t., \quad i = 2, 4, \quad c = a_i \text{ or } b_i.\end{aligned}$$

For $i = 1, 3$, the domains are fixed so

$$Y_i^{ex}(c) - Y_i^{ap}(c) = Y_i(c), \quad i = 1, 3, \quad c = \alpha_i \text{ or } \beta_i.$$

At the two end points of $I_3 = (\alpha_3, \beta_3)$,

$$\begin{aligned}Y_4^{ex}(a_4 + \delta a_4) &= Y_3^{ex}(\beta_3), & Y_3^{ex}(\alpha_3) &= Y_2^{ex}(a_2 + \delta a_2), \\ Y_4^{ap}(a_4) &= Y_3^{ap}(\beta_3) + \mathcal{J}_Y^{3,4}, & Y_3^{ap}(\alpha_3) &= Y_2^{ap}(a_2 + \delta a_2) + \mathcal{J}_Y^{2,3}.\end{aligned}$$

Let $J^{i,i+1} = -\mathcal{J}^{i,i+1}$ be the jump for the correction terms and let $J_U^{i,i+1}, J_Y^{i,i+1}$ be its U and Y components. Subtracting the two lines and dropping the higher order terms, we have

$$\begin{aligned}Y_4(a_4) + \dot{Y}_4^0(a_4)\delta a_4 &= Y_3(\beta_3) + J_Y^{3,4}, \\ Y_3(\alpha_3) &= Y_2(a_2) + \dot{Y}_2^0(a_2)\delta a_2 + J_Y^{2,3}.\end{aligned}$$

Here we replaced $\dot{Y}_i^{ap}, i = 2, 4$ by \dot{Y}_i^0 with errors that are of higher order terms.

Similar results hold for the end points of I_1 . Thus the jump conditions for correction functions $Y_i, i = 1, 2, 3, 4$ and variation of junction times $\delta a_i, \delta b_i, i = 2, 4$, are :

$$\begin{aligned}Y_{i+1}(a_{i+1}) - Y_i(\beta_i) + \dot{Y}_{i+1}^0(a_{i+1})\delta a_{i+1} &= J_Y^{i,i+1}, \quad i = 1, 3, \\ Y_{i+1}(\alpha_{i+1}) - Y_i(b_i) - \dot{Y}_i^0(b_i)\delta b_i &= J_Y^{i,i+1}, \quad i = 2, 4,\end{aligned}\quad (5.5)$$

where we have dropped the higher order terms and used the convention $Y_{i+4} = Y_i$.

Similarly, the linear variational systems for the correction functions $U_i, i = 1, 2, 3, 4$ are

$$\begin{aligned} U_{i+1}(a_{i+1}) - U_i(\beta_i) + \dot{U}_{i+1}^0(a_{i+1})\delta a_{i+1} &= J_U^{i,i+1}, \quad i = 1, 3, \\ U_{i+1}(\alpha_{i+1}) - U_i(b_i) - \dot{U}_i^0(b_i)\delta b_i &= J_U^{i,i+1}, \quad i = 2, 4. \end{aligned}$$

Since $\dot{U}_i^0(a_i) = -F_U^{-1}F_Y\dot{Y}_i^0(a_i)$, $\dot{U}_i^0(b_i) = F_U^{-1}F_Y\dot{Y}_i^0(b_i)$ by $F(U_i^0, Y_i^0, 0) = 0$, then

$$\begin{aligned} U_{i+1}(a_{i+1}) - U_i(\beta_i) - F_U^{-1}F_Y\dot{Y}_{i+1}^0(a_{i+1})\delta a_{i+1} &= J_U^{i,i+1}, \quad i = 1, 3, \\ U_{i+1}(\alpha_{i+1}) - U_i(b_i) + F_U^{-1}F_Y\dot{Y}_i^0(b_i)\delta b_i &= J_U^{i,i+1}, \quad i = 2, 4. \end{aligned} \quad (5.6)$$

Define the jump for the approximation solution between two slow intervals as:

$$\mathcal{J}_Y^{2,4} = Y_4^{ap}(a_4) - Y_2^{ap}(b_2), \quad \mathcal{J}_Y^{4,2} = Y_2^{ap}(a_2) - Y_4^{ap}(b_4). \quad (5.7)$$

