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Spectral Approximation Based on the Pressure Stabilization Mathod for Unsteady Navier-Stokes Equations *

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Abstract A Legendre spectral approximation based on the pressure stabilization method for non-periodic, unsteady Navier-Stokes equations is considered. The generalized stability and the convergence are proved strictly. The approximation results in this paper are also useful for other non-linear problems.

Keywords Legendre spectral approximation, pressure stabilization method, Navier-Stokes equation.

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1. Introduction

Let $x = (x_1, \cdots, x_n)$ ($n = 2$ or 3), $I = (-1, 1)$ and $\Omega = I^n$ with the boundary $\partial\Omega$. The velocity and the pressure are denoted by $U(x,t)$ and $P(x,t)$ respectively. $\nu > 0$ is the kinetic viscosity. $U_0(x)$ and $f(x,t)$ are given functions. Let $T > 0$ and consider the numerical approximation of the unsteady Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial U}{\partial t} + (U \cdot \nabla)U + \nabla P - \nu \nabla^2 U = f, & \text{in } \Omega \times (0, T], \\ \nabla \cdot U = 0, & \text{in } \Omega \times (0, T], \\ U(x, 0) = U_0(x), & \text{in } \Omega. \end{cases} \tag{1.1}$$

We consider only the non-slip boundary. It means that $U = 0$ for all $x \in \partial\Omega$ and $t \in [0, T]$. For fixing the pressure P, we also require that

$$\int_\Omega P(x,t)\,\mathrm{d}x = 0, \quad t \in [0, T].$$

It is well known that one of the main difficulties in numerical approximation of the unsteady Navier-Stokes equations is how to treat the incompressibility constraint "div $U = 0$". So far, various methods to overcome this difficulty come out (see [1–4]). A fundamental idea of these methods is to relax the incompressibility constraint in an appropriate

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way. One of them is the pressure stabilization method, initially introduced by Brezzi and Pitkäranta (see [3]) for the approximation of the steady Stokes equations. When applied to the unsteady Navier-Stokes equations, this method takes the form (see [4]):

$$\begin{cases} \dfrac{\partial U^\beta}{\partial t} + (U^\beta \cdot \nabla)U^\beta + \nabla P^\beta - \nu\nabla^2 U^\beta = f, \\ \nabla \cdot U^\beta - \beta\nabla^2 P^\beta = 0, \qquad \dfrac{\partial P^\beta}{\partial \vec{n}} = 0, \end{cases} \tag{1.2}$$

where $\beta > 0$ is a small parameter and $\vec{n}$ is unit outer normal on $\partial\Omega$.

The generalized solution of (1.2) is the pair $(U^\beta, P^\beta) \in (H_0^1(\Omega))^n \times (H^1(\Omega) \bigcap L_0^2(\Omega))$ satisfying

$$\begin{cases} (\dfrac{\partial U^\beta}{\partial t}, v) + ((U^\beta \cdot \nabla)U^\beta, v) + (\nabla P^\beta, v) + \nu(\nabla U^\beta, \nabla v) = (f, v), \quad \forall v \in (H_0^1(\Omega))^n, \\ (\nabla \cdot U^\beta, v) + \beta(\nabla P^\beta, \nabla v) = 0, \qquad \forall v \in H^1(\Omega) \bigcap L_0^2(\Omega). \end{cases} \tag{1.3}$$

Since we approximate the incompressibility constraint by the second formula of (1.2), we avoid the very difficult job of choosing the trial function space in which the divergence of every element vanishes everywhere. Furthermore, the trial function spaces for the velocity and the pressure need not to satisfy the Babuska-Brezzi inf-sup condition.

Shen Jie analyzed the semi-discretization of (1.2) by using a finite element method (see [4]). Since spectral approximations have high accuracy, we consider Legendre spectral approximation of (1.2) in this paper. We construct the fully discrete scheme in the next section. Then we list some Lemmas in section 3, and prove the generalized stability and the convergence strictly in the last two sections.

