

# QUADRATIC HARVESTING DOMINATED OPTIMAL STRATEGY FOR A STOCHASTIC SINGLE-SPECIES MODEL

Dianli Zhao<sup>1,†</sup>, Haidong Liu<sup>2</sup>, Yanli Zhou<sup>3</sup> and Sanling Yuan<sup>1</sup>

**Abstract** A stochastic population model with the mixed harvesting strategy is formulated and studied in this paper. Sufficient and necessary conditions for survival of the species are derived firstly. Then, based on the ergodic stationary distribution, the optimal strategy is identified. Results show that the linear harvesting effort threatens to the survival of the species; the quadratic harvesting strategy occupies an absolute advantage in the harvesting and excludes the linear part out of the optimal harvesting strategy. It's interest to see all these occur only in the random environments. Computer simulations are carried out to support the obtained results.

**Keywords** Stochastic single-species model, ergodicity, optimal harvesting strategy, threshold.

**MSC(2010)** 60H10, 60H30, 34K15, 92D25.

## 1. Introduction

Mathematical modeling and its research in harvesting of species was started by Clark [3, 4]. The study of harvesting is related to the optimal management of renewable resources based on the concept of maximal sustainable yield [21], since a suitable amount of harvesting of predator can control the chaotic dynamics and make the system stable. Recently, many researchers paid attention to the impacts of the environmental noises on dynamics and harvesting of the real systems, such as, Beddington and May [2] considered a stochastic logistic model, and May etc [1] and Shepherd and Hotwood [20] compared several different models. Most recently, Li and Wang [11] generalized the optimal harvesting problem in [4] to a more general stochastic logistic model by solving the Fokker-Planck equation. Zou and Wang [26] developed the ergodic technical to deal with the harvesting problem, and obtained the equivalency between the time averaging yield and sustainable yield, which overcome the difficulty in solving the complex Fokker-Planck equation. Liu and his coauthors [5, 12, 13] further developed the studying method by using the results in [17], and got the optimal harvesting strategy for several multi-population stochastic biological model with and without delay. For more about the related harvesting problem, we refer to the cited literatures in [5, 11–13, 26]. To be pointed

<sup>†</sup>the corresponding author. Email address: [Tc\\_zhaodianli@usst.edu.cn](mailto:Tc_zhaodianli@usst.edu.cn) (D. Zhao)

<sup>1</sup>College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China

<sup>2</sup>School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China

<sup>3</sup>College of Arts and Science, Shanghai University of Medicine and Health Sciences, Shanghai 201318, China

out that the harvesting in fore-mentioned literatures are all the proportionate harvesting like  $hx$ . However, many researchers have shown that the harvesting term may take a nonlinear form, see [6,7,10,19] for example. Here we highlighted Gupta's work [6], the harvesting effort is assumed to be proportional to the number of prey  $hx$  and a quadratic harvesting  $hx^2$  is formulated and studied. Panja etc [19] considered the quadratic harvesting in a plankton-fish system and they discussed the bifurcation behavior of the system by taking the harvest constant as bifurcation parameter. Motivated by these, in this paper, we suggest a complex type of harvesting (a mixture of the linear harvesting and quadratic harvesting) as:

$$Y(x) = xH(x) = hx + h_0x^2,$$

where  $H(x) = (h + h_0x)$  is defined as the harvesting effort function. Then the studied single-species model with the mixed harvesting is like

$$dx(t) = x(t)(r - ax(t) - (h + h_0x(t)))dt + \sigma x(t)dB(t), \tag{1.1}$$

where  $x(t)$  represents the size of population at time  $t$ ,  $r$  denotes the intrinsic growth rate and  $r/a$  is the carrying capacity of the environment.  $B(t)$  is a standard Brownian motion defined on a complete probability space  $(\Omega, F, \{\mathfrak{F}_t\}_{t \geq 0}, P)$ .  $\sigma > 0$  is the intensity of white noise.