Then the corrections to the jumps satisfy

$$J_Y^{i,i+2} = J_Y^{i,i+1} + J_Y^{i+1,i+2} + \int_{-\ell}^{\ell} \epsilon q_{i+1}(\tau) d\tau, \quad i = 2, 4.$$

To summarize, the correction terms $w_i, i = 1, 2, 3, 4$ must satisfy the following system:

$$w'_i(\tau) - A_i(\tau)w_i(\tau) = f_i(\tau), \quad i = 1, 3, \quad (5.8)$$

$$\dot{w}_i(t) - A_i(t)w_i(t) = f_i(t), \quad i = 2, 4, \quad (5.9)$$

$$w_{i+1}(a_{i+1} + \delta a_{i+1}) - w_i(\beta_i) = J^{i,i+1}, \quad i = 1, 3, \quad (5.10)$$

$$w_{i+1}(\alpha_{i+1}) - w_i(b_i + \delta b_i) = J^{i,i+1}, \quad i = 2, 4, \quad (5.11)$$

$$d^i \cdot w_i(0) = 0, \quad i = 1, 2, 3, 4, \quad (5.12)$$

where $A_i(\tau)$ and $A_i(t)$ are 4x4 matrices, $f_i(\tau)$ and $f_i(t)$ are the nonhomogeneous terms of the linear system. The vector $d^i \perp \dot{w}_{ap}^i(0)$ determines the phase condition of w^i . Since solutions of linear the system are determined by the jump conditions, we call the system a **linear non-homogeneous jump value problem**.

From (5.2), dropping the h.o.t, the precise form of (5.8) is:

$$U'_i(\tau) = F_U U_i + F_Y Y_i + p_i, \quad Y'_i(\tau) = \epsilon q_i \quad i = 1, 3.$$

Naturally we can solve the 2nd equation then plug into the first:

$$\begin{aligned} Y_i(\tau) &= Y_i(0) + \int_0^\tau \epsilon q_i(\tau) d\tau, \quad -\ell \leq \tau \leq \ell, \\ U'_i(\tau) &= F_U U_i + p_i + F_Y Y_i(0) + F_Y \int_0^\tau \epsilon q_i(\tau) d\tau, \quad i = 1, 3. \end{aligned} \quad (5.13)$$

From (5.3), dropping the h.o.t, the precise form of (5.9) is:

$$\begin{aligned} \epsilon \dot{V}_i(t) &= F_U(t)V_i(t) + p_i + \epsilon F_U^{-1}F_Y q_i, \\ \dot{Y}_i(t) &= (G_Y - G_U F_U^{-1}F_Y)Y_i + G_U V_i + q_i, \quad i = 2, 4. \end{aligned} \quad (5.14)$$

In the rest of the sciton we look for solutions of (5.13), (5.14) with jump conditions (5.10), (5.11).

5.2. Non-homogeneous boundary value problems that determine (U_i, V_i)

Let $\ell = \epsilon^{-1/2}$, which is a large constant if $\epsilon > 0$ is small. Let $A(\tau)$ be defined as in (3.4) where $u(\tau) = u_i^0(\tau)$, $i = 2, 4$. Consider (3.5) with boundary conditions:

$$P_s(-\ell)U(-\ell) = \phi_s, \quad P_u(\ell)U(\ell) = \phi_u. \quad (5.15)$$

~ **Melnikov's method for boundary value problems on a finite domain.**

We have the following lemma for the existence of a solution $U \in C^1(-\ell, \ell)$.

Lemma 5.1. (Lemma 2.3 of [18]) *For any $H \in C[-\ell, \ell]$ and $\phi_s \in (RP_s(-\ell))$, $\phi_u \in RP_u(\ell)$, the linear system (3.5) with boundary conditions (5.15) has a unique solution $U \in C^1(-\ell, \ell)$ with $U(0) \perp (u'(0), v'(0))^T$ if and only if*

$$\int_{-\ell}^{\ell} \langle \Psi(\tau), H(\tau) \rangle d\tau + \langle \Psi(-\ell), \phi_s \rangle - \langle \Psi(\ell), \phi_u \rangle = 0. \quad (5.16)$$