2. The Scheme

Let $r \geq 0$, we will use the standard notations $L^2(\Omega)$, $H^r(\Omega)$ and $H_0^r(\Omega)$ to denote the Sobolev spaces over Ω. The norm and inner product of $(L^2(\Omega))^n$ are $\|\cdot\|$ and $(\cdot,\cdot)$. The norm and semi- norm of $(H^r(\Omega))^n$ are denoted by $\|\cdot\|_r$ and $|\cdot|_r$. Besides, $L^\infty(\Omega)$ is also the usual Sobolev space with the norm $\|\cdot\|_\infty$. We also define

$$L_0^2(\Omega) = \{v \in L^2(\Omega) / \int_\Omega v \,\mathrm{d}x = 0\}.$$

In order to approximate the nonlinear term, we introduce a trilinear form $J(\cdot,\cdot,\cdot)$: $((H^1(\Omega))^n)^3 \to R^1$ as follows:

$$J(\eta, \varphi, \xi) = \frac{1}{2}\{((\varphi \cdot \nabla)\eta, \xi) - ((\varphi \cdot \nabla)\xi, \eta)\}.$$

Clearly

$$J(\eta, \varphi, \xi) + J(\xi, \varphi, \eta) = 0, \tag{2.1}$$

and if $\nabla \cdot \varphi = 0$, then

$$J(\eta, \varphi, \xi) = ((\varphi \cdot \nabla)\eta, \xi).$$

Therefore the weak formulation of (1.1) is to find $(U,P)\in (H_0^1(\Omega))^n\times (H^1(\Omega)\bigcap L_0^2(\Omega))$ such that

$$\begin{cases} (\frac{\partial U}{\partial t},v)+J(U,U,v)+(\nabla P,v)+\nu(\nabla U,\nabla v)=(f,v), & \forall v\in (H_0^1(\Omega))^n,\\ (\nabla\cdot U,v)=0, & \forall v\in L_0^2(\Omega),\\ U(x,0)=U_0(x). \end{cases} \tag{2.2}$$

Now we construct the scheme. For any positive integer N, we denote by $\mathcal{P}_N$ the space of all polynomials of degree up to N in each variable x_i, for $i=1,\cdots,n$. Define

$$V_N=\{v(x)\in\mathcal{P}_N/v(x)=0,\forall x\in\partial\Omega\},\qquad W_N=\mathcal{P}_N\bigcap L_0^2(\Omega).$$

Let $P_N^{1,0}:(H_0^1(\Omega))^n\to (V_N)^n$ be a projection operator such that for any $v\in (H_0^1(\Omega))^n$,

$$(\nabla(v-P_N^{1,0}v),\nabla w)=0,\qquad \forall w\in (V_N)^n.$$

Let τ be the mesh size in time t and

$$S_\tau=\{t=l\tau/0\le l\le [\frac{T}{\tau}]\}.$$

For simplicity, $u(x,t)$ is denoted by $u(t)$ or u usually. Let

$$u_t(t)=\frac{1}{\tau}(u(t+\tau)-u(t)).$$

A fully discrete Legendre spectral scheme for (2.2) is to seek $(u,p)\in (V_N)^n\times W_N$ for all $t\in S_\tau$ such that

$$\begin{cases} (u_t,v)+J(u+\delta\tau u_t,u,v)+\nu(\nabla(u+\sigma\tau u_t),\nabla v)\\ \qquad +(\nabla(p+\theta\tau p_t),v)=(f,v), & \forall v\in (V_N)^n, & (2.3)_1\\ \beta(\nabla p,\nabla v)+(\nabla\cdot u,v)=0, & \forall v\in W_N, & (2.3)_2\\ u(0)=P_N^{1,0}U_0, \end{cases}$$

where $\delta,\sigma\ge 0$ and $\theta>\frac{1}{2}$ are parameters.