Let  $h_0 = 0$ , then (1.1) reduces to the classical logistic model with linear harvesting. By solving the Fokker-Planck equation, the authors [2] have proven that the optimal harvesting effort is  $h^* = \frac{1}{2} \left( r - \frac{\sigma^2}{2} \right)$  if  $r > \frac{\sigma^2}{2}$ , and the maximum of expectation of sustainable yield (ESY) is

$$Y^* = \max_{h>0} \left\{ \lim_{t \rightarrow \infty} E(hx(t)) \right\} = \frac{1}{4a} \left( r - \frac{\sigma^2}{2} \right)^2. \tag{1.2}$$

This result has been verified and developed in [11,12,26]. However, when  $h_0 \neq 0$ , to the best of our knowledge, the optimal harvesting for model (1.1) has not been studied by any scholars up to now. So, in this paper, we will focus our attention on showing how the harvesting term exerts an influence on dynamics of (1.1), and deriving the optimal harvesting strategy  $(h^*, h_0^*)$  such that:

- (i) The expectation of sustainable yield  $\lim_{t \rightarrow \infty} E(hx(t) + h_0x^2(t))$  gets its maximum value.
- (ii) The population  $x$  modeled by (1.1) is persistent.

The rest of this paper is organized as follows. In Section 2, a optimal harvesting strategy with sufficient and necessary criteria for its existence is proposed by using the stationary properties of the solution, and then we analyze the impact of the harvesting. Together with the harvesting effort, Section 3 discusses the effectiveness of the optimal harvesting strategy. The last part summarizes the main results.

## 2. Ergodicity and Optimal harvesting strategy

In this section, we will deduce the optimal harvesting strategy for  $x$  in model (1.1). To begin with, let's firstly prepare some lemmas. The proofs of these lemmas are given in Section 4.

**Lemma 2.1.** For any given positive initial value  $x(0) = x_0$ , model (1.1) admits a unique global and positive solution  $x(t)$  on  $R_+$ . Moreover,

- (i) if  $(r - h) - \frac{\sigma^2}{2} < 0$ , then  $x$  goes to extinction almost surely:  $\lim_{t \rightarrow \infty} x(t) = 0$  a.s.;
- (ii) if  $(r - h) - \frac{\sigma^2}{2} = 0$ , then  $x$  goes to extinction in time mean almost surely:  $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t x(s) ds = 0$  a.s.;
- (iii) if  $(r - h) - \frac{\sigma^2}{2} > 0$ , then  $x$  is persistent in the mean, i.e.,  $\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t x(s) ds \geq \frac{(r-h) - \frac{\sigma^2}{2}}{a+h_0}$  a.s.

**Lemma 2.2.** For any  $p > 0$ ,  $\lim_{t \rightarrow \infty} E[x^p(s)] \leq L(p)$  holds for some positive constant  $L(p)$ .

**Lemma 2.3.** If  $(r - h) - \frac{\sigma^2}{2} > 0$ , then (1.1) has a unique stationary distribution  $\pi_0(\cdot)$  with ergodicity such that for any  $\pi_0(\cdot)$ -integrable function  $f(\cdot)$

$$P\left(\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(x(s)) ds = \int_0^{+\infty} f(x) \pi_0(dx)\right) = 1.$$

**Remark 2.1.** From Lemma 2.2 and Lemma 2.3, it holds that

$$\lim_{t \rightarrow \infty} E[x^p(t)] = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t x^p(s) ds \text{ a.s.} \quad (2.1)$$

Now we are in the position to state our main result on the optimal harvesting strategy. From Lemma 2.1, we note that even if  $r - \frac{\sigma^2}{2} > 0$  holds, as long as  $h$  is large enough such that  $(r - h) - \frac{\sigma^2}{2} \leq 0$ ,  $x$  will not persist in mean. Noting that biological survival is the prerequisite for optimal harvesting, so we consider  $r - \frac{\sigma^2}{2} > 0$  and  $h < r - \frac{\sigma^2}{2}$  in the following.

**Theorem 2.1.** Let  $x(t)$  be solution of (1.1) with a positive initial value  $x_0$ , if  $r - \frac{\sigma^2}{2} > 0$  and  $h < r - \frac{\sigma^2}{2}$ , then the optimal harvesting strategy  $(h^*, h_0^*) = (0, a)$  and the maximum of ESY is  $Y^* = \frac{r(r - \frac{\sigma^2}{2})}{4a}$ .

