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SOME SIMPLE GROUPS OF LIE TYPE CONSTRUCTED BY THE INNER-AUTOMORPHISM

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C. Chevalley constructed some simple groups and obtained for them the structure theorems[1]. Using the graph automorphism of simple Lie algebra L and the automorphism of the field $\mathscr{K}$, R. Steinberg and R. Ree constructed some twisted simple group of Lie type[2,3]. It is analogous to the construction of real form of L by the outer-automorphism of L and the conjugation of complex number field $\mathscr{C}$.

From the theory of Symmetric Rieman Space[4], we know that the real form of L can be constructed by the involutive inner-automorphism of L and the conjugation of the field $\mathscr{C}$. In this paper, we denote by L the simple Lie algebra C_l where l is even and $l\geqslant 4$ and denote by $\mathscr{K}$ the field $\mathscr{K}_0(\sqrt{-1})$ where $\mathscr{K}_0$ is an ordering field. Using the involutive inner-automorphism of L and the involutive automorphism of the field $\mathscr{K}$, we construct the simple group of Lie type which is denoted by ${}^2C_l\ (\mathscr{K})$. The construction of ${}^2C_l(\mathscr{K})$ is connected with the theory of the classification of the real form of L[5] and the theory of the conjugation between the cartan subalgebra of the real form of L[6]***.

Let Φ be the system of roots of L, Φ can be expressed in the form: Φ: $\{\pm e_i \pm e_j, \pm 2e_i, 1\leqslant i<j\leqslant l\}$. Let $\Gamma=(\Phi)_{\mathscr{R}}$ be the vector space generated by Φ over the real number field $\mathscr{R}$. Obviously, $\mathscr{B}$: $\{e_1, e_2, \cdots, e_l\}$is a basis of Γ, we define an ordering $\mathscr{E}$ of the space Γ, let $r\in\Gamma$, the first non-zero coefficient of r for the basis $\mathscr{B}$ is positive, then r is called positive, i. e., $r>0$. For the ordering $\mathscr{E}$, he positive system of roots Φ^+: $\{e_i\pm e_j, 2e_i, 1\leqslant i<j\leqslant l\}$ and the fundamental system of roots Π: $\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$, where $\alpha_i=e_i-e_{i+1}$ for $i=1, 2, \cdots, l-1$ and $\alpha_l=2e_l$. Let $\mathscr{D}$: $\{1, 3, \cdots, l-1\}$ and let $w_0=w_{\alpha_1}w_{\alpha_3}\cdots w_{\alpha_{l-1}}$, obviously, $w_0e_i=e_{i+1}$, $w_0e_{i+1}=e_i$ for each $i\in\mathscr{D}$. Let Π°: $\{\alpha_1, \alpha_3, \cdots, \alpha_{l-1}\}$and let $\Phi^*=\Phi^+\backslash\Pi^\circ$.

Lemma 1.1 *For each $r\in\Phi^+$, $r\in\Pi^\circ$ if and only if $w_0\, r\in\Phi^-$.*

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*** In this paper, we use the definitions and the symbols given by [7] without explanation.

Proof If $\alpha_i, \alpha_j \in \Pi^\circ$, $j\neq i$, then $(\alpha_i, \alpha_j)=0$. Thus, $w_0 r=-r\in\Phi^-$ for each $r\in\Pi^\circ$. Conversely, let $W_i=w_{\alpha_i}w_{\alpha_{i+2}}\cdots w_{\alpha_{l-1}}$ and let $W_{l+1}=I$, then there exists an element k of $\{\mathscr{D}, l+1\}$such that $W_k r=\beta\in\Phi^+$ and $W_{k-2}r=W_{\alpha_{k-2}}\beta\in\Phi^-$. It follows from $\alpha_{k-2}\in\Pi$ that $\beta=\alpha_{k-2}\in\Pi^\circ$. Obviously, $W_k^2=I$, therefore, $r=W_k\beta=W_k\alpha_{k-2}$. If $j\neq k-2$, then $(\alpha_j, \alpha_{k-2})=0$, thus $w_{\alpha_j}\alpha_{k-2}=\alpha_{k-2}$. Therefore, $W_k\alpha_{k-2}=\alpha_{k-2}$, hence, $r=\alpha_{k-2}\in\Pi^\circ$.

Since $w_0^2=I$, $\Gamma=\Gamma^++\Gamma^-$ where Γ^+: $\{r\in\Gamma\,|\,w_0r=r\}$ and Γ^-: $\{r\in\Gamma\,|\,w_0r=-r\}$. Obviously, $(\Gamma^+, \Gamma^-)=0$. For each $r\in\Gamma$, we denote by r' the projection of r onto the subspace Γ^+. Clearly, $\mathscr{B}^+$: $\{e'_1, e'_3, \cdots, e'_{l-1}\}$ is a basis of the subspace Γ^+ where $e'_i=\frac{1}{2}(e_i+e_{i+1})$, $i\in\mathscr{D}$. We can define an ordering $\mathscr{E}'$ of the subspace Γ^+: let $r\in\Phi^*$. The first non-zero coefficient of r for the basis $\mathscr{B}$ is positive, then r' is called positive, i.e., $r'>0$. and if $r\in\Phi$, $r>0$ for the ordering $\mathscr{E}$, then $r'\geqslant 0$ for the ordering $\mathscr{E}'$, moreover if $\alpha\in\Pi^\circ$, then $\alpha'=0$ and $\Gamma^-=(\Pi^\circ)_{\mathscr{R}}$. Let Φ', $\Phi^{+\prime}$ and Π' be the projections of Φ, Φ^+ and Π onto the subspace Γ^+ respectively. It is easy to verify that Φ' is a system of roots and $\Phi^{+\prime}$ (Π') is a positive (fundamental) system of roots of Φ'. Obviously, $\Phi^{+\prime}=\Phi^{*\prime}$ and $\Pi'=\Pi^{*\prime}$ where $\Pi^{*\prime}$ is the pvojection of $\Pi^*=\Pi\backslash\Pi^\circ$ onto the subspace Γ^+. It is easy to verify:

Lemma 1.2 1) *If* $\alpha\in\Pi^\circ$ *and* $r\in\Phi^*$ *and if* $\alpha\pm r\in\Phi$, *then* $\alpha\pm r\in\Phi^*$; 2) Π' and $\Pi(C_m)$ *are isomorphic where* $\Pi(C_m)$ *is the fundamental system of roots of* C_m, $m=l/2$.

Let us denote by $\bar r$ the w_0r for each $r\in\Phi$. It follows from 1.1 that if $r\in\Phi^*$, then $\bar r\in\Phi^*$. Let Φ_1^*: $\{r\in\Phi^*\,|\,r\leqslant\bar r\}$ and let J_r: $\{r, \bar r\}$ for each $r\in\Phi_1^*$. For each $r\in\Phi_1^*$, let $\overline{W}_r^1=w_r$ or $\overline{W}_r^1=w_rw_{\bar r}$ according as $r=\bar r$ or $r\neq\bar r$. For each $\alpha\in\Pi^\circ$, let J_α: $\{\alpha\}$ and let $\overline{W}_\alpha^1=W_\alpha$. Obviously, for each $r\in\Phi_1^\dagger=\Phi_1^*\cup\Pi^\circ$, $w_rw_{\bar r}=w_{\bar r}w_r$, $(\overline{W}_r^1)^2=I$ and $\overline{W}_r^1J_r=-J_r$: $\{-r, -\bar r\}$.

Definition 1.3 W^1: $\{w\in w\,|\,w_0w=ww_0\}$.

It is easy to verify the following lemma:

Lemma 1.4 *For each* $r\in\Phi_1^\dagger$, *then* $W_r^1\in W^1$.

For each $W\in W^1$, then $w\Gamma^+=\Gamma^+$ and $w\Gamma^-=\Gamma^-$. Let w' be the restriction of w onto T^+, it is easy to verify that for each $r\in\Phi_1^*$, $\overline{W}_r^{1\prime}=w_{r'}$, where $w_{r'}$ is the reflection in the hyperplane orthogonal to the root r' of Φ' operating on the subspace Γ^+, If $r\in\Pi^\circ$, then $\overline{W}_r^{1\prime}=I'$ where I' is the identity mapping on the subspace Γ^+.

Theorem 1.5 $W^1=\langle\overline{W}'_r, r\in\Pi\rangle$.

Proof We denote by W'_1 the restriction of W' operating on the subspace Γ^+ and let $W(\Phi')$: $\{w_{r'}, r'\in\Phi'\}$. It follows from 2) of 1.2 that $W'_1=W(\Phi')$. Therefore, by 2.1.8 of [7], every element w' of W'_1 can be expressed in the form

$$w'=w_{r'_1}w_{r'_2}\cdots w_{r'_p}\quad\text{where } r'_i\in\Pi',\ 1\leqslant i\leqslant p.$$

Let $w\in W^1$and $W_1=W^{-1}\overline{W}_{r_1}^1\overline{W}_{r_2}^1\cdots\overline{W}_{r_p}^1$ where $r_i\in\Pi^*$ and the projection of r_i onΓ^+

is r'_i, $1 \leqslant i \leqslant p$. Clearly, for each $r \in \Phi_1^*$, $\overline{W}_r^{1\prime} = W_{r'}$, therefore, $W'_1 = I'$, where w'_1 is the restriction of w_1 onto the subspace Γ^+. Clearly, for each $\alpha \in \Pi^\circ$, $w_1\alpha = \pm\beta$, where $\beta \in \Pi^\circ$. Therefore, there exist $\beta_1, \beta_2, \cdots, \beta_q \in \Pi^\circ$ such that $w_2 = w_1 \overline{W}^1_{\beta_1} \overline{W}^1_{\beta_2} \cdots \overline{W}^1_{\beta_q}$ leaves Π° invariant. Obviously, $w'_\alpha = I'$ for all $\alpha \in \Pi^\circ$, then $w'_2 = I'$, where w'_2 is the restriction of w_2 onto the subspace Γ^+. In the following, we shall prove that W_2 leaves all elements of Γ^- invariant. Assume the contrary, there exist $\alpha = e_i - e_{i+1}$ and $\beta = e_j - e_{j+1}$, $i, j \in \mathscr{D}$, $i \neq j$ such that $w_2\alpha = \beta$. Let $r_i = 2e_i$, $r_j = 2e_j$ and $r_{j+1} = 2e_{j+1}$, then either $w_2 r_i = r_j$ or $w_2 r_i = -r_{j+1}$. Obviously, $r_i, r_j, r_{j+1} \in \Phi^*$ and $r'_i \neq r'_j$, $r'_i \neq -r'_{j+1}$. Hence we have a contradiction. Therefore, w_2 leaves all elements of Γ^- invariant. Thus, $w = \overline{W}^1_{r_1} \overline{W}^1_{r_2} \cdots \overline{W}^1_{r_n}$, where $r_i \in \Pi$, $1 \leqslant i \leqslant n$, $n = p+q$. This proves the theorem.

Corollary 1.6 *If $w \in W'$, then $w\Pi^\circ \subset \{\Pi^\circ, -\Pi^\circ\}$; $w\Phi^* \subset \{\Phi^*, -\Phi^*\}$.*

Obviously, we can separate all elements of Φ_1^* into the following three types of the partition $\mathscr{A}$: $\mathscr{A}$: ① $r = \bar{r}$; ② $r \neq \bar{r}$, r is long; ③ $r \neq \bar{r}$, r is short.