It is obvious that for each fixed $t\in S_\tau$, $(2.3)_1$–$(2.3)_2$ is a linear system for unknown variable $(u(t+\tau),p(t+\tau))$. First of all, we assert that for each fixed $t\in S_\tau$, there exists a unique solution to $(2.3)_1$–$(2.3)_2$. Since the matrix of this linear system is square, and therefore it suffices to prove either uniqueness or existence. In this case, the uniqueness is clear. Indeed, using $(2.3)_1$, we obtain

$$\begin{aligned} &(u(t+\tau),v)+\delta\tau J(u(t+\tau),u(t),v)+\theta\tau(\nabla p(t+\tau),v)+\\ &\nu\sigma\tau(\nabla u(t+\tau),\nabla v)=R(t)(v), \qquad \forall v\in (V_N)^n, \end{aligned} \tag{2.4}$$

where $R(t)$ is a linear form defined on $(V_N)^n$ and depends only on $u(t),p(t)$ and $f(t)$. Clearly, if $R(t)=0$, then by putting $v=u(t+\tau)$ in (2.4), we get from (2.1) that

$$\|u(t+\tau)\|^2+\theta\tau(\nabla p(t+\tau),u(t+\tau))+\nu\sigma\tau|u(t+\tau)|_1^2=0. \tag{2.5}$$

It follows from $(2.3)_2$ that

$$\beta|p(t+\tau)|_1^2 - (u(t+\tau), \nabla p(t+\tau)) = 0. \tag{2.6}$$

The combination of (2.5) and (2.6) leads to

$$\|u(t+\tau)\|^2 + \beta\theta\tau|p(t+\tau)|_1^2 + \nu\sigma\tau|u(t+\tau)|_1^2 = 0.$$

Hence $(u(t+\tau), p(t+\tau)) = 0$. This implies the uniqueness.

On the other hand, using $(2.3)_2$ we can express $p(t+\tau)$ by $u(t+\tau)$, and then by substituting this relational expression into $(2.3)_1$, we obtain $u(t+\tau)$ firstly. Using this relational expression again, we get $p(t+\tau)$ immediately. In this case, we can solve the velocity and the pressure separately. This is one of the advantages of the pressure stabilization treatment.

3. Some Lemmas

Throughout the paper, C will denote various positive constants independent of N, τ and any functions.

To analyze the convergence, we introduce the operator $\tilde{P}_N^1$ from $H^1(\Omega)\bigcap L_0^2(\Omega)$ to W_N such that for any $\eta \in H^1(\Omega)\bigcap L_0^2(\Omega)$,

$$(\nabla(\eta - \tilde{P}_N^1\eta), \nabla w) = 0, \quad \forall w \in W_N.$$

Lemma 1 *If $v \in H^1(\Omega)\bigcap L_0^2(\Omega)$ with $r \geq 1$, then*

$$\|v - \tilde{P}_N^1 v\|_\mu \leq CN^{\mu - r}\|v\|_r, \quad \mu = 0, 1. \tag{3.1}$$

Proof By Poincaré inequality, $H^1(\Omega)\bigcap L_0^2(\Omega)$ is a Hilbert space for the inner product $(\nabla u, \nabla v)$, and $|v|_1 = (\nabla v, \nabla v)^{\frac{1}{2}}$ is a norm equivalent to the standard norm of $\|v\|_1$.

Let $P_N^1 : H^1(\Omega) \to \mathcal{P}_N$ be the operator of orthogonal projection for the inner product of $H^1(\Omega)$, that is, for each $\eta \in H^1(\Omega)$

$$(\eta - P_N^1\eta, \varphi) + (\nabla(\eta - P_N^1\eta), \nabla\varphi) = 0, \quad \forall \varphi \in \mathcal{P}_N.$$

We define the polynomial

$$v^N = d + P_N^1 v.$$

The constant d is chosen in such a way that $v^N \in W_N$. By (9.7.14) of [5], we have

$$|v - v^N|_1 = |v - P_N^1 v|_1 \leq CN^{1-r}\|v\|_r. \tag{3.2}$$

Thus the result (3.1) for $\mu = 1$ follows, noting that

$$|v - \tilde{P}_N^1 v|_1 \leq |v - \phi|_1, \quad \forall \phi \in W_N.$$

In order to prove (3.1) for $\mu = 0$, we indicate the following regularity result.