**Proof.** By using the Itô formula to (1.1), we have

$$\begin{aligned} d \ln x(t) &= \left[ (r - ax(t) - (h + h_0x(t))) - \frac{\sigma^2}{2} \right] dt + \sigma dB(t) \\ &= \left[ (r - h) - \frac{\sigma^2}{2} - (a + h_0)x(t) \right] dt + \sigma dB(t). \end{aligned} \quad (2.2)$$

Taking integrations on both sides shows that

$$\ln \frac{x(t)}{x(0)} = \left( (r - h) - \frac{\sigma^2}{2} \right) t - (a + h_0) \int_0^t x(s) ds + \sigma B(t). \quad (2.3)$$

By applying the ergodicity in Lemma 2.3 and the Strong Law of Large Numbers, (2.3) yields

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{x(t)}{x(0)} = 0 \text{ and } \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t x(s) ds = \frac{(r - h) - \frac{\sigma^2}{2}}{a + h_0} \text{ a.s.} \quad (2.4)$$

At the same time, from (1)

$$x(t) = x(0) + (r - h) \int_0^t x(s) ds - (a + h_0) \int_0^t x^2(s) ds + \sigma \int_0^t x(s) dB(s).$$

In view of Lemma 2.2 and Lemma 2.3, by letting  $t \rightarrow \infty$  we get

$$(r - h) \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t x(s) ds = (a + h_0) \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t x^2(s) ds.$$

By using Remark 2.1, we get

$$\lim_{t \rightarrow \infty} E[x^2(t)] = \frac{(r - h)}{a + h_0} \lim_{t \rightarrow \infty} E[x(t)]. \tag{2.5}$$

Combining (2.4)-(2.5) with Remark 2.1 shows that the expectation of sustainable yield

$$Y = \lim_{t \rightarrow \infty} E(hx(t) + h_0x^2(t)) = \frac{\left(r - h - \frac{\sigma^2}{2}\right)(ah + rh_0)}{(a + h_0)^2}. \tag{2.6}$$

To get the the maximum of ESY, we calculate partial derivatives of (2.6):

$$\frac{\partial Y}{\partial h} = \frac{a\left(r - \left(2h + \frac{r}{a}h_0\right) - \frac{\sigma^2}{2}\right)}{(a + h_0)^2} \text{ and } \frac{\partial Y}{\partial h_0} = \frac{a\left(r - h - \frac{\sigma^2}{2}\right)\left(r - \left(2h + \frac{r}{a}h_0\right)\right)}{(a + h_0)^3}. \tag{2.7}$$

Obviously,  $\frac{\partial Y}{\partial h} = 0$  and  $\frac{\partial Y}{\partial h_0} = 0$  can not hold together in  $R_+^2$ . So we give the following analysis:

Table 1. Classification of the states of two partial derivatives.	
$2h + \frac{r}{a}h_0 < r - \frac{\sigma^2}{2}$	$\frac{\partial Y}{\partial h} > 0$ & $\frac{\partial Y}{\partial h_0} > 0$
$2h + \frac{r}{a}h_0 = r - \frac{\sigma^2}{2}$	$\frac{\partial Y}{\partial h} = 0$ & $\frac{\partial Y}{\partial h_0} > 0$
$r - \frac{\sigma^2}{2} < 2h + \frac{r}{a}h_0 < r$	$\frac{\partial Y}{\partial h} < 0$ & $\frac{\partial Y}{\partial h_0} > 0$
$2h + \frac{r}{a}h_0 = r$	$\frac{\partial Y}{\partial h} < 0$ & $\frac{\partial Y}{\partial h_0} = 0$
$2h + \frac{r}{a}h_0 > r$	$\frac{\partial Y}{\partial h} < 0$ & $\frac{\partial Y}{\partial h_0} < 0$

From this table, it's easy to deduce the optimal harvesting effort  $(h^*, h_0^*) = (0, a)$ , and then we can get the desired maximum of ESY.  $\square$

**Remark 2.2.** From Theorem 2.1 and statement of the optimal harvesting strategy,  $r - \frac{\sigma^2}{2} > 0$  is the sufficient and necessary condition for existence of the optimal harvesting strategy.