Lemma 1.7 1) *Any two elements r and s of the same type in $\mathscr{A}$ are conjugate with each other under W^1, moreover, there exists r_i belonging to ⓘ type of $\mathscr{A}$ such that $(r, r_i) \neq 0$, $i = 1, 2, 3$.* 2) *Any two elements α and β of Π° are conjugate with each other under W^1.*

Proof 1)① If r and s belong to ① type of $\mathscr{A}$, then r and s can be expressed in the form: $r = e_i + e_{i+1}$, $s = e_j + e_{j+1}$, $i, j \in \mathscr{D}$. If $i \neq j\,(i<j)$, then $\overline{W}^1_\delta r = s$, where $\delta = e_{i+1} - e_j$. Moreover, let $r_1 = e_i + e_{i+1}$, $r_2 = 2e_{i+1}$ and $r_3 = e_{i+1} - e_{i+2}$ or $r_3 = e_{i-1} - e_i$, then $(r, r_i) \neq 0$ for $i = 1, 2, 3$. ② If r and s belong to ② type of $\mathscr{A}$, then r and s can be expressed in the form: $r = 2e_{i+1}$, $s = 2e_{j+1}$, $i, j \in \mathscr{D}$. If $i \neq j\,(i<j)$, then $\overline{W}^1_\delta r = s$, where $\delta = e_{i+1} - e_j$. Moreover, r_i, $i = 1, 2, 3$ mentioned in ① satisfy $(r, r_i) \neq 0$, $i = 1, 2, 3$. ③ If r belongs to ③ type of $\mathscr{A}$ and $r = e_{i+1} + e_{j+1}$, $i, j \in \mathscr{D}$, $i \neq j$, then $\overline{W}^1_{\delta_0} r = e_{i+1} - e_j$ where $\delta_0 = 2e_{j+1}$. Therefore, if r and s belong to ③ type of $\mathscr{A}$, under the conjugation of W^1, r and s can be expressed in the form $r = e_{i+1} - e_j$, $s = e_{k+1} - e_h$, $i, j, k, h \in \mathscr{D}$. Let $w_1 = I$ or $w_1 = \overline{W}^1_{\delta_1}$, where $\delta_1 = e_{i+1} - e_k$ according as $i = k$ or $i \neq k\,(i<k)$. Similarly, let $w_2 = I$ or $w_2 = \overline{W}^1_{\delta_2}$, where $\delta_2 = e_{j+1} - e_h$ according as $j = h$ or $j = h\,(j<h)$. Clearly, $w_1 w_2 r = s$. Let $r_1 = e_i + e_{i+1}$, $r_2 = 2e_{i+1}$, $r_3 = r$, then $(r_1 r_i) \neq 0$ for $i = 1, 2, 3$.

2) If $\alpha, \beta \in \Pi^\circ$, then α and β can be expressed in the form: $\alpha = e_i - e_{i+1}$, $\beta = e_j - e_{j+1}$, $i, j \in \mathscr{D}$. If $i \neq j\,(i<j)$, then $\overline{W}^1_\delta \alpha = \beta$, where $\delta = e_{i+1} - e_j$.

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Let $G = L(\mathscr{K})$ be the Chevalley group for the Chevalley basis C_b: $\{h_r,\ r \in \Pi;\ \tilde{e}_r,\ r \in \Phi\}$. For each $\alpha \in \Pi^\circ$, let $n_\alpha = \exp ad\tilde{e}_\alpha \exp(-ad\tilde{e}_{-\alpha}) \exp ad\tilde{e}_\alpha$, let $n_0 = n_{\alpha_1} n_{\alpha_2} \cdots n_{\alpha_{l-1}}$, then n_0 corresponds to w_0 under the natural homomorphism from N onto W. Obviously, $n_0^2 = I$, where I is the identity element of G.

Lemma 2.1 *There is a Chevalley basis C_b: $\{h_r,\ r\in\Pi;\ e_r,\ r\in\Phi\}$ such that $n_0e_r=k_re_{\bar r}$, $r\in\Phi$ where k_r, $r\in\Phi$ satisfy the following relations: a) If $r\in\Phi^*$, $r=\bar r$ or $r\in\Pi^\circ$, then $k_r=-1$; b) If $r\in\Phi^*$, $r\neq\bar r$, then $k_r=1$.*

Proof We shall prove that there are $e_r=\eta_r\tilde e_r$ for all $r\in\Phi$, where $\eta_r=\pm1$, by appropriate sign changes of the η_r, such that C_b: $\{h_r,\ r\in\Pi;\ e_r,\ r\in\Phi\}$ is a Chevalley basis of L and the relations a) and b) hold.

a) 1) For each $\alpha\in\Pi^\circ$, $n_0\tilde e_\alpha=-\tilde e_{-\alpha}$, $n_0\tilde e_{-\alpha}=-\tilde e_\alpha$. Thus, let $e_\alpha=\tilde e_\alpha$, $e_{-\alpha}=\tilde e_{-\alpha}$, for all $\alpha\in\Pi^\circ$, we have $k_r=-1$ for all $r\in\Pi^\circ$, 2) If $r\in\Phi^*$, $r=\bar r$, then $r=e_i+e_{i+1}$, $i\in\mathscr{D}$, hence $n_{\alpha_i}\tilde e_r=(-1)\epsilon_0/\epsilon_0\tilde e_r$, where $\alpha_i=e_i-e_{i+1}$ by 6.4.2 of [7]. It is easy to verify that $n_{\alpha_j}\tilde e_r=\tilde e_r$, where $\alpha_j=e_j-e_{j+1}$, $j\in\mathscr{D}$, $j\neq i$. Therefore, $n_0\tilde e_\alpha=-\tilde e_\alpha$. Let $e_r=\tilde e_r$, $e_{-r}=\tilde e_{-r}$ for all $r\in\Phi^*$, $r=\bar r$, then we have $k_r=-1$ for each $r\in\Phi^*$, $r=\bar r$.

b) Let $\Phi^{(*)}$: $\{r\in\Phi^*\,|\,\bar r\neq r\}$. Clearly, $\Pi^*\subset\Phi^{(*)}\subset\Phi^+$. Let $e_r=\tilde e_r$ for all $r\in\Pi^*$, we can choose the appropriate $\eta_{\bar r}$ so that $k_r=1$ for all $r\in\Pi^*$. Suppose that for all r of Φ^*, of which the height $h(r)<n\ (n>1)$ the η_r and the $\eta_{\bar r}$ have been chosen so that $k_r=1$. For each element r of $\Phi^{(*)}$ of which the height $h(r)=n$. If η_r and $\eta_{\bar r}$ have been chosen so that $k_{\bar r}=1$, then $k_r=1$ by $n_0^2=I$; if η_r and $\eta_{\bar r}$ have not been chosen, then we can choose the appropriate η_r and $\eta_{\bar r}$ so that $k_r=1$. By the steps along the lines of the preceding discussion, we obtain $e_r=\eta_r\tilde e_r$ for all $r\in\Phi^{(*)}$, such that $k_r=1$. Moreover, for each $r\in\Phi^{(*)}$, we can choose e_{-r} such that $[e_r,\ e_{-r}]=h_r$. Thus, we obtain the set C_b: $\{h_r,\ r\in\Pi;\ e_r,\ r\in\Phi\}$. It follows from p. 58 of [7] that C_b is a Chevalley basis, and the relations a) and b) hold obviously.

Clearly, the chevalley group for the Chevalley basis C_b is $G=L(\mathscr{K})$ as was mentioned in [7] and for each $\alpha\in\Pi^\circ$ $n_\alpha=\exp a\,de_\alpha\exp-a\,de_{-\alpha}\exp a\,de_\alpha$. Let f be an involutive automorphism of the field $\mathscr{K}$ which satisfies $f(a)=a$ for all $a\in\mathscr{K}_0$, and $f(\sqrt{-1})=-\sqrt{-1}$. The mapping $x_r(t)\to x_r(f(t))$ can be extended to an involutive automorphism of G (denoted by f also). By $f(1)=1$ and $f(-1)=1$. We have $fn_0=n_0$.

Definition 2.2 *For all $x\in G$, let $\sigma x=f\rho x=f(n_0xn_0^{-1})$. Obviously, $\sigma^2x=x$ for all $x\in G$, i., e., $\sigma^2=I$.*

Definition 2.3 U^1: $\{u\in U\,|\,\sigma u=u\}$.

If $r\in\Phi_1^*$, $r=\bar r$, then let $X_r^1(t)=x_r(t)$, where $t\in\mathscr{K}$ and $\bar t=f(t)=-t$; if $r\in\Phi_1^*$, $r\neq\bar r$, then let $X_r^1(t)=x_r(t)x_{\bar r}(\bar t)$, $t\in\mathscr{K}$, $f(t)=\bar t$.

Lemma 2.4 *For each $r\in\Phi_1^*$, $X_r^1(t)\in U^1$, $t\in\mathscr{K}$.*

Lemma 2.5 *Every element u of U^1 can be expressed in the following form*

$$u=x_{r_1}(t_1)x_{r_2}(t_2)\cdots x_{r_n}(t_n),\ r_i\in\Phi^*,\ t_i\in\mathscr{K}^*=\mathscr{K}\backslash 0,\ 1\leqslant i\leqslant n,\ r_1<r_2<\cdots<r_n.$$

Proof It follows from 5.3.3 of [7] that u can be expressed in the form

$$u=x_{s_1}(t'_1)x_{s_2}(t'_2)\cdots x_{s_p}(t'_p),\ \text{where } s_i\in\Phi^+,\ t_i\in\mathscr{K}^*,\ 1\leqslant i\leqslant p,\ s_1<s_2<\cdots<s_p.$$

In fact, $s_i\in\Phi^*$ for all $1\leqslant i\leqslant p$. Suppose the contrary, it follows from 1.2 and 5.2.2

of [7] that

$$u=x_{\beta_1}(a_1)x_{\beta_2}(a_2)\cdots x_{\beta_k}(a_k)x_{r_1}(t_1)x_{r_2}(t_2)\cdots x_{r_n}(t_n),$$

where $\beta_i\in\Pi^\circ$, $a_i\in\mathscr{K}^*$, $1\leqslant i\leqslant k$; $r_i\in\Phi^*$, $t_i\in\mathscr{K}^*$, $1\leqslant i\leqslant n$, $r_1<r_2<\cdots<r_n$. Obviously, σu can be expressed in the form

$$\sigma u=x_{-\beta_1}(-\bar{a}_1)x_{-\beta_2}(-\bar{a}_2)\cdots x_{-\beta_k}(-\bar{a}_k)x_{\bar{r}_1}(\tilde{t}_1)x_{\bar{r}_2}(\tilde{t}_2)\cdots x_{\bar{r}_n}(\tilde{t}_n),$$

where $\tilde{t}_i\in\mathscr{K}^*$, $1\leqslant i\leqslant n$ and if $\bar{r}_i\neq r_i$, then $\tilde{t}_i=\bar{t}_i$; if $\bar{r}_i=r_i$, then $\tilde{t}_i=-\bar{t}_i$, $1\leqslant i\leqslant n$. Hence, it follows from $\sigma u=u$ and 7.1.2 of [7] that $a_i=0$ for all $1\leqslant i\leqslant k$. Thus, we have a contradication. Therefore, $s_i\in\Phi^*$ for all $1\leqslant i\leqslant p$ and we have

$$u=x_{s_1}(t'_1)x_{s_2}(t'_2)\cdots x_{s_p}(t'_p)=x_{r_1}(t_1)x_{r_2}(t_2)\cdots x_{r_n}(t_n),$$

where $r_i\in\Phi^*$, $t_i\in\mathscr{K}^*$, $1\leqslant i\leqslant n$, $r_1<r_2<\cdots<r_n$.

