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Dirichlet Shift of Finite Multiplicity

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Abstract In this paper, we show that a multiplication operator on the Dirichlet space $\mathcal{D}$ is unitarily equivalent to Dirichlet shift of multiplicity $n+1$ ($n \geq 0$) if and only if its symbol is $c\,z^{n+1}$ for some constant c. The result is very different from the cases of both the Bergman space and the Hardy space.

Keywords Dirichlet space; Dirichlet shift; multiplication operator; unitary equivalence.

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1. Introduction

Let $\mathbb{D}$ be the open unit disk and dA denote the normalized Lebesgue area measure on $\mathbb{D}$. The Dirichlet space $\mathcal{D}$ consists of analytic function f on $\mathbb{D}$ with finite Dirichlet integral

$$D(f) = \int_{\mathbb{D}} |f'|^2 \mathrm{d}A < \infty.$$

Endow $\mathcal{D}$ with norm $\|f\| = \big(|f(0)|^2 + D(f)\big)^{\frac{1}{2}}$, $f \in \mathcal{D}$. $\mathcal{D}$ is a Hilbert space with inner product

$$\langle f, g\rangle = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\mathrm{d}A(z),\ \ f, g \in \mathcal{D}.$$

It is well known that $\mathcal{D}$ is a reproducing function space with reproducing kernel

$$K_\lambda(z) = 1 + \log\frac{1}{1-\bar{\lambda}z},\quad \lambda, z \in \mathbb{D}.$$

In recent years, the Dirichlet space has received a lot attention from the analysts. We refer readers to the survey paper [1] for more information about the Dirichlet space.

A function ϕ on $\mathbb{D}$ is called a multiplier of $\mathcal{D}$ if $\phi\mathcal{D} \subset \mathcal{D}$. Denote by $\mathcal{M}$ the multiplier space of $\mathcal{D}$. For $\phi \in \mathcal{M}$, a simple application of the closed graph theorem shows that the multiplication operator $M_\phi : f \to \phi f$, $f \in \mathcal{D}$, is bounded.

The multiplication operator M_z known as the Dirichlet shift is an important operator and has been studied deeply [2–5]. In this paper, we study when a multiplication operator M_ϕ on $\mathcal{D}$ is

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essentially the Dirichlet shift, i.e., M_ϕ is unitarily equivalent to M_z. More generally, we study the multiplication operator on $\mathcal{D}$ which is unitarily equivalent to $M_{z^{n+1}}$ $(n \geq 0)$, the Dirichlet shift of multiplicity $n+1$. Recall that two operators A, B on Hilbert spaces H and K respectively are called unitarily equivalent if there exists a unitary operator $U : H \to K$ such that $UAU^* = B$. To characterize the condition for two operators to be unitarily equivalent is an important topic in the operator theory [6]. For the unitary equivalence of Toeplitz operators or multiplication operators on the Hardy space or the Bergman space, see [7–9].

On the Hardy space, every finite Blaschke product is a unilateral shift of finite multiplicity [7]. On the Bergman space, Sun, Zheng and Zhong [10] completely characterized the multiplication operators which are unitarily equivalent to a weighted unilateral shift of finite multiplicity.

On the Dirichlet space, the author [11] characterized the unitarily equivalent multiplication operators to M_{z^2} by the characterization of reducing subspaces of such operators. In [12], the unitary equivalence of the multiplication operator defined by finite Blaschke product of order two is considered. In this paper, we will show that a multiplication operator is unitarily equivalent to the Dirichlet shift of multiplicity $n+1$ $(n \geq 0)$ if and only if its symbol is a constant multiple of z^{n+1}.

Theorem 1.1 *Let $\phi \in \mathcal{M}$. Then M_ϕ is unitarily equivalent to $M_{z^{n+1}}$ $(n \geq 0)$ if and only if $\phi(z) = cz^{n+1}$ for some constant c with $|c| = 1$.*

2. Proof of the main result

Since the proof of the main result depends on a representation formula for the Dirichlet integral given by Carleson [13], here we give some discussion about the Carleson formula.