For any $g \in L_0^2(\Omega)$, there exists a unique $\psi \in H^1(\Omega) \bigcap L_0^2(\Omega)$ such that

$$(\nabla\psi, \nabla\phi) = (g, \phi), \quad \forall \phi \in H^1(\Omega) \bigcap L_0^2(\Omega). \tag{3.3}$$

Moreover, $\psi \in H^2(\Omega)$ and

$$\|\psi\|_2 \leq C\|g\|. \tag{3.4}$$

We use a well-known duality argument, based on the identity

$$\|v - \tilde{P}_N^1 v\| = \sup_{\substack{g\in L_0^2(\Omega)\\ g\neq 0}} \frac{(g, v - \tilde{P}_N^1 v)}{\|g\|}. \tag{3.5}$$

Let ψ be the solution of (3.3) corresponding to a given g. Then choosing $\phi = v - \tilde{P}_N^1 v$ in (3.3) and recalling the definition of $\tilde{P}_N^1$, we get

$$(g, v - \tilde{P}_N^1 v) = (\nabla\psi, \nabla(v - \tilde{P}_N^1 v)) = (\nabla(\psi - \tilde{P}_N^1\psi), \nabla(v - \tilde{P}_N^1 v)).$$

Estimate (3.1) with $\mu = 1$ and (3.4) yield

$$|(g, v - \tilde{P}_N^1 v)| \leq CN^{-1}\|g\|\ |v - \tilde{P}_N^1 v|_1.$$

Again using (3.1) with $\mu = 1$, we obtain the desired result from (3.5).

Lemma 2 *For any $v \in \mathcal{P}_N$,*

$$\|v\|_\infty \leq CN^{\alpha_n}\|v\|_1, \tag{3.6}$$

$$\alpha_n = \begin{cases} \alpha, & \alpha > 0 \text{ is an arbitrary small constant, if } n = 2, \\ 1, & \text{if } n = 3. \end{cases}$$

Proof By (9.4.3) of [5], we have that for each $v \in \mathcal{P}_N$,

$$\|v\|_\infty \leq CN^{\frac{2n}{q}}\|v\|_{L^q(\Omega)}, \quad \forall q \in [1, \infty]. \tag{3.7}$$

By the Sobolev's imbedding theorems, the following two inclusions hold

$$H^1(\Omega) \hookrightarrow L^{q'}(\Omega), \qquad \forall q' \in [1, \infty), \qquad \text{if } n = 2, \tag{3.8}$$

$$H^1(\Omega) \hookrightarrow L^6(\Omega), \qquad \text{if } n = 3. \tag{3.9}$$

The combination of (3.7) with $n = 2$ and (3.8) leads to

$$\|v\|_\infty \leq CN^{\frac{4}{q'}}\|v\|_1, \quad q' \in [1, \infty).$$

This implies (3.6) for $n = 2$, due to $q' \in [1, \infty)$.

On the other hand, (3.6) for $n = 3$ follows from (3.7) with $n = 3$ and $q = 6$ together with (3.9).

Lemma 3 *For each $v \in \mathcal{P}_N$,*

$$\|v\|_1 \leq \frac{4}{3}N^4\|v\|.$$

Proof The conclusion comes from (2.3.15) of [5].