Lemma 2.6 *For each $s\in\Phi_1^*$, there exists $\tilde{s}\in\Phi_1^*$ such that $s'=\tilde{s}'$ and $s\neq\tilde{s}$. Let $r=\mathrm{Min}\{s,\tilde{s}\}$, and $r^*=\mathrm{Max}\{s,\tilde{s}\}$, then*

1) *If $r_1\in\Phi_1^*$ and $r'=r'_1$, then either $r_1=r$ or $r_1=r^*$;*

2) *If $r_1, r_2\in\{J_r, J_{r^*}\}$, then $r_1+r_2\notin\Phi$;*

3) *There is no $r_1\in\Phi_1^*$ such that $r<r_1<r^*$. Moreover, $r<r^*\leqslant\bar{r}^*<\bar{r}$.*

Proof ① If $s\in\Phi_1^*$ and $s=\bar{s}$, then $s=e_i+e_{i+1}$, $i\in\mathscr{D}$. Let $\tilde{s}=2e_{i+1}$, then $\tilde{s}'=s'$ and $\tilde{s}\neq s$. ② If $s\in\Phi_1^*$, $s\neq\bar{s}$ and s is long, then $s=2e_{i+1}$, $i\in\mathscr{D}$. Let $\tilde{s}=e_i+e_{i+1}$, then $\tilde{s}'=s'$ and $\tilde{s}\neq s$. ③ If $s\in\Phi_1^*$, $s\neq\bar{s}$ and s is short, then $s=e_{i+1}-e_j$ or $s=e_{i+1}+e_{j+1}$, $i, j\in\mathscr{D}$, $i<j$. Let $\tilde{s}=e_{i+1}-e_{j+1}$ or $\tilde{s}=e_{i+1}+e_j$ respectively, then $\tilde{s}'=s'$ and $\tilde{s}\neq s$. It is easy to verify that the relations 1), 2) and 3) hold.

We arrange the roots of Φ^* and Φ_1^* in increasing order relative to $<$, then by 2.6 we have

$$\Phi^*:\ \{r_1, r_1^*, \bar{r}_1^*, \bar{r}_1, r_2, r_2^*, \bar{r}_2^*, \bar{r}_2, \cdots, r_q, r_q^*, \bar{r}_q^*, \bar{r}_q\}\text{ and}$$
$$\Phi_1^*:\ \{r_1, r_1^*, r_2, r_2^*, \cdots, r_q, r_q^*\}.$$

Lemma 2.7 *Every element u of U^1 can be expressed in the following form uniquely*

$$u=X^1_{r_1}(t_1)X^{1*}_{r_1^*}(t_1^*)X^1_{r_2}(t_2)X^{1*}_{r_2^*}(t_2^*)\cdots X^1_{r_q}(t_q)X^{1*}_{r_q^*}(t_q^*),\ t_i,\ t_i^*\in\mathscr{K}.$$

Proof It follows from 5.3.3 of [7] that u can be expressed in the following form uniquely

$$u=x_{r_1}(t_1)x_{r_1^*}(t_1^*)x_{\bar{r}_1^*}(\tilde{t}_1)x_{\bar{r}_1}(t'_1)\cdots x_{r_q}(t_q)x_{r_q^*}(t_q^*)x_{\bar{r}_q^*}(\tilde{t}_q)x_{\bar{r}_q}(t'_q),$$

where $t_i, t_i^*, \tilde{t}_i, t'_i\in\mathscr{K}$ and if $r_i^*=\bar{r}_i^*$, then $\tilde{t}_i=0$, $1\leqslant i\leqslant q$.

From $\sigma u=u$, we have $t'_i=\bar{t}_i$ and if $r_i^*=\bar{r}_i^*$, then $\bar{t}_i^*=-t_i^*$, $\tilde{t}_i=0$; if $r_i^*\neq\bar{r}_i^*$, then $\tilde{t}_i=\bar{t}_i^*$ for $1\leqslant i\leqslant q$. Therefore, for each $1\leqslant i\leqslant q$, $x_{r_i}(t_i)x_{r_i^*}(t_i^*)x_{\bar{r}_i^*}(\tilde{t}_i)x_{\bar{r}_i}(t'_i)$ can be expressed in the form $X^1_{r_i}(t_i)X^{1*}_{r_i^*}(t_i^*)$ uniquely. This proves the lemma.

By 2.7, we obtain the following theorem immediately:

Theorem 2.8 *Every element u of U^1 can be expressed in the following form uniquely*

$$u=X^1_{s_1}(t_1)X^1_{s_2}(t_2)\cdots X^1_{s_p}(t_p)\ \text{where } s_i\in\Phi_1^*,\ t_i\in\mathscr{K}^*,\ 1\leqslant i\leqslant p,\ s_1<s_2<\cdots<s_p.$$

In the following, let $J_1(u)=s_1$ and $J(u)=\{s_1, s_2, \cdots, s_p\}$, for the expression of the element u of U^1 mentioned above.

Definition 2.9 V^1: $\{v\in V \mid \sigma v=v\}$.

If $r\in\Phi_1^*$, $r=\bar{r}$, then let $X^1_{-r}(t)=x_{-r}(t)$, where $t\in\mathscr{K}$ and $\bar{t}=-t$; if $r\in\Phi_1^*$, $r\neq\bar{r}$, then let $X^1_{-r}(t)=x_{-r}(t)x_{-\bar{r}}(\bar{t})$, $t\in\mathscr{K}$. Similarly, we have

Theorem 2.10 *Every element v of V^1 can be expressed in the following form uniquely*

$v=X^1_{-s_1}(t_1)X^1_{-s_2}(t_2)\cdots X^1_{-s_n}(t_n)$, where $s_i\in\Phi_1^*$, $t_i\in\mathscr{K}^*$, $1\leqslant i\leqslant n$, and $s_1<s_2<\cdots<s_n$.

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If $r\in\Phi_1^*$ and $r=\bar{r}$, then let $N_r^1(t)=n_r(t)$, where $t\in\mathscr{K}^*$ and $\bar{t}=-t$, and let $N_r^1=N_r^1(\sqrt{-1})$; if $r\in\Phi_1^*$ and $r\neq\bar{r}$, then let $N_r^1(t)=n_r(t)n_{\bar{r}}(\bar{t})$, $t\in\mathscr{K}^*$ and let $N_r^1=N_r^1(1)$.

Definition 3.1 N^1: $\langle N_r^1(t), r\in\Phi_1^*, t\in\mathscr{K}^*\rangle$, $H^1=N^1\cap H$.

Lemma 3.2 1) *If $n\in N^1$, $h\in H^1$, then $\sigma n=n$, $\sigma h=h$.* 2) *For each $\alpha\in\Pi^\circ$, there exists $N_\alpha^1\in N^1$ such that N_α^1 corresponds to $\overline{W}_\alpha^1=w_\alpha$ under the natural homomorphism.*

Proof 1) The assertion of 1 is obvious. 2) Let $\alpha=e_i-e_{i+1}$, $i\in\mathscr{D}$, then $N_\alpha^1=N_r^1N_{r^*}^1\in N^1$, where $r=2e_{i+1}$, and $r^*=e_i+e_{i+1}$. Obviously, N_α^1 corresponds to $\overline{W}_\alpha^1=w_\alpha$ under the natural homomorphism.

Corollary 3.3 1) *For each $r\in\Phi_1^*$, then $h_r^1(t)=N_r^1(t)(N_r^1)^{-1}\in H^1$.* 2) $H^1\subset N^1\subset\langle U^1, V^1\rangle$.

Proof 1) is obvious. 2) For each $r\in\Phi_1^*$, $r\pm\bar{r}\notin\Phi$, hence

$$N_r^1(t)=X_r^1(t)X^1_{-r}(-t^{-1})X_r^1(t). \text{ Thus } H^1\subset N^1\subset\langle U^1, V^1\rangle.$$

Theorem 3.4 *N^1 corresponds to W^1 under the natural homomorphism from N onto W and $N^1/H^1\cong W^1$.*

Proof Let w be an element of W^1, then it follows from 1.5 that W can be expressed in the following form

$$w=\overline{W}_{r_1}^1\overline{W}_{r_2}^1\cdots\overline{W}_{r_n}^1, \text{ where } r_i\in\Pi,\ 1\leqslant i\leqslant n.$$

Let $n=N_{r_1}^1N_{r_2}^1\cdots N_{r_n}^1$, then n corresponds to w under the natural homomorphism from N onto W.

Conversely, if $n\in N^1$, then $n=N_{s_1}^1(t_1)N_{s_2}^1(t_2)\cdots N_{s_p}^1(t_p)$, where $s_i\in\Phi_1^*$, $t_i\in\mathscr{K}^*$, $1\leqslant i\leqslant p$. Obviously, n corresponds to $w=\overline{W}_{s_1}^1\overline{W}_{s_2}^1\cdots\overline{W}_{s_p}^1$ under the natural homomorphism from N onto W. It follows from 1.4 that $w\in W^1$. This proves that N^1 corresponds to W^1 under the natural homomorphism from N onto W.

Obviously, the kernel of the natural homomorphism from N onto W is H, hence the kernel of the natural homomorphism from N^1 onto W^1 is $H^1=H^1\cap N^1$.

Thus, $N^1/H^1 \cong W^1$.

Corollary 3.5 *Let $N_1^1 = \langle N_r^1, r \in \Phi_1^+ \rangle$, then the image of N^1 under the natural homomorphism is W^1.*

By 3.5, for each $w \in W^1$, we can choose an element of N_1^1 denoted by n_w, such that n_w corresponds to w under the natural homomorphism. The elements n_w for all $w \in W^1$ form a set N_0^1.

Corollary 3.6 *If $n_1, n_2 \in N^1$, and the images of n_1 and n_2 under the natural homomorphism are w, then $n_1 = h(\chi) n_2$, where $h(\chi) \in H^1$.*

Corollary 3.7 *If $u \in U^1$, $n \in N^1$, and $h(\chi) \in H^1$, then $h(\chi) u\, h(\chi)^{-1} \in U'$, and $n\, h(\chi) n^{-1} \in H^1$.*

Corollary 3.8 *If $\alpha = e_i - e_{i+1} \in \Pi^\circ$, $i \in \mathscr{D}$, and $r = 2e_{i+1}$, then $n_\alpha \in N^1$; and if $h(\chi_1) = h_r(\mu) h_{\bar{r}}(\mu')$, $\mu, \mu' \in \mathscr{K}^*$ and if $\sigma h(\chi) = h(\chi)$, then $h(\chi) \in H^1$.*

Proof Obviously, $h_\alpha(\sqrt{-1}) = h_r^1(-\sqrt{-1}) = h_r(-\sqrt{-1})\, h_{\bar{r}}(\sqrt{-1}) \in H^1$, hence, $n_\alpha = h_{\bar{\alpha}}(-\sqrt{-1}) N_\alpha^1 \in N^1$. Let $s = e_{i+1} - e_j$, $j \in \mathscr{D}$, $j \neq i$, and $e_s \in C_{b}$, then $h(\chi) e_s = \mu e_s$ and $\sigma h(\chi) e_s = \bar{\mu}' e_s$. It follows from $\sigma h(\chi) = h(\chi)$ that $\bar{\mu}' = \mu$, thus, $h(\chi) = h_r^1(\mu) = h_r(\mu)\, h_{\bar{r}}(\bar{\mu}) \in H^1$.