If (5.16) does not hold then denote the left hand side by g . There exists a unique piecewise C^1 solution $U \in C^1[-\ell, 0] \cap C^1[0, \ell]$ with $U(0\pm) \perp (u'(0), v'(0))^T$ such that

$$U(0+) - U(0-) = g\Psi(0). \quad (\text{Assume } |\Psi(0)| = 1).$$

There exists $\alpha > 0$, independent of ℓ such that following estimates for (g, U) hold:

$$|g| \leq C(|\phi_s| + |\phi_u|)e^{-\alpha\ell} + |H|, \quad (5.17)$$

$$|U| \leq C(|\phi_s| + |\phi_u| + |H|). \quad (5.18)$$

We now consider the equations for U_i , $i = 1, 3$ and V_i , $i = 2, 4$.

$$\begin{aligned} U_i'(\tau) &= F_U(\tau)U_i + F_Y(\tau)Y_i(0) + p_i + \int_0^\tau \epsilon q_i(\tau) d\tau, & i = 1, 3, \\ \epsilon \dot{V}_i(t) &= F_U(t)V_i(t) + p_i + \epsilon F_U^{-1}F_Y q_i, & i = 2, 4. \end{aligned} \quad (5.19)$$

In the first equation, $Y_i(0) = (y_i^0, z_i^0)^T$ where y_i^0 and z_i^0 are undetermined.

For $i = 1, 3$, from Lemma 2.1, system $U_i'(\tau) = F_U(\tau)U_i$, where $F_U(\tau) = A(\tau)$ as in (3.4), has exponential dichotomies on $(-\ell, 0]$ and $[0, \ell)$ respectively.

For $i = 2, 4$, the system $\epsilon \dot{V}_i(t) = F_U(t)V_i(t)$ has exponential dichotomy on $[a_i, b_i]$ with the exponents being $O(1/\epsilon)$. This can be shown by changing the time t to $\tau = t/\epsilon$ so the system becomes $V_i'(\tau) = F_U(\epsilon\tau)V_i(\tau)$, $i = 2, 4$. For each fixed $\tau \in [a_i/\epsilon, b_i/\epsilon]$, the system has exponential dichotomy. The coefficient is slow varying since $(d/d\tau)F_U(\epsilon\tau) = O(\epsilon)$. Thus the rescaled system has exponential dichotomy in $[a_i/\epsilon, b_i/\epsilon]$ as from Lemma 2.1. After scaling back $\tau \rightarrow t$, the system $\epsilon \dot{V}_i(t) = F_u(t)V_i(t)$, $i = 2, 4$ has an exponential dichotomy and the exponents become $O(1/\epsilon)$.

At the junction point between I_i and I_{i+1} , the vectors w_{ap}^i, w_{ap}^{i+1} are near S , so we split the jump into (V, Y) components. From (5.10), (5.11), the jump conditions for V_i are

$$\begin{aligned} V_{i+1}(a_{i+1} + \delta a_{i+1}) - V_i(\beta_i) &= J_U^{i,i+1} + F_U^{-1}F_Y J_Y^{i,i+1}, & i = 1, 3, \\ V_{i+1}(\alpha_{i+1}) - V_i(b_i + \delta b_i) &= J_U^{i,i+1} + F_U^{-1}F_Y J_Y^{i,i+1}, & i = 2, 4. \end{aligned}$$

For $c = a_i$ or b_i , from the definition of V_i , and

$$F(U_i^0(c), Y_i^0(c), 0) = 0, \quad F(U_i^0(c + \delta c), Y_i^0(c + \delta c), 0) = 0,$$

we have

$$V_i(a_i + \delta a_i) = V_i(a_i) + h.o.t., \quad V_i(b_i + \delta b_i) = V_i(b_i) + h.o.t.$$

Remark 5.1. To simplify the notation, we let $V_i = U_i$ in internal layers $i = 1, 3$.

Dropping the higher order term, the jump condition on $V_i, i = 1, 2, 3, 4$ are

$$V_{i+1}(\alpha_{i+1}) - V_i(\beta_i) = J_V^{i,i+1} := J_U^{i,i+1} + F_U^{-1} F_Y J_Y^{i,i+1}, \quad i = 1, 2, 3, 4. \quad (5.20)$$

We shall solve (5.19) with the jump conditions (5.20) in three steps.