Let $f \in \mathcal{D}$, $f = BSF$ be the canonical factorization of f as a function in the Hardy space, where $B = \prod_{j=1}^{\infty} \frac{\bar{a}_j}{|a_j|} \frac{a_j - z}{1 - \bar{a}_j z}$ is a Blaschke product, S is the singular part of f and F is the outer part of f. Then

$$D(f) = \int_{\mathbb{T}} \sum_{n=1}^{\infty} P_{\alpha_n}(\xi)|f(\xi)|^2 \frac{|\mathrm{d}\xi|}{2\pi} + \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{2}{|\zeta - \xi|^2}|f(\xi)|^2 \mathrm{d}\mu(\zeta)\frac{|\mathrm{d}\xi|}{2\pi} + \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{(e^{2u(\zeta)} - e^{2u(\xi)})(u(\zeta) - u(\xi))}{|\zeta - \xi|^2} \frac{|\mathrm{d}\zeta|}{2\pi}\frac{|\mathrm{d}\xi|}{2\pi},$$

where $u(\xi) = \log|f(\xi)|$, $P_\alpha(\xi)$ is the Poisson kernel and μ is the singular measure corresponding to S.

Let

$$\varphi_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda}z}, \quad \lambda, z \in \mathbb{D}$$

be the Möbius transform. For $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{D}$, $\phi = \varphi_{\lambda_0}\varphi_{\lambda_1}\varphi_{\lambda_2}\cdots\varphi_{\lambda_n}$ is a finite Blaschke product of order $n+1$.

By the Carleson formula, for $f, g \in \mathcal{D}$, and integer $m \geq 1$, $k = 0, 1, 2, 3$, we have

$$D(\phi^m(f + i^k g)) = m\int_{\mathbb{T}} \big((P_{\lambda_0} + P_{\lambda_1} + \cdots + P_{\lambda_n})|f(\xi) + i^k g(\xi)|^2\big)\frac{|\mathrm{d}\xi|}{2\pi} + D(f + i^k g)$$

$$=m\int_{\mathbb{T}}\left((P_{\lambda_0}+P_{\lambda_1}+\cdots+P_{\lambda_n})|f(\xi)+i^kg(\xi)|^2\right)\frac{|\mathrm{d}\xi|}{2\pi}+$$
$$\|f+i^kg\|^2-|f(0)+i^kg(0)|^2,$$

where i is the imaginary unit.

By the polarization identity, we have

$$\begin{aligned}\langle\phi^m f,\phi^m g\rangle &=\sum_{k=0}^{3}\frac{i^k}{4}\|\phi^m(f+i^kg)\|^2\\ &=\sum_{k=0}^{3}\frac{i^k}{4}\left(D(\phi^m(f+i^kg))+|\phi^m(0)(f(0)+i^kg(0))|^2\right)\\ &=m\int_{\mathbb{T}}(P_{\lambda_0}+P_{\lambda_1}+\cdots+P_{\lambda_n})f(\xi)\overline{g(\xi)}\frac{|\mathrm{d}\xi|}{2\pi}+\\ &\quad\langle f,g\rangle-f(0)\overline{g(0)}+|\phi^m(0)|^2f(0)\overline{g(0)}.\end{aligned}\tag{1}$$

To continue, we need the following lemma, which has appeared in [11].

Lemma 2.1 *Let $\phi\in\mathcal{M}$. If M_ϕ is unitarily equivalent to $M_{z^{n+1}}$, then ϕ is a Blaschke product of order $n+1$.*

Proof Let $U:\mathcal{D}\to\mathcal{D}$ be a unitary operator such that $U^*M_\phi U=M_{z^{n+1}}$, and let I be the identity operator and k_λ be the normalization of K_λ for $\lambda\in\mathbb{D}$, that is, $k_\lambda=K_\lambda/\|K_\lambda\|$.