Lemma 4 *For all* $\eta, \xi, \varphi \in (H_0^1(\Omega))^n$,

$$\begin{aligned} |J(\eta,\varphi,\xi)| &\leq C|\eta|_1|\xi|_1\|\varphi\|^{\frac{1}{2}}|\varphi|_1^{\frac{1}{2}}, \\ |J(\eta,\varphi,\xi)| &\leq C|\varphi|_1|\xi|_1\|\eta\|^{\frac{1}{2}}|\eta|_1^{\frac{1}{2}}, \\ |J(\eta,\varphi,\xi)| &\leq C|\eta|_1|\xi|_1|\varphi|_1. \end{aligned}$$

Proof The conclusions can be proved by integration by parts, the inclusion (3.9) and Hölder's inequality.

Lemma 5 ((9.7.14) of [5]). *If* $v \in H_0^1(\Omega)\bigcap H^r(\Omega)$ *with* $r \geq 1$, *then*

$$\|v - P_N^{1,0}v\|_\mu \leq CN^{\mu-r}\|v\|_r, \quad \mu = 0,\ 1.$$

Lemma 6 (Lemma 4.16 of [6]) *Suppose that the following conditions are fulfilled*

(i) $Z(t)$ *is a non-negative function defined on* S_τ, D_1, D_2 *and* ρ *are non-negative constants;*

(ii) $H(\xi)$ *is a real-valued function defined on* R^1, *such that* $H(\xi) \leq 0$ *for* $\xi \leq D_2$;

(iii) for all $t \in S_\tau$ *and* $t > 0$,

$$Z(t) \leq \rho + \tau \sum_{t' \leq t-\tau} (D_1 Z(t') + H(Z(t')));$$

(iv) $Z(0) \leq \rho$ *and* $\rho e^{D_1 t_1} \leq D_2$ *for some* $t_1 \in S_\tau$.

Then for all $t \in S_\tau$ *and* $t \leq t_1$, *we have*

$$Z(t) \leq \rho e^{D_1 t}. \tag{3.10}$$

Remark 3.1 *If* $H(\xi) \leq 0$ *for all* $\xi \in R^1$, *then (3.10) holds for all* t *and any* ρ *satisfying (iii).*

4. The generalized stability

In this section, we analyze the generalized stability of Scheme $(2.3)_1$–$(2.3)_2$. Suppose that the initial values and the right terms of Scheme $(2.3)_1$–$(2.3)_2$ have the errors $\tilde{u}(0)$, $\tilde{f}(t)$ and $\tilde{g}(t)$ respectively, which induce the errors $\tilde{u}$ and $\tilde{p}$ of u and p. They satisfy

$$\begin{cases} (\tilde{u}_t, v) + J(u + \delta\tau u_t, \tilde{u}, v) + J(\tilde{u} + \delta\tau\tilde{u}_t, u + \tilde{u}, v) + \nu(\nabla(\tilde{u} + \sigma\tau\tilde{u}_t), \nabla v) + \\ \quad (\nabla(\tilde{p} + \theta\tau\tilde{p}_t), v) = (\tilde{f}, v), \qquad \forall v \in (V_N)^n, & (4.1)_1 \\ \beta(\nabla(\tilde{p} + \theta\tau\tilde{p}_t), \nabla v) + (\nabla\cdot(\tilde{u} + \theta\tau\tilde{u}_t), v) = (\tilde{g} + \theta\tau\tilde{g}_t, v) \qquad \forall v \in W_N. & (4.1)_2 \end{cases}$$