Step 1: Let $\bar{V}_i, i = 1, 2, 3, 4$ be the solutions of (5.19) with the boundary conditions

$$P_s^i(\alpha_i) \bar{V}_i(\alpha_i) = 0, \quad P_u^i(\beta_i) \bar{V}_i(\beta_i) = 0.$$

For $i = 2, 4$, system (5.19) has exponential dichotomies on I_i . If the right hand side of (5.19) is denoted by $f_i(t)$, the solution

$$\bar{V}_i = \int_{a_i}^t T(t, s) P_s(s) f_i(s) ds + \int_{b_i}^t T(t, s) P_u(s) f_i(s) ds$$

exists, and are bounded by $C|f_i| \leq C(|p_i| + |q_i|)$.

For $i = 1, 3$ system (5.19) has exponential dichotomies on the left and the right half of the interval I_i . From Lemma 5.1, there exists a unique \bar{y}_i^0 such that if $Y_i(0) = (\bar{y}_i^0, \bar{z}_i^0)^T$, then the solution \bar{V}_i exist, and is bounded by $(|p_i| + |q_i|)$ as well.

Step 2: The functions $\tilde{V}_i = V_i - \bar{V}_i, i = 1, 2, 3, 4$ are solutions of (5.19) with $p_i, q_i = 0$. The jump conditions are modified to

$$\tilde{V}_{i+1}(\alpha_{i+1}) - \tilde{V}_i(\beta_i) = J_V^{i,i+1} - (\bar{V}_{i+1}(\alpha_{i+1}) - \bar{V}_i(\beta_i)) \stackrel{def}{=} K_V^{i,i+1}, \quad i = 1, 2, 3, 4. \quad (5.21)$$

Let $Q^{i,i+1}$ be the projection to the space $RP_s(\alpha_{i+1})$ with the null space $NQ^{i,i+1} = RP_u(\beta_i)$. Then the jump conditions are approximated by the boundary conditions:

$$P_s(\alpha_{i+1}) \tilde{V}_{i+1}(\alpha_{i+1}) = Q^{i,i+1} K_V^{i,i+1}, \quad P_u(\beta_i) \tilde{V}_i(\beta_i) = -(I - Q^{i,i+1}) K_V^{i,i+1}.$$

We have finally obtained the boundary conditions for $\tilde{V}_i, i = 1, 2, 3, 4$,

$$\begin{aligned} P_s(\alpha_i) \tilde{V}_i(\alpha_i) &= \phi_s^i, & P_u(\beta_i) \tilde{V}_i(\beta_i) &= \phi_u^i, \\ \text{where } \phi_s^i &= Q^{i-1,i} K_V^{i-1,i}, & \phi_u^i &= -(I - Q^{i,i+1}) K_V^{i,i+1}. \end{aligned} \quad (5.22)$$

The system for \tilde{V}_i with boundary conditions is homogeneous:

$$\begin{aligned} \tilde{V}_i'(\tau) &= F_U \tilde{V}_i + F_Y Y_i(0), & i &= 1, 3, \\ \epsilon \dot{\tilde{V}}_i(t) &= F_U(t) \tilde{V}_i(t), & i &= 2, 4. \end{aligned}$$

The system for $\tilde{V}_i, i = 2, 4$ has an exponential dichotomy in the domain I_i . Solution $\tilde{V}_i(t) = T(t, a_i) \phi_s^i + T(t, b_i) \phi_u^i$ that satisfies the boundary conditions uniquely exists and is bounded by $|\phi_s^i| + |\phi_u^i|$.

The system for $\tilde{V}_i, i = 1, 3$ has exponential dichotomies on $[-\ell, 0]$ and $[0, \ell]$ respectively. There exists a unique \tilde{y}_i^0 such that if $Y_i(0) = (\tilde{y}_i^0, \tilde{z}_i^0)$, then the solution of the boundary value problem \tilde{V}_i exists and is bounded by $|\phi_s^i| + |\phi_u^i|$.

One nice fact is that the uncontrolled boundary terms

$$P_u(\alpha_i)\tilde{V}_i(\alpha_i), \quad P_s(\beta_i)\tilde{V}_i(\beta_i), \quad i = 1, 2, 3, 4$$

are exponentially smaller than the specified boundary terms $|\phi_s^i| + |\phi_u^i|$. Although the jump conditions (5.21) are not be satisfied by the solutions of the boundary value problem \tilde{V}_i , but the errors are smaller than $K_V^{i,i+1}$ by a factor $c < 1$. Using an iteration scheme, the solutions with exact jumps as in (5.21) can be obtained.