It is easy to verify that $M_{z^{n+1}}M^*_{z^{n+1}}-I$ is compact and k_λ weakly converges to 0 as $|\lambda|\to1$. Hence, as $|\lambda|\to1$

$$\langle M_\phi M_\phi^* k_\lambda,k_\lambda\rangle-1=\langle U(M_{z^{n+1}}M^*_{z^{n+1}}-I)U^*k_\lambda,k_\lambda\rangle\to0.$$

As we know

$$\langle M_\phi M_\phi^* k_\lambda,k_\lambda\rangle-1=|\phi(\lambda)|^2-1,$$

which means that $|\phi(\lambda)|^2\to1$ as $|\lambda|\to1$. It follows that ϕ is an inner function. We claim that ϕ is a Blaschke product of finite order. Otherwise we can always find infinitely many $\lambda_m\in\mathbb{D}$ such that $|\lambda_m|\to1$, but $\phi(\lambda_m)\to0$. Since $M_{z^{n+1}}$ has order $n+1$, ϕ must have order $n+1$. □

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1 By Lemma 2.1, without loss of generality, assume $\phi=\varphi_{\lambda_0}\varphi_{\lambda_1}\cdots\varphi_{\lambda_n}$. Let U be a unitary operator on $\mathcal{D}$ such that $U^*M_\phi U=M_{z^{n+1}}$.

Let $E_j(z)=z^j$, $0\le j<\infty$. $\{UE_j\}_{j=0}^n$ is an orthogonal basis of $\mathcal{D}\ominus\phi\mathcal{D}$. Set $f_j=UE_j$. For any integer $m\ge1$, $j=0,1,\ldots,n$, we have

$$UE_{m(n+1)+j}=UM^m_{z^{n+1}}E_j=M^m_\phi UE_j=\phi^m f_j.$$

Therefore

$$\langle\phi^m f_k,\phi^m f_j\rangle=\langle E_{m(n+1)+k},E_{m(n+1)+j}\rangle=0,\quad j\ne k;$$

$$\langle\phi^m f_j,\phi^l f_j\rangle=\langle E_{m(n+1)+k},E_{l(n+1)+j}\rangle=0,\quad m\ne l.$$

By (1),

$$\langle \phi^m f_j, \phi^m f_k \rangle = m \int_{\mathbb{T}} (P_{\lambda_0} + P_{\lambda_1} + P_{\lambda_2} + \cdots + P_{\lambda_n}) f_j(\xi) \overline{f_k(\xi)} \frac{|\mathrm{d}\xi|}{2\pi} + \\ \langle f_j, f_k \rangle + (|\phi^m(0)|^2 - 1) f_j(0) \overline{f_k(0)}.$$

When $j \neq k$, we have

$$0 = \int_{\mathbb{T}} (P_{\lambda_0} + P_{\lambda_1} + P_{\lambda_2} + \cdots + P_{\lambda_n}) f_j(\xi) \overline{f_k(\xi)} \frac{|\mathrm{d}\xi|}{2\pi} + \frac{(|\phi^m(0)|^2 - 1) f_j(0) \overline{f_k(0)}}{m}. \tag{2}$$

Let $m \to \infty$. Then

$$0 = \int_{\mathbb{T}} (P_{\lambda_0} + P_{\lambda_1} + P_{\lambda_2} + \cdots + P_{\lambda_n}) f_j(\xi) \overline{f_k(\xi)} \frac{|\mathrm{d}\xi|}{2\pi}.$$

It follows from (2) that $(|\phi^m(0)|^2 - 1) f_j(0) \overline{f_k(0)} = 0$, and thus $f_j(0) \overline{f_k(0)} = 0$.

If for all $j = 0, 1, \ldots, n$, $f_j(0) = 0$, then $1 \perp \mathcal{D} \ominus \phi\mathcal{D}$ and hence $1 \in \phi\mathcal{D}$. This is impossible. So there exists j in $\{0, 1, \ldots, n\}$ such that $f_j(0) \neq 0$, say $j = 0$, and hence for $j \neq 0$, $f_j(0) = 0$.