Let $\varepsilon > 0$ be a suitably small constant to be chosen below. Using (2.1) and the identity

$$2\,(\tilde{u}(t), \tilde{u}_t(t)) = \left(\|\tilde{u}(t)\|^2\right)_t - \tau\|\tilde{u}_t(t)\|^2, \tag{4.2}$$

we have by taking $v = 2\tilde{u} + 2\theta\tau\tilde{u}_t$ in $(4.1)_1$ that

$$(\|\tilde{u}\|^2)_t + \tau(2\theta - 1 - \varepsilon)\|\tilde{u}_t\|^2 + 2\nu|\tilde{u}|_1^2 + \nu\tau(\sigma + \theta)(|\tilde{u}|_1^2)_t + \nu\tau^2(2\theta\sigma - \sigma - \theta)|\tilde{u}_t|_1^2 - 2(\tilde{p} + \theta\tau\tilde{p}_t, \nabla\cdot(\tilde{u} + \theta\tau\tilde{u}_t)) + \sum_{j=1}^{3} F_j(t) \le \|\tilde{u}\|^2 + (1 + \frac{\tau\theta^2}{\varepsilon})\|\tilde{f}\|^2, \tag{4.3}$$

where

$$\begin{aligned} F_1(t) &= 2J(u + \delta\tau u_t, \tilde{u}, \tilde{u} + \theta\tau\tilde{u}_t),\\ F_2(t) &= 2\tau(\theta - \delta)J(\tilde{u}, u, \tilde{u}_t),\\ F_3(t) &= 2\tau(\theta - \delta)J(\tilde{u}, \tilde{u}, \tilde{u}_t). \end{aligned}$$

By taking $v = 2\tilde{p} + 2\theta\tau\tilde{p}_t$ in $(4.1)_2$ and noting the fact that $\|\tilde{p} + \theta\tau\tilde{p}_t\| \le C|\tilde{p} + \theta\tau\tilde{p}_t|_1$, we get

$$\beta|\tilde{p} + \theta\tau\tilde{p}_t|_1^2 + 2(\nabla\cdot(\tilde{u} + \theta\tau\tilde{u}_t), \tilde{p} + \theta\tau\tilde{p}_t) \le \frac{C}{\beta}\|\tilde{g} + \theta\tau\tilde{g}_t\|^2. \tag{4.4}$$

Adding (4.4) to (4.3), we obtain

$$(\|\tilde{u}\|^2)_t + \tau(2\theta - 1 - \varepsilon)\|\tilde{u}_t\|^2 + 2\nu|\tilde{u}|_1^2 + \nu\tau(\sigma + \theta)(|\tilde{u}|_1^2)_t + \nu\tau^2(2\sigma\theta - \sigma - \theta)|\tilde{u}_t|_1^2 + \beta|\tilde{p} + \theta\tau\tilde{p}_t|_1^2 + \sum_{j=1}^{3} F_j(t) \le \|\tilde{u}\|^2 + (1 + \frac{\tau\theta^2}{\varepsilon})\|\tilde{f}\|^2 + \frac{C}{\beta}\|\tilde{g} + \theta\tau\tilde{g}_t\|^2. \tag{4.5}$$

Let $|||u|||_1 = \max_{t\in S_\tau}\|u(t)\|_1$, etc.. We now estimate $|\ F_j(t)\ |$ (j=1, 2, 3). First we have from Lemma 4 that

$$\begin{aligned} |F_1(t)| &\le C|u + \delta\tau u_t|_1|\tilde{u} + \theta\tau\tilde{u}_t|_1\|\tilde{u}\|^{\frac{1}{2}}|\tilde{u}|_1^{\frac{1}{2}}\\ &\le \varepsilon\nu|\tilde{u}|_1^2 + \varepsilon\nu\tau^2|\tilde{u}_t|_1^2 + \frac{C(1 + \theta^4)(1 + \delta)^4}{\varepsilon^3\nu^3}|||u|||_1^4\|\tilde{u}\|^2.\\ |F_2(t)| &\le C\tau|\theta - \delta|\ |u|_1\|\tilde{u}\|^{\frac{1}{2}}|\tilde{u}|_1^{\frac{1}{2}}|\tilde{u}_t|_1\\ &\le \varepsilon\nu|\tilde{u}|_1^2 + \varepsilon\nu\tau^2|\tilde{u}_t|_1^2 + \frac{C(\theta - \delta)^4}{\varepsilon^3\nu^3}|||u|||_1^4\|\tilde{u}\|^2. \end{aligned}$$