Step 3: If we let $V_i = \bar{V}_i + \tilde{V}_i$, then equations (5.19) and jump conditions (5.20) are satisfied. The parameter $Y_i(0) = (y_i^0, z_i^0)^T$ where $y_i^0 = \bar{y}_i^0 + \tilde{y}_i^0$ has been determined but z_i^0 is arbitrarily.

In summary we have obtained corrections $V_i, i = 1, 2, 3, 4$ to any given approximation. Now suppose the given approximation is V_i^j , then the next approximation

$$V_i^{j+1} = V_i^j + V_i, \quad i = 1, 2, 3, 4, \quad U_i^{j+1} = V_i^{j+1}, \quad i = 1, 3. \quad (5.23)$$

But for $U_i^{j+1}, i = 2, 4$, we need to know $Y_i^{j+1}, i = 2, 4$, which will be obtained in the next subsection.

5.3. Poincare mappings for linear systems and construction of (Y_2, Y_4) .

We have constructed (Y_1, Y_3) by the first equation of (5.13). Let

$$Y_i(0) = (y_i^0, z_i^0), \quad Y_i(-\ell) = (y_i^-, z_i^-), \quad Y_i(\ell) = (y_i^+, z_i^+), \quad i = 1, 3.$$

The values of y_i^0, y_i^\pm have been completely determined. For the z coordinate, only the value of the gap $z_i^+ - z_i^-$ has been determined.

The purpose of this subsection is to construct $Y_i^{j+1}, i = 2, 4$ based on $Y_i^j, i = 2, 4$. We will use a linear version of Poincare mappings to find Y_2, Y_4 . As always, linear Poincare mappings are tricky to define. To illustrate the intuitive idea, we first show how the next iteration Y_i^{j+1} should look like in Fig. 6.

The linear variational system for Y_i is

$$\dot{Y}_i(t) = (G_Y - G_U F_U^{-1} F_Y)Y_i + G_U V_i + q_i, \quad i = 2, 4. \quad (5.24)$$

Let $S^i(t, s)$ be the solution matrix for $\dot{Y}_i(t) = (G_Y - G_U F_U^{-1} F_Y)Y_i$. Then $\bar{Y}_i(t) = \int_0^t S^i(t, s)(G_U V_i + q_i(s))ds$ is a solution to (5.24). By subtracting \bar{Y}_i from (5.24), we only have to look for $\tilde{Y}_i = Y_i - \bar{Y}_i$ that satisfies a homogeneous system. We also update the boundary conditions and jump conditions related to \tilde{Y}_i to include the contributions from $\bar{Y}_i(t)$.

The equation for $\tilde{Y}_i, i = 2, 4$ must satisfy the following system (we rewrite \tilde{Y}_i as Y_i to simplify the notations):

$$\begin{aligned} \dot{Y}_i(t) - (G_Y - G_U F_U^{-1} F_Y)Y_i &= 0, \quad i = 2, 4, \\ y_i(a_i + \delta a_i) &= y_i^-, \quad y_i(b_i + \delta b_i) = y_i^+, \\ z_i(b_{i+2} + \delta b_{i+2}) - z_i(a_i + \delta a_i) &= J_z^{i,i+2}, \end{aligned} \quad (5.25)$$

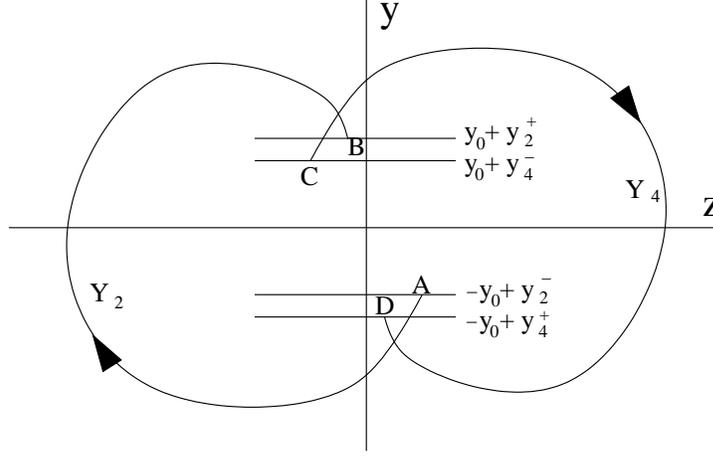


Figure 6. The size of jumps BC and DA are given on the lines $y = y_0 + y_4^-$, $y = y_0 + y_2^+$ and $y = -y_0 + y_2^-$, $y = -y_0 + y_4^+$.