**Remark 2.3.** From the proof of Theorem 2.1, we see that the expectation of sustainable yield

$$Y = \frac{\left(r - h - \frac{\sigma^2}{2}\right)(ah + rh_0)}{(a + h_0)^2}. \tag{2.8}$$

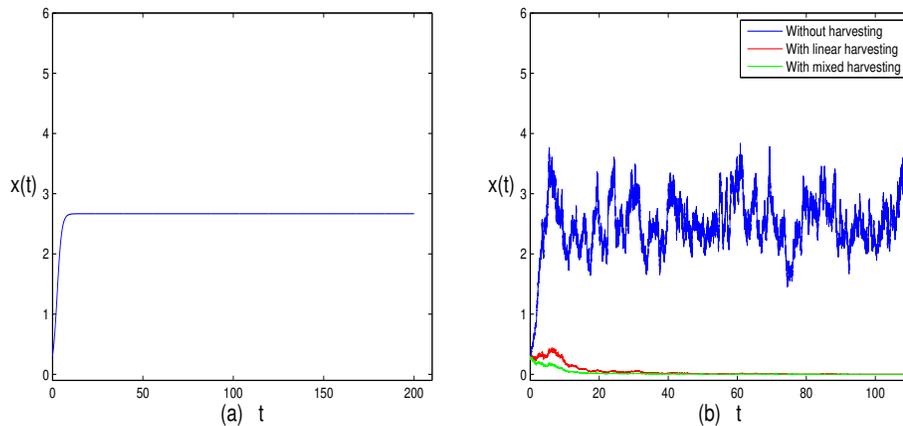
Let  $h_0 = 0$  in (2.8), one can easily get the optimal harvesting effort  $h^* = \frac{1}{2} \left( r - \frac{\sigma^2}{2} \right)$  and the maximum of ESY is  $Y_{Linear}^* = \frac{1}{4a} \left( r - \frac{\sigma^2}{2} \right)^2$ . Clearly, this is the classical results given in [2]. By comparing these results, it's of interest to see that the quadratic harvesting term ( $h_0x^2$ ) occupies an absolute advantage in the harvesting, which results in the exclusion of the linear part ( $hx$ ) from the optimal harvesting strategy. Compute that  $\frac{Y^*}{Y_{Linear}^*} = \frac{r}{r - \frac{\sigma^2}{2}} > 1$ , a novel fact is revealed that when one takes a quadratic harvesting strategy with  $h_0 = a$ , he will get more sustainable yield in sense of expectation than that by taking the linear harvesting strategy.

### 3. Effectiveness of the optimal harvesting strategy

Without restrictions on the effort, the optimal harvesting strategy for model (1.1) is discussed. It's shown that the maximum of ESY with quadratic harvesting strategy is better than that with linear harvesting strategy. However, when considering the harvesting effort, the optimal harvesting problem become more complicated. For example, if one gains more but gets this with the cost of more effort, we do not think he has adopted a more effective strategy. To illustrate this problem more clearly, let's start with an example.

**Example 3.1.** Consider model (1.1) with an initial value  $x_0 = 0.3$ ,  $r = 0.8$ ,  $a = 0.3$ . Without special declaration,  $\sigma = 0.2$ . To show the influence of the harvesting, we choose the parameters as follows: (1) deterministic version:  $\sigma = 0$  and  $(h, h_0) = (0, 0)$ ; (2) linear harvesting strategy  $(h, h_0) = (0.8, 0)$ ; (3) mixed harvesting strategy  $(h, h_0) = (0.8, 1)$ .

Numerical simulations for the solutions of stochastic model (1.1) and its corresponding deterministic version are presented by using the famous Milstein method [8]. From these figures in Figure 1, together with comparison of three sets of pa-



**Figure 1.** Computer simulations of  $x(t)$  for the three cases given in Example 3.1.

rameters, we can see that the small noise has no disadvantage to persistence of the species  $x$ . Excessive harvesting may threaten the survival of species, where

the linear harvesting plays a major role. These results are confirmed theoretically by (i) in Lemma 2.1. In order to meet the requirement of the definition of the optimal harvesting strategy, let's make the harvesting effort smaller and consider the following three cases: (A1) linear harvesting strategy  $(h, h_0) = (0.39, 0)$ ; (A2) mixed harvesting strategy  $(h, h_0) = (0.2, 0.1)$ ; (A3) quadratic harvesting strategy  $(h, h_0) = (0, 0.3)$ . In the following, the blue line represents the path of model (1.1) under condition (A1), the red line represents the path of model (1.1) under condition (A2), and the green line is that under condition (A3).