By (1),

$$\langle \phi^{m+1} f_0, \phi^m f_0 \rangle = m \int_{\mathbb{T}} (P_{\lambda_0} + P_{\lambda_1} + P_{\lambda_2} + \cdots + P_{\lambda_n}) \phi(\xi) |f_0(\xi)|^2 \frac{|\mathrm{d}\xi|}{2\pi} + \\ \langle \phi f_0, f_0 \rangle + (|\phi^m(0)|^2 - 1) \phi(0) |f_0(0)|^2.$$

i.e.,

$$0 = m \int_{\mathbb{T}} (P_{\lambda_0} + P_{\lambda_1} + P_{\lambda_2} + \cdots + P_{\lambda_n}) \phi(\xi) |f_0(\xi)|^2 \frac{|\mathrm{d}\xi|}{2\pi} + \\ (|\phi^m(0)|^2 - 1) \phi(0) |f_0(0)|^2.$$

Reasoning as above, we have $(|\phi^m(0)|^2 - 1) \phi(0) |f_0(0)|^2 = 0$. Consequently,

$$\phi(0) = 0.$$

Without loss of generality, assume $\lambda_0 = 0$.

Since $\phi(0) = 0$, $1 \in \mathcal{D} \ominus \phi\mathcal{D}$. Let $\{1, \mathcal{E}_1, \ldots, \mathcal{E}_n\}$ be an orthonormal basis of $\mathcal{D} \ominus \phi\mathcal{D}$ and

$$f_j = a_{j0} + a_{j1}\mathcal{E}_1 + \cdots + a_{jn}\mathcal{E}_n, \quad j = 0, 1, \ldots, n.$$

For $j = 1, 2, \ldots, n$, we have $a_{j0} = 0$ since $f_j(0) = 0$. So

$$0 = \langle f_j, f_0 \rangle = a_{j1}\bar{a}_{01} + a_{j2}\bar{a}_{02} + \cdots + a_{jn}\bar{a}_{0n}, \quad j = 1, 2, \ldots, n. \tag{3}$$

Since

$$a_{00} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0n} \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1n} \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0, \tag{4}$$

it follows from (3) that $a_{01}=a_{02}=\cdots=a_{0n}=0$. Hence $f_0=a_{00}$. By the formula (1), for $j=1,2,\ldots,n$,

$$0=\langle\phi f_j,\phi f_0\rangle=\int_{\mathbb{T}}(1+P_{\lambda_1}+\cdots+P_{\lambda_n})f_j(\xi)\overline{f_0(\xi)}\frac{|\mathrm{d}\xi|}{\pi}.$$

Hence

$$f_j(\lambda_1)+\cdots+f_j(\lambda_n)=0.$$

In other words, for $j=1,2,\ldots,n$,

$$a_{j1}(\mathcal{E}_1(\lambda_1)+\mathcal{E}_1(\lambda_2)+\cdots+\mathcal{E}_1(\lambda_n))+\cdots+a_{jn}(\mathcal{E}_n(\lambda_1)+\mathcal{E}_n(\lambda_2)+\cdots+\mathcal{E}_n(\lambda_n))=0.$$

By (4), for $l=1,2,\ldots,n$,

$$\mathcal{E}_l(\lambda_1)+\mathcal{E}_l(\lambda_2)+\cdots+\mathcal{E}_l(\lambda_n)=0,$$

i.e.,

$$\langle\mathcal{E}_l,K_{\lambda_1}+K_{\lambda_2}+\cdots+K_{\lambda_n}\rangle=0.$$

Since $K_{\lambda_1},K_{\lambda_2},\ldots,K_{\lambda_n}\in\mathcal{D}\ominus\phi\mathcal{D}$, we have

$$K_{\lambda_1}+K_{\lambda_2}+\cdots+K_{\lambda_n}=\gamma$$

for some constant γ. Obviously $\gamma=n$. Then for any integer $m\geq 1$,

$$\langle z^m,K_{\lambda_1}+K_{\lambda_2}+\cdots+K_{\lambda_n}\rangle=\lambda_1^m+\lambda_2^m+\cdots+\lambda_n^m=0.$$

So $\lambda_1=\lambda_2=\cdots=\lambda_n=0$, and thus $\phi(z)=c\,z^{n+1}$ for some constant c. The proof is completed. □

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