where the boundary conditions in y and the jump conditions in z for $i = 2, 4$ are:

$$\begin{aligned} y_i^- &= y_{i-1}^+ + J_y^{i-1,i} - \bar{y}_i(a_i), \\ y_i^+ &= y_{i+1}^- - J_y^{i,i+1} - \bar{y}_i(b_i), \\ J_z^{i,i+2} &= J_z^{i,i+1} + J_z^{i+1,i+2} + (z_{i+1}^+ - z_{i+1}^-). \end{aligned}$$

The line $\{(y, z) : y = \bar{y}, z \in \mathbb{R}\}$ is called a y section and denoted by $\Sigma(\bar{y})$. Consider the three y sections:

$$\Sigma(0) = \{y = 0\}, \quad \Sigma(y_i^-) = \{y = y_i^-\}, \quad \Sigma(y_i^+) = \{y = y_i^+\}.$$

Define the linear affine map (linear Poincare map) from $t = 0$ to $t = b_i$, $\Sigma(0) \rightarrow \Sigma(y_i^+)$ as follows. Given $(0, z_i)^T \in \Sigma(0)$, there exists a unique δb_i such that

$$S(b_i, 0)(0, z_i)^T + \delta b_i \cdot \dot{Y}_i^0(b_i) \in \Sigma(y_i^+), \quad i = 2, 4.$$

Also define the backwards linear affine map from $t = 0$ to $t = a_i < 0$, $\Sigma(0) \rightarrow \Sigma(y_i^-)$ as

$$S(a_i, 0)(0, z_i)^T + \delta a_i \cdot \dot{Y}_i^0(a_i) \in \Sigma(y_i^-), \quad i = 2, 4.$$

Let $\delta b_i = \delta \bar{b}_i + \delta \tilde{b}_i$, $\delta a_i = \delta \bar{a}_i + \delta \tilde{a}_i$ such that

$$\delta \bar{b}_i \cdot \dot{Y}_i^0(b_i) \in \Sigma(y_i^+), \quad \delta \bar{a}_i \cdot \dot{Y}_i^0(a_i) \in \Sigma(y_i^-).$$

The remaining part $\delta \tilde{b}_i, \delta \tilde{a}_i$ determined from the linear homogeneous Poincare maps $\Sigma(0) \rightarrow \Sigma(0)$:

$$\begin{aligned} S(b_i, 0)(0, z_i)^T + \delta \tilde{b}_i \cdot \dot{Y}_i^0(b_i) &\stackrel{def}{=} (0, z_i^+)^T \in \Sigma(0), & i = 2, 4, \\ S(a_i, 0)(0, z_i)^T + \delta \tilde{a}_i \cdot \dot{Y}_i^0(a_i) &\stackrel{def}{=} (0, z_i^-)^T \in \Sigma(0), & i = 2, 4. \end{aligned} \quad (5.26)$$

The first equation induces a linear mapping $\pi_{2i}^+ : z_{2i} \rightarrow z_{2i}^+$. The second equation induces a linear mapping $\pi_{2i}^- : z_{2i} \rightarrow z_{2i}^-$. We look for a sequence $\{z_{2i}\}_{-\infty}^{\infty}$ such that

$$\pi_{2i+2}^-(z_{2i+2}) - \pi_{2i}^+(z_{2i}) = \tilde{J}_z^{2i,2i+2},$$

where $\tilde{J}_z^{2i,2i+2} = z_{2i+2}^- - z_{2i}^+$ is given, and must include the contributions from $\delta \bar{b}_i \cdot \dot{Y}_i^0(b_i)$ and $\delta \bar{a}_i \cdot \dot{Y}_i^0(a_i)$. We look for a sequence $\{z_{2i}\}_{-\infty}^{\infty} \in \ell^\infty$ such that

$$[(\pi_{2i+2}^-)^{-1} \circ \pi_{2i}^+](z_{2i}) + (\pi_{2i+2}^-)^{-1} \tilde{J}_z^{2i,2i+2} = z_{2i+2}, \quad i \in \mathbb{Z}. \quad (5.27)$$