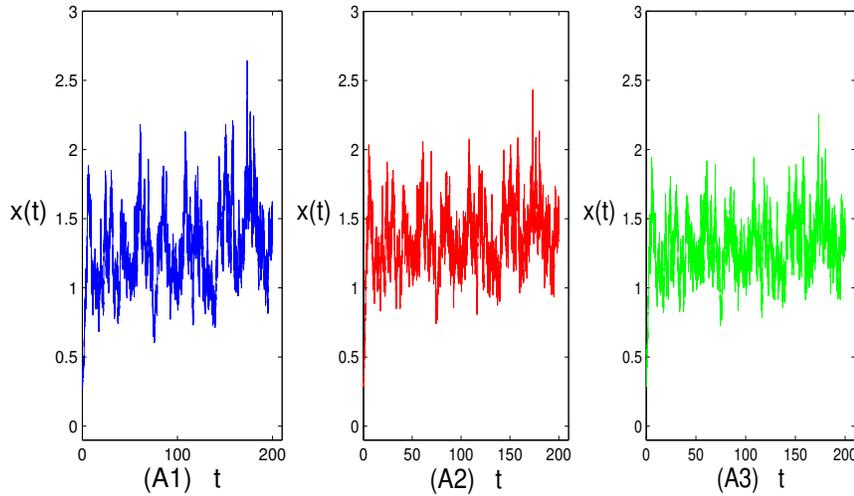


Figure 2. Sample trajectories of model (1.1) for the cases (A1)-(A3).

Figure 2 shows the paths of model (1.1) with the parameters from (A1)-(A3) separately. All the populations in these three cases are persistent, since the harvesting is small such that  $(r - h) - \frac{\sigma^2}{2} > 0$ , which has been shown theoretically by (iii) of Lemma 2.1. So a small amount of harvesting can not threaten the survival of species. To give the limit behaviors of the system, we compute the time-averaged yield  $\frac{1}{t} \int_0^t [hx(s) + h_0x^2(s)] ds$  and the related harvesting effort  $\frac{1}{t} \int_0^t [h + h_0x(s)] ds$ .

From (b) of Figure 3, it is clear that the path of (A3) is on top of other curves, since  $(h, h_0) = (0, 0.3)$  is the best harvesting strategy which has been confirmed by Theorem 2.1 theoretically. Further, (a) and (b) of Figure 4, in turn, support results of Theorem 2.1. To consider the optimal harvesting problem more perfectly, we analyze the harvesting effort in (a) of Figure 3. The harvesting effort values of (A1) and (A3) are almost the same provided the time is sufficiently long, and the related value of (A2) is bigger than the other two. That is to say, when one take the mixed harvesting strategy (A2), he will pay more efforts than that by taking one of the other two. However, back to (b) of Figure 3 again, the yields of the mixed harvesting strategy (A2) is more than that of (A1). A natural conclusion is that the more you give and the more you gain. So when we consider both the sustainable yield and harvesting effort, it is difficult to distinguish which is better between (A1) and (A2).

From the above analysis, a new standard is proposed for the optimal harvesting

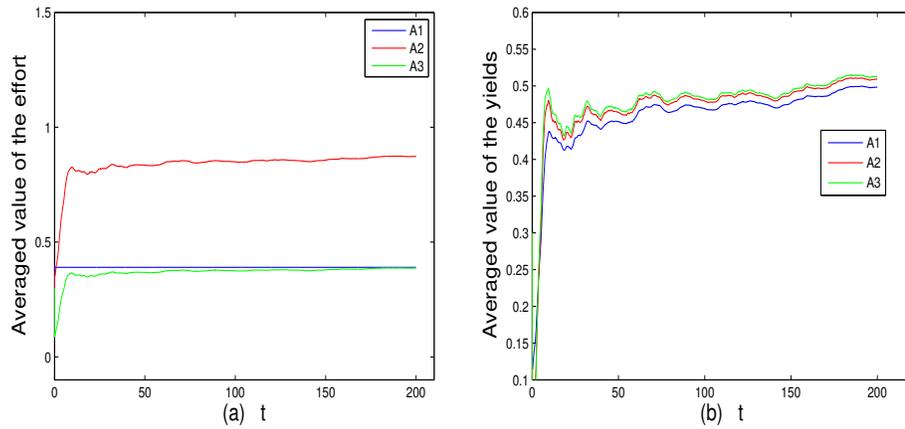


Figure 3. Time-averaged harvesting yield and the related harvesting effort for the cases (A1)-(A3).