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Let $\alpha = e_i - e_{i+1} \in \Pi^\circ$, $i \in \mathscr{D}$ and $r = 2e_{i+1}$, $\bar{r} = 2e_i$ and let K_α: $\langle x_\alpha(t), x_{-\alpha}(t'), t, t' \in \mathscr{K} \rangle$, moreover, let $h_\alpha^1(\mu) = h_{\bar{r}}(\mu)$, $h_{\bar{\alpha}}^1(\mu') = h_r(\mu')$, where $\mu, \mu' \subset \mathscr{K}^*$ and let H_α: $\{h_{\bar{\alpha}}^1(\mu) h_{\bar{\alpha}}^1(\mu'), \mu, \mu' \in \mathscr{K}^*\}$. We denote by G_α the set $H_\alpha K_\alpha$. Obviously, G_α is a subgroup of G. Clearly, an element g of G belongs to G_α if and only if g can be expressed in the form: $g = h(\chi) x_\alpha(a) n_\alpha x_\alpha(b)$, where $a, b \in \mathscr{K}$ and $h(\chi) \in H_\alpha$.

Definition 4.1 *For each $\alpha = e_i - e_{i+1} \subset \Pi^\circ$, $i \in \mathscr{D}$, if $b \in \mathscr{K}$ and $b = 0$, then let $y_\alpha(b) = N_\alpha^1$; if $b \in \mathscr{K}^*$, then let $y_\alpha(b) = h_{\bar{\alpha}}^1(a^{-1}\bar{b}) x_\alpha(a) n_\alpha x_\alpha(b)$ where $a \in \mathscr{K}^*$ and satisfies $a^{-1} = b + \bar{b}^{-1}$.*

Obviously, if $a, b \in \mathscr{K}^*$ and $a^{-1} = b + \bar{b}^{-1}$, then the following relation $\mathscr{P}$ holds:

$\mathscr{P}$, 1) $\bar{a} = (1 - ab) b$; $a = (1 - \bar{a}\bar{b})\bar{b}$;

2) a and $a^{-1}b$ are determined by b uniquely;

3) $ab = (b + \bar{b}^{-1}) b = (1 + \bar{b}^{-1} b^{-1})^{-1} = (\bar{b} + b^{-1})\bar{b} = \bar{a}\bar{b}$;

4) $a\bar{b}^{-1} = (1 - \bar{a}\bar{b})\bar{b}\bar{b}^{-1} = (1 - ab) b b^{-1} = \bar{a} b^{-1}$; $a^{-1}\bar{b} = \bar{a} b^{-1}$;

Proof The assertions of 2, 3, 4 are obvious. We shall verify 1). Obviously, $\bar{b}b^2 + b + b + \bar{b}^{-1} - \bar{b}b^2 - b = b + \bar{b}^{-1}$. Hence $(\bar{b}b + 1)(b + \bar{b}^{-1}) - (\bar{b}b + 1) b = b + \bar{b}^{-1}$. Therefore, it follows from $b + \bar{b}^{-1} \neq 0$ that $b\bar{b} + 1 \neq 0$ and we have $1 - ab = (\bar{b}b + 1)^{-1} = \bar{a} b^{-1}$. Hence $(1 - ab) b = \bar{a}$ and $(1 - \bar{a}\bar{b})\bar{b} = a$.

Lemma 4.2 *For each $\alpha \in \Pi^\circ$ and $b \in \mathscr{K}$, $\sigma y_\alpha(b) = y_\alpha(b)$ and $y_\alpha(b) \in G_\alpha$.*

Proof If $b = 0$, then $\sigma y_\alpha(b) = \sigma N_\alpha^1 = N_\alpha^1 = y_\alpha(b)$. If $b \neq 0$ and $a^{-1} = b + \bar{b}^{-1}$, then

$c=\bar{b}(1-\bar{a}\bar{b})=a\neq 0$ by 1) of $\mathscr{P}$. Hence we have

$$\begin{aligned}\sigma y_\alpha(b)&=h_{\bar{\alpha}}^1(\bar{a}^{-1}b)x_{-\alpha}(-\bar{a})n_\alpha x_{-\alpha}(-\bar{b})\\&=h_{\bar{\alpha}}^1(\bar{a}^{-1}b)x_{-\alpha}(-\bar{a})n_\alpha x_\alpha(-\bar{b}^{-1})h_\alpha(\bar{b}^{-1})n_\alpha x_\alpha(-\bar{b}^{-1})\\&=h_{\bar{\alpha}}^1(\bar{a}^{-1}b)h_\alpha(-\bar{b})x_{-\alpha}(c)x_\alpha(-\bar{b}^{-1})\\&=h_{\bar{\alpha}}^1(\bar{a}^{-1}b)h_\alpha(-\bar{b})x_\alpha(a^{-1})h_\alpha(-a^{-1})n_\alpha x_\alpha(a^{-1})x_\alpha(-\bar{b}^{-1})\\&=h_{\bar{\alpha}}^1(\bar{a}^{-1}b)h_\alpha(a^{-1}\bar{b})x_\alpha(a)n_\alpha x_\alpha(a^{-1}-\bar{b}^{-1})\\&=h_{\bar{\alpha}}^1(\bar{a}^{-1}b)h_\alpha(a^{-1}\bar{b})x_\alpha(a)n_\alpha x_\alpha(b).\end{aligned}$$

From $\mathscr{P}$, we have $a^{-1}\bar{b}=\bar{a}^{-1}b$. Hence, $h_{\bar{\alpha}}^1(\bar{a}^{-1}b)h_\alpha(a^{-1}\bar{b})=h_{\bar{\alpha}}^1(\bar{a}^{-1}b)h_\alpha(\bar{a}^{-1}b)=h_\alpha^1(\bar{a}^{-1}b)$ $=h_\alpha^1(a^{-1}\bar{b})$. Therefore, $\sigma y_\alpha(b)=y_\alpha(b)$. From 3.8, we have $h_\alpha(\sqrt{-1})\in H_\alpha$. Hence $N_\alpha^1=h_\alpha(\sqrt{-1})n_\alpha\in G_\alpha$. If $b\neq 0$, then $y_\alpha(b)\in G_\alpha$ obviously.

Corollary 4.3 *If $g\in G_\alpha$, then g can be expressed in the form $g=h(\chi)x_\alpha(a)n_\alpha x_\alpha(b)$, where $h(\chi)\in H_\alpha$ and a, $b\in\mathscr{K}$. Moreover, if $\sigma g=g$, then $g=h_1(\chi)y_\alpha(b)$, where $h_1(\chi)\in H^1$.*

proof Let $\sigma h(x)=h'(x)$ and $c=\bar{b}(1-\bar{a}\bar{b})$. If $b\neq 0$ and $c\neq 0$, then by the argument used in the proof of 4.2, we have

$\sigma g=h'(x)h_\alpha(c^{-1}b)x_\alpha(c)n_\alpha x_\alpha(c^{-1}-\bar{b}^{-1})$.

It follows from $\sigma g=g$ and 8.4.4 of [7] that $c=a$ and $c^{-1}=a^{-1}=b+\bar{b}^{-1}$. Therefore, g can be expressed in the form $g=h_1(x)y_\alpha(b)$, where $h_1(x)\in H_\alpha$ and $\sigma h_1(x)=h_1(x)$. From 3.8, we have $h_1(x)\in H^1$. We shall now prove that if $b\neq 0$, then $c\neq 0$. Suppose the contrary, then $\sigma g=h'(x)h_\alpha(-\bar{b})x_\alpha(-\bar{b})\neq g$. We have a contrdication. If $b=0$ and $a\neq 0$, then $\sigma g=h'(x)n_\alpha x_\alpha(\bar{a})\neq g$, this contradicts the fact that $\sigma g=g$. Therefore, if $b=0$, then $g=h_1(x)N_\alpha^1=h_1(x)y_\alpha(b)$, where $h_1(x)\in H^1$ and $b=0$.

Lemma 4.4 1a) *Let α, $\beta\in\Pi^\circ$ and $\alpha\neq\beta$, then $y_\alpha(b)y_\beta(b')=y_\beta(b')y_\alpha(b)$;*

1b) *$y_\alpha(b_1)y_\alpha(b_2)=h_1(\chi)y_\alpha(b)$, where b_1, b_2, $b\in\mathscr{K}$ and $h_1(x)\in H^1$;*

2) *$y_\alpha(b)^{-1}=h_1(\chi)y_\alpha(b')$, where b, $b'\in\mathscr{K}$ and $h_1(\chi)\in H'$;*

3) *Let $n\in N^1$, then $ny_\alpha(b)n^{-1}=h_1(\chi)y_\beta(b')$, where b, $b'\in\mathscr{K}$, $\beta\in\Pi^\circ$ and $h_1(\chi)\in H^1$.*

proof 1a) The assertion of 1a) is obvious. 1b) Let $g=y_\alpha(b_1)y_\alpha(b_2)$, then $g\in G_\alpha$ and $\sigma g=g$. Hence it follows from4.3 that the assertion of 1b) is established. 2) Let $g=y_\alpha(b)^{-1}$, then $g\in G_\alpha$ and $\sigma g=g$. Hence the assertion of 2) is established by 4.3. 3) Let w be the image of n under the natural homomorphism and $w\alpha=\pm\beta$, where $\beta\in\Pi^\circ$, then $nG_\alpha n^{-1}=G_\beta$. Let $g=n\,y_\alpha(b)n^{-1}$, then $g\in G_\beta$ and $\sigma g=g$. It follows from 4.3 that the assertion of 3) is established.

Definition 4.5 Y^1: $\{I,\ y_{\beta_1}(b_1)y_{\beta_2}(b_2)\cdots y_{\beta_k}(b_k),\ \beta_i\in\Pi^\circ,\ i\neq j,\ \beta_i\neq\beta_j,\ b_i\in\mathscr{K},\ 1\leqslant i,\ j\leqslant k\}$. $B^1=U^1H^1Y^1$.

Lemma 4.6 1) *Let $h(\chi)\in H^1$, then $h(\chi)y_\alpha(b)h(\chi)^{-1}=h_1(\chi)y_\alpha(b')$, where b, $b'\in\mathscr{K}$ and $h_1(\chi)\in H^1$. 2) Let $r\in\Phi_1^*$ and $\alpha\in\Pi^\circ$, then $y_\alpha(b)X_r^1(t)=u\,y_\alpha(b)$, where t, $t'\in\mathscr{K}$ and $u\in U^1$.*

Proof 1) obviously, $g=h(\chi)y_a(b)h(\chi)^{-1}=h(\chi)h'(\chi)g'$, where $g'\in G_a$, and $h'(\chi)=n_a h(\chi)^{-1}n_a^{-1}\in H^1$. Clearly, $\sigma h'(\chi)=h'(\chi)$ and $\sigma g=g$, hence $\sigma g'=g'$. From 4.3 we have $g'=h_1'(\chi)y_a(b')$, where $h_1'(\chi)\in H^1$. Therefore, $h(\chi)y_a(b)h(\chi)^{-1}=h_1(\chi x)y_a(b)$, where $h_1(\chi)=h(\chi)h'(\chi)h_1'(\chi)\in H^1$. 2) Let $r\in\Phi^*$ and $a\in\Pi^\circ$, it follows from 5.2.2 of [7] and 1.2 that $x_a(c)x_r(t)=u'x_a(c)$, where c, $t\in\mathscr{K}$ and $u'\in U^*$: $\langle x_r(t), r\in\Phi^*, t\in\mathscr{K}\rangle$. If $r\in\Phi^*$ and $t\in\mathscr{K}$, then $n_a x_r(t)n_a^{-1}=x_s(\eta_{a,r}t)$, where $s=W_a r\in\Phi^*$, hence $g=y_a(b)X_r^1(t)=uy_a(b)$, where $u\in U^*\subset U$. Obviously, $\sigma g=g$ and $\sigma y_a(b)=y_a(b)$, thus $\sigma u=u$. Therefore, $u\in U^1$.