The mapping $[(\pi_{2i+2}^-)^{-1} \circ \pi_{2i}^+]$ is the linearized Poincare mapping around Y_{2i}^0 and Y_{2i+2}^0 , from $\Sigma(0)$ to $\Sigma(0)$. The basic assumption of this subsection, from (H2) is **(H2')** The linear homogeneous Poincare mapping $[(\pi_{2i+2}^-)^{-1} \circ \pi_{2i}^+] : \Sigma(0) \rightarrow \Sigma(0), i \in \mathbb{Z}$ is a contraction mapping.

There exists $0 < c < 1$ such that $|[(\pi_{2i+2}^-)^{-1} \circ \pi_{2i}^+]| < c < 1$ uniformly for all i . The left hand side of (5.27) is a contraction mapping in ℓ^∞ composed with a shift of indices. Therefore there exists a unique sequence $\{z_{2i}\}_{-\infty}^{\infty}$ that satisfies (5.27).

Finally with the sequence initial points $Y_i(0) = (0, z_i)^T, i = 2, 4$, we can obtain the corrections $Y_i(t), i = 2, 4$ in the domain $[a_i + \delta a_i, b_i + \delta b_i]$.

In the previous subsection, the solutions $Y_i, i = 1, 3$ has been constructed by integrals with one degree of freedom in the choice of $z_i(0), i = 1, 3$. It is now fully determined too. If $Y_i^{ap}, i = 2, 4$ is the j th approximation Y_i^j in the interval I_i , then the $(j + 1)$ th approximation $Y_i^{j+1}(t) = Y_i^{ap} + Y_i(t), i = 1, 2, 3, 4$ has been constructed. Moreover, using $Y_i, i = 2, 4$, we obtain U_i^{j+1} as

$$U_i = V_i - F_U^{-1} F_Y Y_i, \quad U_i^{j+1} = U_i^j + U_i, \quad i = 2, 4. \quad (5.28)$$

5.4. The exact periodic traveling wave to the original nonlinear system

Before concluding that $w^j \rightarrow w^{ex}$, as $j \rightarrow \infty$, in some suitable topology, we point out that the sequence of functions $\{w^{j+1}(t), t \in \mathbb{R}\}_{j=0}^{\infty}$ do not satisfy the Cauchy criterion in the super norm of t since the period of each approximation is different. To complete the proof we can show that orbitally the sequence is convergent, see [19]. Alternatively, using the local coordinates we have a natural way to show in what sense the approximations approach the exact solution.

First, based on our construction, if we compare w_i^{j+1} to w_i^j , the following residual errors and jump errors decay by a factor $0 < c < 1$

$$\sup\{(|f_i| + |J^{i,i+1}|) : i \in \mathbb{Z}\}.$$

The correction terms w_i and change of boundary points $\delta a_i, \delta b_i$ at each iteration are controlled by the error terms and jump terms as in (5.8) to (5.11). So

$$\sup\{(|w_i| + |\delta a_i| + |\delta b_i|) : i \in \mathbb{Z}\}$$

also decays by a factor $0 < c < 1$ when j increases to $j + 1$.

Then the existence of an exact periodic solution w^{ex} which is the concatenation of $w_i^{ex}, i \in \mathbb{Z}$ can be proved in 5 easy steps. Details will not be presented.

- (1) The domain $[a_i^j, b_i^j]$ approaches the limit $[a_i^{ex}, b_i^{ex}]$ as $j \rightarrow \infty$.

- (2) The function w_i^{ex} is well defined on $[a_i^{ex}, b_i^{ex}]$.
- (3) The function w_i^{ex} satisfies the nonlinear system with no residual error.
- (4) The sequence $w_i^{ex}, i \in \mathbb{Z}$ has no jump between two successive intervals, i.e., $w_i^{ex}(b_i^{ex}) = w_{i+1}^{ex}(a_{i+1}^{ex})$.
- (5) The function w^{ex} is periodic in t due to the fact $w_i^{ex} = w_{i+4}^{ex}$.