By Lemma 2, we have

$$\begin{aligned} |F_3(t)| &\le C\tau|\theta - \delta|(\|\tilde{u}\|\ |\tilde{u}|_1\|\tilde{u}_t\|_\infty + \|\tilde{u}\|\ \|\tilde{u}\|_\infty|\tilde{u}_t|_1)\\ &\le C\tau|\theta - \delta|N^{\alpha_n}\|\tilde{u}\|\ |\tilde{u}|_1|\tilde{u}_t|_1\\ &\le \varepsilon\nu\tau^2|\tilde{u}_t|_1^2 + \frac{C(\theta - \delta)^2N^{2\alpha_n}}{\varepsilon\nu}\|\tilde{u}\|^2|\tilde{u}|_1^2. \end{aligned}$$

By substituting the above estimations into (4.5), we have

$$(\|\tilde{u}\|^2)_t + \tau(2\theta - 1 - \varepsilon)\|\tilde{u}_t\|^2 + \nu(1 - 2\varepsilon)|\tilde{u}|_1^2 + \nu\tau(\sigma + \theta)(|\tilde{u}|_1^2)_t + \beta|\tilde{p} + \theta\tau\tilde{p}_t|_1^2 + \nu\tau^2(2\sigma\theta - \sigma - \theta - 3\varepsilon)|\tilde{u}_t|_1^2 \le M_1\|\tilde{u}\|^2 + B(\|\tilde{u}\|)|\tilde{u}|_1^2 + G_1(t) \tag{4.6}$$

where

$$
\begin{aligned}
M_1 &= 1+\frac{C\{(1+\theta^4)(1+\delta)^4+(\theta-\delta)^4\}}{\varepsilon^3\nu^3}|||u|||_1^4,\\
B(\|\tilde{u}\|) &= -\nu+\frac{C(\theta-\delta)^2N^{2\alpha_n}}{\varepsilon\nu}\|\tilde{u}\|^2,\\
G_1(t) &= (1+\frac{\tau\theta^2}{\varepsilon})\|\tilde{f}\|^2+\frac{C}{\beta}\|\tilde{g}+\theta\tau\tilde{g}_t\|^2.
\end{aligned}
$$

We are now in a position to choose the constant ε. Let $r_0\geq 0$ be sufficiently small. If $\sigma>\frac{1}{2}$ and $\theta>\frac{\sigma}{2\sigma-1}$, we take ε and r_0 such that

$$
2\theta\geq \max\left(1+\varepsilon+r_0,\frac{2(\sigma+3\varepsilon)}{2\sigma-1}\right),
$$

thus we have

$$
\tau(2\theta-1-\varepsilon)\|\tilde{u}_t\|^2+\nu\tau^2(2\sigma\theta-\sigma-\theta-3\varepsilon)|\tilde{u}_t|_1^2\geq r_0\tau\|\tilde{u}_t\|^2. \tag{4.7}
$$

If $\sigma\leq\frac{\theta}{2\theta-1}$, and

$$
\frac{4\nu\tau}{3}N^4<\frac{2\theta-1}{\sigma+\theta-2\sigma\theta}, \tag{4.8}
$$

we take ε and r_0 such that

$$
2\theta>1+r_0+\varepsilon+\frac{4\nu\tau}{3}(\sigma+\theta-2\sigma\theta+3\varepsilon)N^4.
$$

In this case, (4.7) also holds, due to Lemma 3. Consequently, we have in both cases that

$$
\begin{aligned}
&(\|\tilde{u}\|^2)_t+r_0\tau\|\tilde{u}_t\|^2+\frac{\nu}{2}|\tilde{u}|_1^2+\nu\tau(\sigma+\theta)(|\tilde{u}|_1^2)_t+\beta|\tilde{p}+\theta\tau\tilde{p}_t|_1^2\\
&\quad\leq M_1\|\tilde{u}\|^2+B(\|\tilde{u}\|)|\tilde{u}|_1^2+G_1(t).
\end{aligned} \tag{4.9}
$$