Theorem 4.7 *B^1 is a subgroup of G.*

Proof 1) Let $b=uhy\in B^1$, where $u\in U^1$, $h\in H^1$ and $y\in Y^1$, then $b^{-1}=y^{-1}h^{-1}u^{-1}$. It follows from 4.4, 4.7 and 3.7 that $b^{-1}=h_1y'h^{-1}u^{-1}=h_1h'y''u^{-1}=h''u'y''=u''h''y''$, where u', $u''\in U^1$, h_1, h', $h''=h_1h'\in H^1$ and y', $y''\in Y^1$. Thus, $b^{-1}\in B^1$. 2) Let $b_1=u_1h_1y_1$ and $b_2=u_2h_2y_2$, where u_1, $u_2\in U^1$, h_1, $h_2\in H^1$ and y_1, $y_2\in Y^1$, then

$$b_1b_2=u_1h_1y_1u_2h_2y_2=u_1h_1u_2'y_1h_2y_2=u_1h_1u_2'h_2'y_1'y_2=u_1u_2''h_1h_2'y=uhy,$$

where u_2', u_2'', $u=u_1u_2''\in U^1$, $h=h_1h_2'\in H^1$ and y_1', $y=y_1'y_2\in Y^1$. Hence $b_1b_2=uhy=b\in B^1$. It follows from 1) and 2) that B^1 is a subgroup of G.

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For each $w\in W^1$, let $J_{\bar{w}}$: $\{s\in\Phi_1^*\mid ws\in\Phi^-\}$and $\Phi_{\bar{w}}$: $\{r\in\Phi^*\mid wr\in\Phi^-\}$. It is easy to verify the following lemma

Lemma 5.1 *For each $r\in\Pi^*$, then* $J_{\bar{w}_r}$: $\{r, r^*\}$ and $\Phi_{\bar{w}_r}$: $\{r, r^*, \bar{r}^*, \bar{r}\}$.

If $r\in\Pi^*$, then either $r=2e_l$ or $r=e_{i+1}-e_j$, where $i\in\mathscr{D}$, $j=i+2$, $i<l-1$.

1) Let $r=2e_l$, then $r^*=e_{l-1}+e_l$, $\bar{r}^*=r$, and $\bar{r}=2e_{l-1}$. Clearly, the structure constants $N_{\alpha,r}=\eta_1$ and $N_{\alpha,r^*}=2\eta_2$, where $\eta_1=\pm1$ and $\eta_2=\pm1$. From 4.1.2 of [7], we have

$$\frac{1}{2}N_{-r^*,\alpha}=N_{\alpha,r}=N_{r,-r^*}=\eta_1;\ N_{-r^*,\bar{r}}=N_{\bar{r},-\alpha}=\frac{1}{2}N_{-\alpha,-r^*}=-\eta_2.$$

Obviously, $n_0e_r=\eta_1\eta_2e_{\bar{r}}=e_{\bar{r}}$, hence $\eta_1\eta_2=1$.

2) Let $r=e_{i+1}-e_j$, $i\in\mathscr{D}$, $i<l-1$, $j=i+2$, then $r^*=e_{i+1}-e_{j+1}$, $\bar{r}^*=e_i-e_j$ and $\bar{r}=e_i-e_{j+1}$. Clearly, let $\alpha=e_i-e_{i+1}$, $\beta=e_j-e_{j+1}$, then the structure constants $N_{\alpha,r}=\xi_1$, $N_{\beta,r}=\xi_2$ and $N_{\beta,\bar{r}^*}=\xi_3$, where $\xi_i=\pm1$, $i=1, 2, 3$. From 4.1.2 of [7], we have

$$N_{\alpha,-\bar{r}^*}=N_{-\bar{r}^*,r}=N_{r,\alpha}=-\xi_1;\quad N_{r,-r^*}=N_{-r^*,\beta}=N_{\beta,r}=\xi_2;$$

$$N_{\bar{r},-\bar{r}^*}=N_{-\bar{r}^*,-\beta}=N_{-\beta,\bar{r}}=\xi_2;\quad N_{-\alpha,-r^*}=N_{-r^*,\bar{r}}=N_{\bar{r},-\alpha}=\xi=-\xi_1\xi_2/\xi_3.$$

Obviously, $n_0e_{r^*}=\xi\xi_3e_{\bar{r}^*}$ and $n_0e_r=\xi_1\xi_3e_{\bar{r}}$ Hence $\xi\xi_3=1$ and $\xi_1\xi_3=1$. Therefore, $\xi\xi_1=1$.

Lemma 5.2 *Let $r\in\Pi^*$ and t', $u'\in\mathscr{K}$, then* $x=N_r^1X_r^1(t')X_{r^*}^1(u')(N_r^1)^{-1}\in B^1N_r^1B^1$ *or* $x=1$.

Proof 1) Let $r=2e_l$, then x can be expressed in the form: $x=X_{-r}^1(t)X_{-r^*}^1(u)=$

$x_{-r}(t)x_{-\bar r}(\bar t)x_{-r^*}(u)$, t, $u\in\mathscr{K}$, $\bar u=-u$. First, we discuss the case of $t\neq 0$. Since $\mathscr{K}_0$ is an ordering field, hence $d=t-\eta_1\eta_2u^2\bar t^{-1}=t+u\bar u\bar t^{-1}\neq 0$. Let $d^{-1}=u\bar t^{-1}d$, it is easy to verify the following formula: $t^{-1}+\eta_1\eta_2u^2\bar t^{-2}d^{-1}=\bar d^{-1}$; $\bar t^{-1}d^{-1}=t^{-1}\bar d^{-1}$ and $\bar d'=-d'$. Using the formula mentioned above. We have

$$\begin{aligned}
x&=x_{-r^*}(u)x_{\bar r}(\bar t^{-1})h_{\bar r}(-\bar t^{-1})n_{\bar r}x_{\bar r}(\bar t^{-1})x_{-r}(t)\\
&=x_{\bar r}(\bar t^{-1})x_{-r^*}(u)x_\alpha(-\eta_2u\bar t^{-1})x_{-r}(\eta_1\eta_2u^2\bar t^{-1})x_{-r}(t)h_{\bar r}(-\bar t^{-1})n_{\bar r}x_{\bar r}(\bar t^{-1})\\
&=x_{\bar r}(\bar t^{-1})x_\alpha(-\eta_2u\bar t^{-1})x_{-r^*}(u)x_{-r}(t-\eta_1\eta_2u^2\bar t^{-1})h_{\bar r}(-\bar t^{-1})n_{\bar r}x_{\bar r}(\bar t^{-1})\\
&=x_{\bar r}(\bar t^{-1})x_\alpha(-\eta_2u\bar t^{-1})x_r(d^{-1})h_r(-d^{-1})h_{\bar r}(-\bar t^{-1})n_rn_{\bar r}x_r(d^{-1})x_\alpha(\eta_2u\bar t^{-1})x_{\bar r}(\bar t^{-1})\\
&=x_{\bar r}(\bar t^{-1})x_r(d^{-1})x_\alpha(-\eta_2u\bar t^{-1})x_{\bar r^*}(-\eta_1\eta_2d')x_{\bar r}(-\eta_1\eta_2u\bar t^{-1}d')h_r(-d^{-1})h_{\bar r}(-\bar t^{-1})\\
&\qquad N_r^1x_r(d^{-1})x_{\bar r}(\bar t^{-1})x_\alpha(\eta_2\bar t^{-1}u)\\
&=x_{\bar r}(\bar t^{-1})x_r(d^{-1})x_{r^*}(-d')x_\alpha(-\eta_2u\bar t^{-1})x_{\bar r}(\eta_1\eta_2u\bar t^{-1}d')h_r(-d^{-1})h_{\bar r}(-\bar t^{-1})\\
&\qquad N_r^1x_r(d^{-1})x_{\bar r}(\bar t^{-1})x_\alpha(\eta_2u\bar t^{-1})\\
&=X_r^1(d^{-1})X_{r^*}^1(-d')N_r^1x_{-\alpha}(\eta,\ u\bar t^{-1})x_{\bar r}(\bar t)x_r(d)h_r(-d)h_{\bar r}(-\bar t)x_\alpha(\eta_2u\bar t^{-1})\\
&=X_r^1(d^{-1})X_{r^*}^1(-d')N_r^1x_{\bar r}(\bar t)x_{-\alpha}(\eta_1u\bar t^{-1})x_{r^*}(\eta_1\eta_2u)x_r(-\eta_1\eta_2u^2\bar t^{-1})x_r(d)h_r(-d)\\
&\qquad h_{\bar r}(-\bar t)x_\alpha(\eta_2u\,\bar t^{-1})\\
&=X_r^1(d^{-1})X_{r^*}^1(-d')N_r^1x_{\bar r}(\bar t)x_{r^*}(u)x_{-\alpha}(\eta_1u\bar t^{-1})x_r(\eta_1\eta_2u^2\bar t^{-1})\\
&\qquad x_r(d)h_r(-d)h_{\bar r}(-\bar t)x_\alpha(\eta_2u\bar t^{-1})\\
&=X_r^1(d^{-1})X_{r^*}^1(-d')N_r^1X_r^1(t)X_{r^*}^1(u)h_r(-d)h_{\bar r}(-\bar t)\\
&\qquad x_\alpha(\eta_1u^{-1}d)h_\alpha(-\eta_1u^{-1}d)n_\alpha x_\alpha(\eta_1u^{-1}d)x_\alpha(\eta_2u\bar t^{-1})\\
&=X_r^1(d^{-1})X_{r^*}^1(-d')N_r^1X_r^1(t)X_{r^*}^1(u)h_r(-d)h_{\bar r}(-\bar t)\\
&\qquad\quad h_\alpha(-\eta_1u^{-1}d)x_\alpha(\eta_1ud^{-1})n_\alpha x_\alpha(\eta_2u^{-1}t)\\
&=X_r^1(d^{-1})X_{r^*}^1(-d')N_r^1X_r^1(t)X_{r^*}^1(u)h_r'(-\eta_1u)y_\alpha(\eta_1u^{-1}t).
\end{aligned}$$

Thus, if $t\neq 0$, then $x\in B^1N_r^1B^1$. If $t=0$, $u\neq 0$, then we have $x=x_{r^*}(u^{-1})h_{r^*}(-u^{-1})n_{r^*}x_{r^*}(u^{-1})=X_{r^*}^1(u^{-1})N_r^1X_{r^*}^1(u')h(\chi)N_\alpha^1\in B^1N_r^1B^1$, where $u'\in\mathscr{K}$, $\bar u'=-u'$ and $h(\chi)\in H^1$. If $t=0$, $u=0$, then $x=1$.