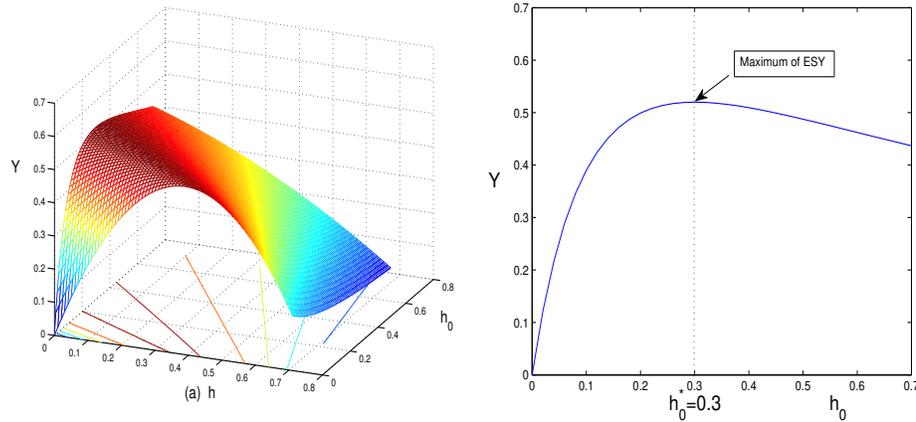


Figure 4. Computer simulations of the harvesting function images related to the harvesting parameters, and the marginal curve for maximum of the ESY.

strategy:

- Let  $A$  and  $B$  be two harvesting strategies,  $A$  is said to be better than  $B$  provided that (1) the maximum of (ESY):  $Y_A^* > Y_B^*$  and (2) the harvesting effort in time mean:  $H_A^* \leq H_B^*$ .

Fortunately, a simple computation shows that the harvesting effort in mean for model (1.1) is

$$H^* = \lim_{t \rightarrow \infty} E(h^* + h_0^* x(t)) = \frac{1}{2} \left( r - \frac{\sigma^2}{2} \right) = h_{Linear}^*$$

This and Remark 2.3 illustrate that the optimal harvesting strategy given in Theorem 2.1 is better than the linear optimal harvesting strategy given in [2]. (a) and

(b) in Figure 3 confirm this result.

### 4. Proofs of Lemma 2.1-2.3

Proofs for Lemma 2.1-Lemma 2.3 are similar to those in [12, 26] and [9, 16]. To keep the integrity of this paper, we list the main proofs in the sequel. Consider the one-dimensional time-homogeneous stochastic differential equation;

$$dX = b(X) dt + \alpha(X) dB(t) \text{ with } X(0) \in R_+, \tag{4.1}$$

**Lemma 4.1** (See [25]). *Let  $X(t)$  be a time-homogeneous solution of (4.1) on  $E_1$  (1-dimensional Euclidean space). Assume that:*

(U.1) *In the domain  $U \subset E_1$  and some neighborhood thereof, the diffusion  $\sigma(X)$  is bounded away from zero;*

(U.2) *If for all  $X \in E_1 \setminus U$  the mean time  $\tau_X$  at which a path emerging from  $X$  reaches the set  $U$  is finite, and  $\sup_{X \in K} E(\tau_X) < \infty$  for every compact subset  $K \subset E_1$ .*

*Then the Markov process  $X(t)$  has a stationary distribution  $\pi(x)$ , and for an integrable function  $f(\cdot)$*

$$P \left( \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f[x(s)] ds = \int_{-\infty}^{\infty} f(x) \pi(dx) \right) = 1.$$

**Lemma 4.2** (See [23, Lemma 2.6]). *Let  $\lambda_0 \geq 0$  and  $\lambda > 0$  be constants and  $F(t)$  be a function such that  $\lim_{t \rightarrow \infty} \frac{F(t)}{t} = 0$ . If there is a positive function  $f(t)$  satisfying*

$$\log f(t) \leq \lambda_0 - \lambda \int_0^t f(s) ds + F(t),$$

*then  $\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(s) ds \leq \frac{\lambda_0}{\lambda}$ . If*

$$\log f(t) \geq \lambda_0 - \lambda \int_0^t f(s) ds + F(t),$$

*then  $\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(s) ds \geq \frac{\lambda_0}{\lambda}$ .*

**Lemma 4.3** (See e.g. [1, Lemma 3.1]). *Let  $M(t)$ ,  $t \geq 0$  be a local martingale vanishing at time 0 and define*

$$\rho_M(t) := \int_0^t \frac{d\langle M, M \rangle(s)}{(1+s)^2}, \quad t \geq 0$$