References

- [1] A. Acosta, *Traveling wave solutions in coupled Chua's circuits*, preprint, 1998, 1-39.
- [2] J. Billingham and D. J. Needham, *A note on the properties of a family of travelling-wave solutions arising in cubic autocatalysis*, *Dynamics and Stability of Systems*, 6(1991), 33-49.
- [3] L. O. Chua, *CNN: A paradigm for complexity*, World Scientific Series on Non-linear Science, 31(1998).
- [4] S. N. Chow and X. B. Lin, *Bifurcation of a homoclinic orbit with a saddle-node equilibrium*, *Differential and Integral Equations*, 3(1990), 435-466.
- [5] W. A. Coppel, *Dichotomies in stability theory*, *Lectures notes in Math.* Springer-Verlag, Berlin, 629(1978).
- [6] A. Doelman and V. Rottschäfer, *Singularly perturbed and nonlocal modulation equations for systems with interacting instability mechanisms*, *J. Nonlinear Sci*, 7(1997), 371-409.
- [7] N. Fenichel, *Persistence and smoothness of invariant manifolds for flows*, *Indiana Univ. Math. Journal*, 21(1971), 193-226.
- [8] N. Fenichel, *Geometric singular perturbation theory for differential equations*, *J. Differential Equations*, 31(1979), 53-98.
- [9] J. Guckenheimer, J. Moser and S. E. Newhouse, *Dynamical Systems*, C.I.M.E. Lectures, Bressanone, Italy (1978), Birkhäuser, 1980.
- [10] J. K. Hale and X. B. Lin, *Multiple internal layer solutions generated by spatially oscillatory perturbations*, *J. Differential Equations*, 154(1999), 364-418.
- [11] C. Jones, *Geometric singular perturbation theory*, *Lectures notes in Math.* Springer-Verlag, Berlin, 106(1995), 44-118.
- [12] C. Jones and J. Rubin, *Existence of standing pulse solutions of an inhomogeneous reaction-diffusion system*, *J. Dyn. Diff. Eqns*, 10(1998), 1-35.
- [13] G. Kovacic and S. Wiggins, *Orbits homoclinic to resonances, with an application to chaos in a model of the forced and damped sine-Gordon equations*, *Physica*, 57(1992), 185-225.
- [14] M. Krupa, B. Sandstede and P. Szmolyan, *Fast and slow waves in the FitzHugh-nagumo equation*, *J. Diff. Eqns*, 133(1997), 49-97.
- [15] J. Keener and J. Sneyd, *Mathematical physiology: II: Systems physiology*, Springer, 2008.
- [16] John K-J. Li, *Arterial system dynamics*, New York University Press, 1987.

- [17] X. B. Lin, *Shadowing lemma and singularly perturbed boundary value problems*, SIAM J. Appl. Math, 49(1989), 26-54. 38
- [18] X. B. Lin, *Using Melnikov's methods to solve Silnikov's problems*, Proc. Royal Soc. Edinburgh, 116(1990), 295-325.
- [19] X. B. Lin, *Heteroclinic bifurcation and singularly perturbed boundary value problems*, J. Diff. Eqns, 84(1990), 319-382.
- [20] X-B. Lin, *Construction and asymptotic stability of structurally stable internal layer solutions*, Transactions of the American Mathematical Society, 353(2001), 2983-3043.
- [21] X-B. Lin, *Lin's method* ,[http://www.scholarpedia.org/article/Lin's method](http://www.scholarpedia.org/article/Lin's_method).
- [22] K. J. Palmer, *Exponential dichotomies and transversal homoclinics points*, J. Differential Equations, 55(1984), 225-256.
- [23] V. Perez-Munuzuri, V. Perez-Villar and L. O. Chua, *Traveling wave front and its failure in a one-dimensional array of Chua's circuit, Chua's circuit, A paradigm for Chaos*, World Scientific Singapore, 1993, 336-350.
- [24] D. H. Sattinger, *On the stability of waves of nonlinear parabolic systems*, Advances in Mathematics, 22(1976), 312-355.
- [25] S. Schecter, *Exchange lemmas 2: General exchange lemma*, J. Differential Equations, 245(2008), 411-441.
- [26] Fu Zhang, *Existence of a periodic solution in a Chua's circuit with smooth nonlinearity*, Differential and Integral Equations, 18(2005), 83-120.