2) Let $r=e_{i+1}-e_j$, $i\in\mathscr{D}$, $i<l-1$, $j=i+2$, then x can be expressed in the form $x=X_{-r}^1(t)X_{-r^*}^1(u)=x_{-\bar r}(\bar t)x_{-r}(t)x_{-r^*}(u)x_{-\bar r^*}(\bar u)$, t, $u\in\mathscr{K}$. We first discuss the case of $t\neq 0$. Since $\mathscr{K}_0$ is an ordering field $d=t+\xi\xi_1u\bar u\bar t^{-1}=t+u\bar u\bar t^{-1}\neq 0$. Let $d'=\bar u\bar t^{-1}d^{-1}$, we have

$$\begin{aligned}
x&=x_{-r^*}(u)x_{\bar r}(\bar t^{-1})h_{\bar r}(-\bar t^{-1})n_{\bar r}x_{\bar r}(\bar t^{-1})x_{-r}(t)x_{-\bar r^*}(\bar u)\\
&=x_{\bar r}(\bar t^{-1})x_{-r^*}(u)x_\alpha(\xi u\bar t^{-1})h_{\bar r}(-\bar t^{-1})n_{\bar r}x_{-r}(t)x_{\bar r}(\bar t^{-1})x_{-\bar r^*}(\bar u)\\
&=x_{\bar r}(\bar t^{-1})x_\alpha(\xi u\bar t^{-1})h_{\bar r}(-\bar t^{-1})n_{\bar r}x_{-r}(t)x_{\bar r}(\bar t^{-1})x_{-\bar r^*}(\bar u)x_\alpha(-\xi u\bar t^{-1})x_{-r}(\xi\xi_1\,u\bar u\bar t^{-1})\\
&=x_{\bar r}(\bar t^{-1})x_\alpha(\xi u\bar t^{-1})h_{\bar r}(-\bar t^{-1})x_r(d^{-1})h_{\bar r}(-d^{-1})n_rn_{\bar r}x_r(d^{-1})x_{-r^*}(\bar u)x_{\bar r}(\bar t^{-1})\\
&\qquad x_\beta(\xi_2\bar u\bar t^{-1})x_\alpha(-\xi u\bar t^{-1}).
\end{aligned}$$

Using the formula $\bar d^{-1}=\bar t^{-1}-\xi\xi_1\ u\bar u\bar t^{-2}d^{-2}$ and $\bar t^{-1}d^{-1}=t^{-1}\bar d^{-1}$, by a similar argument as that used in the preceding proof. We obtain

$$x=X_r^1(d^{-1})X_{r^*}^1(d')N_r^1X_r^1(t)X_{r^*}^1(\bar u)h(\chi)y_\alpha(\xi_1\bar u^{-1}t)y_\beta(\xi_2u^{-1}t)\in B^1N_r^1B^1,$$

where $h(\chi)=h_{s_1}^1(\xi u)h_{s_2}^1(\xi\bar d')\in H^1$, $s_1=2e_{i+1}$, $s_2=2e_{j+1}$. If $t=0$, $u\neq 0$, then

$x=X^1_{r*}(u^{-1})h^1_{r*}(-u^{-1})N^1_{r*}X^1_{r*}(u^{-1})\in B^1N^1_rB^1$. If $t=0$, $u=0$, then $x=1$.

Corollary 5.3 *If $b\in\mathscr{K}^*$, then $b+\bar{b}^{-1}\neq 0$, and $y_a(b)\in\langle U^1, V^1\rangle$.*

Lemma 5.4 *If $r\in\Pi$, then $B^1\cup B^1N^1_rB^1$ is a subgroup of G.*

Proof 1) Obviously, $(N^1_r)^{-1}=h_rN^1_r$, where $h_r\in H^1$. From 4.8, we have $(B^1)^{-1}=B^1$. Therefore, $(B^1N^1_rB^1)^{-1}=(B^1)^{-1}(N^1_r)^{-1}(B^1)^{-1}=B^1N^1_rB^1$.

2) We shall prove that $N^1_rB^1N^1_r\subseteq B^1\cup B^1N^1_rB^1$. a) If $r\in\Pi^*$, it follows from 5.1 that if $b\in B^1$, then $b=X^1_r(t')X^1_{r*}(u')u_rhy$, where $h\in H^1$, $y\in Y^1$, t', $u'\in\mathscr{K}$ and $u_r\in U^1$, satistying $N^1_ru_r(N^1_r)^{-1}\in U^1$. From 5.2, 4.7 and 3.7, we have $x=N^1_rbN^1_r=b_1N^1_rb'_2u'_rh'y'h_r^{-1}$, where b_1, $b'_2\in B^1$, $u'_r=N^1_ru_r(N^1_r)^{-1}\in U^1$, $h'=N^1_rh(N^1_r)^{-1}\in H^1$ and $y'=N^1_ry(N^1_r)^{-1}\in Y^1$. Hence $x=b_1N^1_rb_2$, where $b_2\in B^1$. Thus $x\in B^1N^1_rB^1$, b) If $r\in\Pi^\circ$, it follows from 3.3 and 4.7 that if $b=uhy\in B^1$, where $u\in U^1$, $h\in H^1$ and $y\in Y^1$, then $x=N^1_rbN^1_r=u'h'y'h_r^{-1}$, where $u'\in U^1$, $h'\in H^1$ and $y'\in Y^1$. Hence $x\in B^1$.

Lemma 5.5 *Let $r\in\Pi$ and $n\in N^1$, then $B^1nB^1B^1N^1_rB^1\subseteq B^1nN^1_rB^1\cup B^1nB^1$.*

Proof Let w be the image of n under the natural homomorphism, if $r\in\Phi^*$ and $wr\in\Phi^+$, then $wr^*\in\Psi^*$. 1) Let $r\in\Pi^\circ$, from 5.4 we have $N^1_rB^1N^1_r\subseteq B^1$ thus, $B^1nB^1B^1N^1_rB^1\subseteq B^1nN^1_rB^1$. 2) Let $r\in\Pi^*$, a) If $wr\in\Phi^{+1}$ and b_1, b_2, $b_3\in B^1$, it follows from 3.8 and 4.4 that $b_2=X^1_r(t)X^1_{r*}(u)u_rhy$, where $h\in H^1$, $y\in Y^1$, t, $u\in\mathscr{K}$ and $u_r\in U^1$ satisfying $N^1_ru_r(N^1_r)^{-1}\in U^1$. Obviously, if $x\in B^1_nB^1B^1N^1_rB^1$, then

$$x=b_1nX^1_r(t)X^1_{r*}(u)n^{-1}nN^1_r(N^1_r)^{-1}u_rN^1_r(N^1_r)^{-1}hN^1_r(N^1_r)^{-1}yN^1_rb_3$$
$$=b_1u'nN^1_ru'_rh'y'b_3=b'_1nN^1_rb'_3,$$

where u', $u'_r=(N^1_r)^{-1}u_rN^1_r\in U^1$, $h'=(N^1_r)^{-1}hN^1_r\in H^1$, $y'=(N^1_r)^{-1}y(N^1_r)^{-1}\in Y^1$ and b'_1, $b'_3\in B^1$. Thus $x\in B^1nN^1_rB^1$ by 4.8. Therefore, $B^1nB^1B^1N^1_rB^1\subseteq B^1nN^1_rB^1$.

b) If $wr\in\Phi^-$, let $n_1=nN^1_r$ and w_1 is the image of n_1 under the natural homomorphism, then $w_1r\in\Phi^+$ obviously. From a) we have $B^1n_1B^1B^1N^1_rB^1\subseteq B^1n_1N^1_rB^1$. On the other hand, we have $B^1n_1N^1_rB^1\subseteq B^1n_1B^1B^1N^1_rB^1$. Therefore, $B^1n_1B^1B^1N^1_rB^1=B^1n_1N^1_rB^1$. From 5.4, we have

$$B^1nB^1B^1N^1_rB^1=B^1n_1N^1_rB^1N^1_rB^1=B^1n_1B^1B^1N^1_rB^1N^1_rB^1\subseteq B^1n_1B^1(B^1\cup B^1N^1_rB^1)$$
$$=B^1n_1B^1\cup B^1n_1B^1B^1N^1_rB^1=B^1nN^1_rB^1\cup B^1nB^1.$$

Definition 5.6 $G^1=\bigcup\limits_{w\in W^1}B'n_wB^1$, $n_w\in N^1_0$; *G^1 is called ${}^2C_l(\mathscr{K})$.*

Theorem 5.7 *G^1 is a subgroup of G.*

Proof 1) Obviously, $(n_w)^{-1}=hn_w^{-1}$, where $h\in H^1$. Thus from 4.9, we have $(B^1n_wB^1)^{-1}=(B^1)^{-1}(n_w)^{-1}(B^1)^{-1}\subseteq B^1n_w{}^{-1}B^1\subseteq G^1$.

2) Let n_{w_1} and $n_{w_2}\in N^1_0$, where w_1, $w_2\in W^1$. From 3.4, we have $n_{w_2}=hN^1_{r_1}N^1_{r_2}\cdots N^1_{r_t}$, where $h\in H^1$ and $r_i\in\Pi$, $1\leqslant i\leqslant t$. Therefore, from 5.5, we have

$$B'n_{w_1}B^1B^1n_{w_2}B^1=B^1n_{w_1}B^1B^1N^1_{r_1}B^1B^1N^1_{r_2}B^1\cdots B^1N^1_{r_t}B^1$$
$$=(B^1n_{w_1}N^1_rB^1\cup B^1n_{w_1}B^1)B^1N^1_{r_2}B^1\cdots B^1N^1_{r_t}B^1$$
$$=B^1n_{w_1}N^1_{r_1}B^1B^1N^1_{r_2}B^1\cdots B^1N^1_{r_t}B^1\cup B^1n_{w_1}B^1B^1N^1_{r_2}B^1\cdots B^1N^1_{r_t}B^1.$$

By an argument along the lines of the preceding discussion and 3.6, we have

$$B^1 n_{w_1} B^1 B^1 n_{w_2} B^1 \subseteq \bigcup_{w\in W^1} B^1 n_w B^1 = G^1.$$

It follows from 1) and 2) mentioned above that G^1 is a subgroup of G.

Theorem 5.8 *Let $x=b_1 n_w b_2 \in G^1$, where b_1, $b_2\in B^1$ and $n_w\in N_0^1$, then x can be expressed uniquely in the following form*

(II): $x=u\,h\,h_1 h_2\cdots h_k x_{r_1}(a_1)x_{r_2}(a_2)\cdots x_{r_k}(a_k)n_w^* x_{\beta_1}(b_1')x_{\beta_2}(b_2')\cdots x_{\beta_k}(b_k)u'$, *where u, $u'\in U^1$, $h\in H^1$, $h_i\in H$, a_i, $b_i'\in \mathscr{K}^*$, $1\leqslant i\leqslant k$ and the following relations hold (γ_i, $\beta_i\in \Pi^\circ$, $1\leqslant i\leqslant k$).*

a) Let w^* be the image of n_{w^*} under the natural homomorphism, then $w^*=w_1^* w w_2^*$, where $w_i^*\in W_0^1$: $\langle w_\alpha,\ \alpha\in\Pi^\circ\rangle$, $i=1$, 2 and $u^1\in U_{w^*}^-$, $w^*\beta_i\in\Phi^-$, $1\leqslant i\leqslant k$; b) h_i and a_i are determined by b_i' and w^*, $1\leqslant i\leqslant k$.

proof Let $x=b_1 n_w b_2\in G^1$, where b_1, $b_2\in B^1$ and $n_w\in N_0^1$, then x can be expressed in the form $x=uhn_w u'y$, where $h\in H^1$, u, $u'\in U^1$, satisfying $u'\in U_w^-$ and $y=y_{\beta_1}(b_1)y_{\beta_2}(b_2)\cdots y_{\beta_k}(b_k)$, $b_i\in\mathscr{K}^*$, $1\leqslant i\leqslant k$. From 4.8, we have $x=uhn_w y u_1'$, where $u_1'\in U^1$. If $w\beta_1=\gamma_1\in\Pi^\circ$, then we have

$$x=uhh_{\gamma_1}^1(a_1^{-1}\bar{b}_1)x_{\gamma_1}(a_1)n_w n_{\beta_1}x_{\beta_1}(b_1)y_{\beta_2}(b_2)\cdots y_{\beta_k}(b_k)u_1'$$