*where  $\langle M, M \rangle(t)$  is Meyers angle bracket process. Then  $\lim_{t \rightarrow \infty} \frac{M(t)}{t} = 0$  a.s. provided that  $\lim_{t \rightarrow \infty} \rho_M(t) < \infty$  a.s.*

**Proof of Lemma 2.1.** Existence of the global solution is obviously true, so we only prove the critical condition for persistence and extinction of the species. By applying Itô formula to model (1.1), we get

$$d \ln x(t) = \left[ (r - h - (a + h_0)x(t)) - \frac{\sigma^2}{2} \right] dt + \sigma dB(t). \tag{4.2}$$

With help of Lemma 4.3, (i) in Lemma 2.1 is proved. By Lemma 4.2 and Lemma 4.3, from (4.2), we get (ii) and (iii) directly.  $\square$

**Proof of Lemma 2.2.** From (1.1), for any  $p > 0$ ,

$$\begin{aligned} \mathfrak{L}x^p &= px^p \left[ (r - h - (a + h_0)x) - \frac{(p-1)\sigma^2}{2} \right] \\ &\leq -px^p + px^p \left[ \left( r + 1 + \frac{\sigma^2}{2} - (a + h_0)x \right) \right] \\ &\leq -px^p + H(p), \end{aligned}$$

where  $H(p) = \sup_{x>0} \left\{ p \left( r + 1 + \frac{\sigma^2}{2} \right) x^p - (a + h_0)x^{p+1} \right\}$ . By taking expectations, we have  $E(x^p(t)) \leq x^p(0) + \frac{1}{p}H(p) := L(p)$ . The proof is complete.  $\square$

**Proof of Lemma 2.3.** Let  $V(x) = c(x - 1 - \ln x) \geq 0$  for some positive constant  $c$ , by Itô formula

$$\begin{aligned} \mathfrak{L}V(x) &= -c \left( (r - h) - \frac{\sigma^2}{2} \right) + c(a + h_0)x + cx(r - ax(t) - (h + h_0x)) \\ &\leq -c \left( (r - h) - \frac{\sigma^2}{2} \right) + c(r + a + h_0)x. \end{aligned}$$

Let  $U = [\frac{1}{k}, k]$ , then  $\mathfrak{L}V(x) < -1$  for suitable large numbers  $c$  and  $k$ . The left proof follows Theorem 2.1 in [9], see also [16, Theorem 1]. The proof is complete.  $\square$

## 5. Conclusions

This paper studies the impacts of harvesting on dynamics of two stochastic population models, and derives the optimal harvesting strategy. Sufficient and necessary condition for survival of the species modeled by (1.1) is derived. It's shown that the linear harvesting effort threatens to the survival of the species. Large linear harvesting effort may directly lead the population to extinction. However, the quadratic harvesting strategy has nothing to do with the survival threshold of the species, and has only impacts on the number of sustainable population in time mean.

For comparing the effectiveness of different harvesting strategies, we make a new measure, that is, to gain more benefits with less harvesting effort.

The most valuable finding of this paper is that when one takes a mixed strategy of the linear harvesting and quadratic harvesting, the quadratic harvesting term ( $h_0x^2$ ) will occupy an absolute advantage in the harvesting, and exclude the linear part ( $hx$ ) out of the optimal harvesting strategy. Let the noise intensities tend to zero, then the models and the results in this paper are all reduced to the deterministic version. From Theorem 2.1, the maximum values of expectation of sustainable yield are the same for the linear harvesting strategy and the quadratic harvesting, i.e.,  $\frac{Y^*}{Y_{Linear}^*} = 1$ . This means that the advantages of the quadratic harvesting emerge only in the random environment. In general, both the random noises and the harvesting terms can change dynamics of the deterministic system significantly.

In this paper, we consider the model with the white noise. What happens if other perturbations are taken into account, such as pure Markov switching [24], Levy noise [1, 23] and mixed noises [14, 24]. From this viewpoint, more general stochastic

models should be constructed and studied, based on literatures [5,9,11–13,15,16,26]. For these models, whether the quadratic harvesting can show its advantages or not, we leave this for future investigation.

## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

All authors contributed equally to the writing of this paper. The authors read and approved the final manuscript.

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