If $w\beta_1=-\gamma_1$, $\gamma_i\in\Pi^\circ$, then we have

$$\begin{aligned}x&=uhh_{\gamma_1}'(a_1^{-1}\bar{b})x_{-\gamma_1}(a_1)n_w n_{\beta_1}x_{\beta_1}(b_1)y_{\beta_2}(b_2)\cdots y_{\beta_k}(b_k)u_1'\\&=uhh_{\gamma_1}'(a_1^{-1}\bar{b})x_{\gamma_1}(a_1^{-1})h_{\gamma_1}(-a_1^{-1})n_{\gamma_1}n_w n_{\beta_1}x_{\beta_1}(-a_1^{-1})x_{\beta_1}(b_1)y_{\beta_2}(b_2)\cdots y_{\beta_k}(b_k)u_1'\\&=uhh_{\gamma_1}'(a_1^{-1}\bar{b})h_{\gamma_1}(-a_1^{-1})x_{\gamma_1}(a_1)n_{\gamma_1}n_w n_{\beta_1}x_{\beta_1}(-\bar{b}^{-1})y_{\beta_2}(b_2)\cdots y_{\beta_k}(b_k)u_1'.\end{aligned}$$

Therefore, x can be expressed in the following form

$$x=uhh_1 x_{\gamma_1}(a_1)n_1' n_w n_1 x_{\beta_1}(b_1')y_{\beta_2}(b_2)\cdots y_{\beta_k}(b_k)u_1',$$

where $n_1=n_{\beta_1}$, if $w\beta_1=\gamma_1\in\Pi^\circ$, then $n_1'=I$, $b_1'=b_1$ and $h_1=h_{\gamma_1}'(a_1^{-1}\bar{b}_1)$; if $w\beta_1=-\gamma_1$, $\gamma_1\in\Pi^\circ$, then $n_1'=n_{r_1}$, $b_1'=-\bar{b}_1^{-1}$ and $h_1=h_{\gamma_1}'(a_1^{-1}\bar{b})h\gamma_1(-a_1^{-1})$. By an argument along the lines of the preceding discussion. It is easy to show that x can be expressed in formula (II) and $w^*=w_1^* w w_2^*$, where $w_i^*\in W_0^1$, $i=1$, 2. In the following, we show that $u_1'\in U_{w^*}^-$. First, we prove $u_1'\in U_w^-$.

Let $r\in J(u_1')$, then $r=\sum_{i=1}^n m_i s_i+\sum n_j\alpha_j$, where $s_i\in J(u')$, $\alpha_j\in\Pi^\circ$, $m_i\geqslant 0$, $n_j\geqslant 0$ and there is at least one coefficient $m_t\neq 0$, $1\leqslant t\leqslant n$. Obviously, $w\,s_i\in\Phi^-$, hence $(w\,s_i)'<0$. Thus $(wr)'=\sum_{i=1}^n m_i(ws_i)'<0$. Therefore, $wr\in\Phi^-$, hence $u_1'\in U_w^-$. Moreover, if $r\in J(u_1')$, then $w^*r=wr+\sum m_i'\beta_i+\sum m_i'' r_i$, where m_i', m_i'' are integers. Obviously, $(w^*r)'=(wr)'<0$, thus $u_1'\in U_{w^*}^-$.

Corollary 5.9 *Let $x=b_1 n_w b_2\in G^1$, where b_1, $b_2\in B^1$ and $n_w\in N_0^1$ and let $x=u_2 h n_{w_1} u_2'$ be the canonical expression of x mentioned in 8.4.4 of [7], then $w_1=w^*$, where w^* as defined in 5.8, belongs to W_0^1.*

6

Lemma 6.1 *Let r, $s\in\Phi_1^*$ and $r\neq s$, $r\neq\bar{s}$, then there exists $h(\chi)=h_r^1(\mu)$ satisfying $\chi(r)=\pm1$ and $\chi(r)\neq\chi(s)$, where $r=\min\{\delta,\bar{\delta}\}$, $\delta\in\Phi^*$, $\mu\in\mathscr{K}^*$.*

Proof Let $E_i=\{e_i,\bar{e}_i\}$, where $\bar{e}_i=w_0e_i$, we discuss separately the following three cases:

1) $r=e_i\pm e_j$, $s=e_k\pm e_h$, $1\leqslant i\leqslant j\leqslant l$, $1\leqslant k<h\leqslant l$.

If $r'=s'$, then we take $\delta=e_i\mp e_j$, $\mu=\sqrt{-1}$. If $r'\neq s'$, then there is either case a) or case b) as follows a) $r\neq\bar{r}$, and e_i(or e_j) $\notin E_k\cup E_h$, then take $\delta=2e_i$(or $\delta=2e_j$), $\mu=-1$; b) e_k(or e_h) $\notin E_i\cup E_j$, then take $\delta=2e_k$(or $\delta=2e_h$), $\mu=2$.

2) $r=e_i\pm e_j$, $s=2e_k$ or $s=e_i\pm e_j$, $r=2e_k$, $1\leqslant i<j\leqslant l$, $1\leqslant k\leqslant l$.

There is either case a) or case b) as follows a) $E_i\neq E_j$ and e_i(or e_j) $\notin E_k$, then take $\delta=2e_i$(or $\delta=2e_j$), $\mu=-1$ b) $e_k\notin E_i\cup E_j$ or $r'=s'$, then take $\delta=2e_k$, $\mu=\sqrt{-1}$.

3) $r=2e_k$, $s=2e_i$, $1\leqslant k$, $i\leqslant l$, then take $\delta=2e_i$, $\mu=2$.

Corollavy 6.2 *Let r, $s\in\Phi_1^*$,* and *$r\neq s$, $r\neq\bar{s}$, then there exists $h(\chi)\in H^1$ such that $\chi(r)\neq\chi(s)$.*

Lemma 6.3 *Let $w\in W^1\backslash W_0^1$, then there exists $h(\chi)\in H^1$ such that $nh(\chi)\neq h(\chi)n$ where $n\in N^1$ and n corresponds to w under the natural homomorphism.*

Proof For each $w\in W^1\backslash W_0^1$, let w' be the restriction of w onto Γ^+, then $w'\neq I'$. Therefore, there exists $r\in\Phi^*$ such that $\eta wr\in\Phi^*$, $\eta=\pm1$ and $(wr)'\neq r'$. If $(wr)'=-r'$, then take $h(\chi)=h_s^1(2)$, where $s=\min\{r,\bar{r}\}$. If $(wr)'\neq-r'$, it follows from 6.2 that there exisfs $h(\chi)\in H^1$ such that $\chi(\eta wr)=\pm1$ and $\chi(\eta wr)\neq\chi(r)$. Thus we have $\chi(wr)\neq\chi(w)$. Therefore, the lemma is established immediately.

Lemma 6.4 *Let $a=e_i-e_{i+1}$, $i\in\mathscr{D}$, $r^*=e_i+e_{i+1}$, $r=2e_{i+1}$, then $\eta_1=N_{a,r}=\pm1$; $2\eta_2=N_{a,r^*}=\pm2$ and $y_a(b)X_{r^*}^1(t)=X_r^1(2\eta_2bt)X_{r^*}^1(t(b\bar{b}-1))y_a(b)$, where b, $t\in\mathscr{K}^*$ and $\bar{t}=-t$.*

Proof Obviously, $\eta_1\eta_2=1$, $a^{-1}\bar{b}(2ab-1)=2b\bar{b}-(b+\bar{b}^{-1})\bar{b}=b\bar{b}-1$ and $2a^{-1}\bar{b}^2(ab-1)=2\bar{b}^2b-2(b+\bar{b}^{-1})\bar{b}^2=2\bar{b}^2b-2b\bar{b}^2-2\bar{b}=-2\bar{b}$. Using the formula mentioned above, we have

$$\begin{aligned}
y_a(b)X_{r^*}^1(t)&=h_a^1(a^{-1}\bar{b})x_a(a)n_ax_a(b)x_{r^*}(t)\\
&=h_a^1(a^{-1}\bar{b})x_a(a)n_ax_{r^*}(t)x_a(b)x_{\bar{r}}(-2\eta_2bt)\\
&=h_a^1(a^{-1}\bar{b})x_a(a)x_{r^*}(-t)x_{\bar{r}}(-2\eta_2bt)n_ax_a(b)\\
&=h_a^1(a^{-1}\bar{b})x_r(-2\eta_2bt)x_a(a)x_{r^*}(2\eta_1\eta_2abt)x_{r^*}(-t)x_{\bar{r}}(-2\eta_2a^2bt)n_ax_a(b)\\
&=x_r(-2\eta_2bt)x_{r^*}(ta^{-1}b(2ab-1))x_{\bar{r}}(2\eta_2ta^{-2}\bar{b}^2(ab-1)a)y_a(b)\\
&=X_r^1(-2\eta_2bt)X_{r^*}^1(t(b\bar{b}-1))y_a(b).
\end{aligned}$$

In the following, we assume that R^1 is a normal subgroup of G^1 which satisfies $|R^1|>1$, i.e., $R^1\neq\{I\}$.

Lemma 6.5 $|R^1\cap B^1|>1$.

Proof It follows from R^1 being a normal subgroup of G^1 and $|R^1|>1$ that there exists an element x of R^1 which can be expressed in the form: $x=n_wb\neq 1$, where $n_w\in N_0^1$ and $b\in B^1$. From 8.4.4 of [7], we know that x can be expressed in the canonical form $x=u_1h_1n_{w_1}u_1'$, where $u_1\in U$, $h_1\in H$, $n_{w_1}\in N$ and $u_1'\in U_{w_1}^-$. If $w_1\notin W_0^1$, it follows from 6.3 that there exists $h(\chi)\in H^1$ such that $n_{w_1}h(\chi)\neq h(\chi)n_{w_1}$. Let $h'(\chi)=n_w^{-1}h(\chi)n_w$, $h''(\chi)^{-1}=n_{w_1}h(\chi)^{-1}n_{w_1}^{-1}$ and let $x'=x^{-1}h(\chi)xh(\chi)^{-1}$, then $x'=b^{-1}h'(\chi)bh(\chi)^{-1}\in R^1\cap B^1$. In the following, we shall show that $x'\neq 1$. If $x'=1$, then $u_1h_1n_{w_1}u_1'=u_2h(\chi)h_1h''(\chi)^{-1}n_{w_1}u_2'$, where $u_2=h(\chi)u_1h(\chi)^{-1}\in U$, $u_2'=h(\chi)u_1'h(\chi)^{-1}\in U_{w_1}^-$. From 8.4.4 of [7], we have $h_1h(\chi)h''(\chi)^{-1}=h_1$, hence $h(\chi)=h''(\chi)=n_{w_1}h(\chi)n_{w_1}^{-1}$. Therefore, we have a contradiction. If $w_1\in W_0^1$, then $w\in W_0^1$ by 5.9. Therefore, $n_w=h_2(\chi)N_{\beta_1}^1N_{\beta_2}^1\cdots N_{\beta_k}^1$, where $\beta_i\in\Pi^\circ$, $1\leqslant i\leqslant k$ and $h_2(\chi)\in H^1$. Thus $x=y'b=b'\in B^1$, where $y'\in Y^1$. This proves the lemma.

Lemma 6.6 *There exists* $b=uhy\in R^1\cap B^1$, *where* $b\in B^1$, $h\in H^1$, $y\in Y^1$ *and* $u\in U^1$, $u\neq 1$.

Proof From 6.5, there exists $b=uhy\in R^1\cap B^1$, where $u\in U^1$, $h\in H^1$, $b\neq 1$ and $y\in Y^1$. If $u\neq 1$, the lemma is obviously true. In the following, we suppose $u=1$, then $hy\neq 1$. If $y=1$, then $h(\chi)\neq 1$. In fact, there exists $r\in\Phi_1^*$ such that $\chi(r)\neq 1$. Suppose the contrary, then there exists $\alpha\in\Pi^\circ$ such that $\chi(\alpha)\neq 1$. For each $\alpha\in\Pi^\circ$, there exists $s\in\Phi^*$ such that $s+\alpha=r\in\Phi_1^*$. Obviously, $\chi(r)=\chi(s)\chi(\alpha)=\chi(\alpha)\neq 1$. Thus we have a contradiction. It follows from $\chi(r)\neq 1$, $r\in\Phi_1^*$ that $X_r^1(t)h(\chi)X_r^1(-t)=u'h(\chi)\in R^1$, $t\neq 0$, where $u'=X_r^1(t(1-\chi(r)))\neq 1$. If $y\neq 1$, then y can be expressed in the form $y=y_{\beta_1}(b_1)y_{\beta_2}(y_2)\cdots y_{\beta_k}(b_k)$, where $\beta_i\in\Pi^\circ$, $b_i\in\mathscr{K}$, $1\leqslant i\leqslant k$. If $b_i=0$ for all $1\leqslant i\leqslant k$ and $\beta_1=e_i-e_{i+1}$, $i\in\mathscr{D}$, then $b'=h_r^1(\sqrt{-1})\,bh_r^1(\sqrt{-1})^{-1}b^{-1}=h_r^1(-1)\in R^1$, where $r=2e_{i+1}\in\Phi_1^*$. From the preceding discussion we know that there is an element $u'b'=u'h_r^1(-1)$ of R^1, where $u'\in U^1$ and $u'\neq 1$. If $b_i\neq 0$, $1\leqslant i\leqslant k$, then there exists an element $b'=u'hy\in R^1$, where $u'\in U^1$ and $u'\neq 1$, by 6.4.

Lemma 6.7 $|R^1\cap U^1|>1$.

Proof From 6.6, there exists $b=uhy\in R^1$, where $u\in U^1$, $u\neq 1$, $h\in H^1$, and $y\in Y^1$. Let $r_1=J_1(u)$ and $h_1(\chi)=h_{r_1}^1(2)$, then for all $\alpha\in\Pi^\circ$, we have $\chi(\alpha)=1$. Therefore, $x'=h_1(\chi)\,b\,h_1(\chi)^{-1}b^{-1}=u'\in R^1\cap U^1$, and $x'=X_{r_1}^1(t_1(-1+4))u_1'$, where $t_1\in\mathscr{K}^*$ and $r_1<r$, $r\in J(u_1')$, $u_1'\in U^1$. Thus $x'\neq 1$.

For each $r\in\Phi_1^*$, let U_r^1: $\{X_r^1(t),\ t\in\mathscr{K}\}$.

Lemma 6.8 *There exists* $r\in\Phi_1^*$ *such that* $|R^1\cap U_r^1|>1$.

Proof From 6.7, there exists $u\in R^1\cap U^1$, where $u\neq 1$.

Let $r_0=\max\{J_1(u),\ u\in R^1\cap U^1\}$, and u_0 be the element of $R^1\cap U^1$ satisfying $J_1(u_0)=r_0$, Suppose $J(u_0)$: $\{r_1, r_2, \cdots, r_p\}$, If $p=1$, then the lemma is established immediately, suppose $p\neq 1$, it follows from 6.3 that there exists an element $h_1(\chi)\in H^1$

such that $\chi(r_1)=\eta$, $\eta=\pm 1$ and $\chi(r_1)\neq\chi(r_2)$. Let $x'=u_0^{-1}h_1(\chi)u_0h_1(\chi)^{-1}$ or $x'=u_0h_1(\chi)u_0h_1(\chi)^{-1}$ according as $\eta=1$ or $\eta=-1$. Obviously, $x'\in R^1$ and x' can be expressed in the following form $x'=X^1_{r_2}(t_2(\chi(r_2)-\chi(r_1))u')$, where $u'\in U^1$, satisfying $r_2<r$, $r\in J(u')$, $t_2\in\mathscr{K}^*$. From $\chi(r_2)\neq\chi(r_1)$, we have $x'\neq I$. Obviously, $r_0=r_1<r_2=J_1(x')$. Therefore, we have a contradiction, hence $p=1$.

Lemma 6.9 *For each $s\in\Phi_1^*$, $U_s^1\subset R^1$.*

Proof From 6.8, there exists $r\in\Phi_1^*$ such that $X_r^1(t)\in R^1$, $t\in\mathscr{K}^*$. Thus $X^1_{-r}(-t)=N_r^1X_r^1(t)(N_r^1)^{-1}\in R^1$. Therefore, $h_r^1(t)X^1_{-r}(-t)h_r^1(t)^{-1}=X^1_{-r}(-t^{-1})\in R^1$, hence $N_r^1(t)=X_r^1(t)X^1_{-r}(-t^{-1})X_r^1(t)\in R^1$, Thus $N_r^1(4t)=h_r^1(2)N_r^1(t)h_r^1(2)^{-1}\in R^1$. hence $h_r^1(4)=N_r^1(4t)N_r^1(t)^{-1}\in R^1$. It follows from 1.7 that there exists an element r_i of Φ_1^* which belongs to ⓘ type of $\mathscr{A}$. such that $(r_1r_i)\neq 0$. Therefore, for $i=1, 2, 3$ $X^1_{r_i}(-u)h_r^1(4)X^1_{r_i}(u)h_r^1(4)^{-1}=X^1_{r_i}(\mu u)\in R^1$, where $\mu=3$ or $\mu=15$, $u\in\mathscr{K}^*$. For an arbitrary element c of $\mathscr{K}$, let $u=c/\mu$, then $X^1_{r_i}(c)\in R^1$. Thus for each ⓘ type of $\mathscr{A}$, we have $U^1_{r_i}\subset R^1$, where $r_i\in\Phi_1^*$ and r_i belongs to ⓘ type of $\mathscr{A}$. It follows from 1.7 that if $s\in\Phi_1^*$ and s belongs to ⓘ type of $\mathscr{A}$, then there exists an element w of W^1 such that $wr_i=s$. Therefore, $n_wU^1_{r_i}n_w^{-1}=U_s^1\subset R^1$.

Corollary 6.10 *$U^1\subset R^1$ and $V^1\subset R^1$.*

Theorem 6.11 *G^1 is simple and $G^1=\langle U^1, V^1\rangle$.*

Proof It follows from 6.11 and 3.3 that H^1, $N_0^1\subseteq\langle U^1, V^1\rangle\subseteq R^1$. From 5.3 and 6.11, we have $Y^1\subseteq\langle U^1, V^1\rangle\subseteq R^1$. It follows from 5.6 that $G^1\subseteq\langle U^1, H^1, N_0^1, Y^1\rangle\subseteq\langle U^1, V^1\rangle\subseteq R^1$, thus $G^1=R^1$. Therefore, G^1 is simple and $G^1=R^1=\langle U^1, V^1\rangle$.

Remarks

1. Let K_0 be the field which satisfies the following properties a) b) c): a) $\sqrt{-1}\notin\mathrm{K}_0$, $(\sqrt{-1})^2=-1$; b) ch $\mathrm{K}_0>3$; c) if a, $b\in\mathrm{K}_0$, then $a^2+b^2\neq-1$, and let $\mathrm{K}=\mathrm{K}_0(\sqrt{-1})$. If we change the ordering field $\mathscr{K}_0$ for the field K_0 and change $\mathscr{K}$ for K, then the all results remain true.

2. Let $A=(a_{ij})$ $(B=(b_{ij}))$ be $2l\times 2l$ matrix, where $a_{ij}(b_{ij})$, $1\leqslant i$, $j\leqslant 2l$ satisfying the relations: if $1\leqslant i\leqslant l$, then $a_{i,l+i}=1$, $a_{l+i,i}=-1$; if $|i-j|\neq l$, then $a_{ij}=0$ (if i is odd and $1\leqslant i\leqslant 2l$, then $b_{ii}=1$, $b_{i+1}b_{i+1}=-1$; if $i\neq j$, then $b_{ij}=0$). Let $PSp_{2l}(\mathscr{K})$: $\{T=(t_{ij}),\ t_{ij}\in\mathscr{K},\ 1\leqslant i,\ j\leqslant 2l\,|\,T'AT=A\}$ and let $PSp_{2l}(\mathscr{K})^1$: $\{T\in PSp_{2l}(\mathscr{K})\,|\,\bar{T}'\ BT=B\}$. Let $PSp_{2l}(\mathscr{K})^1_0$ be the commutator subgroup of $PSp_{2l}(\mathscr{K})^1$, then ${}^2C_l(\mathscr{K})\cong PSp_{2l}(\mathscr{K})^1_0$ and $PSp_{2l}(\mathscr{K})^1/PSp_{2l}(\mathscr{K})^1_0\cong\{I, I'\}$, where I is the identity 1 matrix, $I'=B$.

3. Let $\mathscr{K}_0$ be the real number field and let $\mathscr{G}$ be the group of linear transformations leaving simultaneously invariant the skew-symmetrical bilinear form $x_1x'_{l+1}-x_{l+1}x'_1+x_2x'_{l+2}-x_{l+2}x'_2+\cdots+x_lx'_{2l}-x_{2l}x'_l$ and the indefinite Hermitian form: $x_1\bar{x}_1-x_2\bar{x}_2+\cdots+x_{2l-1}\bar{x}_{2l-1}-x_{2l}\bar{x}_{2l}$. Let $\mathscr{G}_0$ be the connected complement of $\mathscr{G}$ which contains the identity 1 element of $\mathscr{G}$, then ${}^2C_l(\mathscr{C})\cong\mathscr{G}_0(\mathscr{C}=\mathscr{R}_0(\sqrt{-1}))$ and the Lie algebra of $\mathscr{G}_0$ is

the real form of C_l with $\delta=-l$([4]. p. 292).

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由内自同构构造的李型单群

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摘 要

当 $L\cong C_l$, l 为偶数且 $l\geqslant 4$, 域 $\mathscr{K}=\mathscr{K}_0(\sqrt{-1})$, 其中 $\mathscr{K}_0$ 为一有序域(或 $\mathscr{K}_0$ 满足: a) $\sqrt{-1}\notin\mathscr{K}_0$, $(\sqrt{-1})^2=-1$; b) $\mathrm{ch}\mathscr{K}_0>3$; c) 若 a, $b\in\mathscr{K}_0$, 则 $a^2+b^2\neq-1$). 设 Φ 和 Π: $\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$, α_l 为长根分别为 L 的一组根系和素根系. 令 $\{h_r, r\in\Pi, e_r、r\in\Phi\}$ 为 L 的一组 Chevalley 基; $G=L(\mathscr{K})$ 为对于这一组 Chevalley 基在域 $\mathscr{K}$ 上的 L 型 Chevalley 群. 令 $w_0=w_{\alpha_1}w_{\alpha_2}\cdots w_{\alpha_{l-1}}$, 其中 $\alpha_i\in\Pi$ 且 w_{α_i} 为对于垂直于 α_i 的平面的反射, 显然 w_0 为 L 的 Weyl 群中的元素. 设 N 为 G 的单项子群, $n_0\in N$, n_0 的自然同态像为 w_0, 且 $n_0^2=I$. 存在域 $\mathscr{K}$ 的自同构 f: $f(a)=a$, $a\in\mathscr{K}_0$, $f(\sqrt{-1})=-\sqrt{-1}$, f 在 G 中的扩充为 G 的一个域自同构(仍记为 f), 且 $fn_0=n_0$. 令 $U(V)$ 为 G 对于正(负)根生成的么幂子群, 令 U^1: $\{u\in U\,|\,n_0f(u)n_0^{-1}=u\}$; V^1: $\{v\in V\,|\,n_0f(v)n_0^{-1}=v\}$. 本文证明了 ${}^2C_l(\mathscr{K})=\langle U^1, V^1\rangle$ 为一单群.