Let

$$
\begin{aligned}
E(\tilde{u},\tilde{p},t) &= \|\tilde{u}(t)\|^2+\tau\sum_{t'\in S_\tau,t'<t}\left\{r_0\tau\|\tilde{u}_t(t')\|^2+\frac{\nu}{2}|\tilde{u}(t')|_1^2+\beta|\tilde{p}(t')+\theta\tau\tilde{p}_t(t')|_1^2\right\},\\
\rho(t) &= \|\tilde{u}(0)\|^2+\nu\tau(\sigma+\theta)|\tilde{u}(0)|_1^2+\tau\sum_{t'\in S_\tau,t'<t}G_1(t').
\end{aligned}
$$

By summing (4.9) for all $t'\in S_\tau$ and $t'<t$, we obtain

$$
E(\tilde{u},\tilde{p},t)\leq\rho(t)+\tau\sum_{t'\in S_\tau,t'<t}\left\{M_1E(\tilde{u},\tilde{p},t')+B(\|\tilde{u}(t')\|)|\tilde{u}(t')|_1^2\right\}.
$$

Using Lemma 6, we have the following result.

Theorem 1 *Assume that*

(i) $\rho(T)e^{M_1 T} \leq \dfrac{\varepsilon \nu^2}{C(\theta-\delta)^2 N^{2\alpha_n}}$.

In addition, either of the following two conditions is satisfied:

(ii) $\sigma > \dfrac{\theta}{2\theta-1}$,

(iii) $\sigma \leq \dfrac{\theta}{2\theta-1}$ and $\dfrac{4\nu\tau}{3}N^4 < \dfrac{2\theta-1}{\sigma+\theta-2\sigma\theta}$.

Then for all $t \in S_\tau$,

$$E(\tilde{u},\tilde{p},t) \leq \rho(t)e^{M_1 t}.$$

5. The convergence

We now turn to consider the convergence of Scheme $(2.3)_1$–$(2.3)_2$. First we introduce two function spaces which the exact solutions belong to. Let B be a Banach space with the norm $\|\cdot\|_B$. We denote by $C(a,b;B)$ $(a<b)$ the space of strongly continuous functions $w(z)$ from $[a,b]$ to B, and by $L^2(a,b;B)$ the space of measurable functions $v(z)$ from (a,b) to B satisfying respectively

$$\|w\|_{C(a,b;B)} = \max_{z\in[a,b]} \|w(z)\|_B < \infty, \quad \|v\|_{L^2(a,b;B)} = \left(\int_a^b \|v(z)\|_B^2 \,\mathrm{d}z\right)^{\frac{1}{2}} < \infty.$$

Let the pair (U,P) be the solution of (2.2) and

$$U^*(t) = P_N^{1,0}U(t), \quad P^*(t) = \tilde{P}_N^1 P(t).$$

Then we have from (2.2) that

$$\begin{cases} (U_t^*,v) + J(U^*+\delta\tau U_t^*, U^*, v) + (\nabla(P^*+\theta\tau P_t^*),v) + \nu(\nabla(U^*+\sigma\tau U_t^*),\nabla v) \\ \quad = (f,v) + \sum\limits_{l=1}^{6} E_l(v), \qquad \forall v \in (V_N)^n, \\ \beta(\nabla(P^*+\theta\tau P_t^*),\nabla v) + (\nabla\cdot(U^*+\theta\tau U_t^*),v) = \sum\limits_{l=7}^{9} E_l(v), \quad \forall v \in W_N, \\ U^*(0) = P_N^{1,0}U_0, \end{cases} \tag{5.1}$$

where

$$\begin{aligned} &E_1(v) = (U_t^* - \frac{\partial U}{\partial t}, v), && E_2(v) = J(U^*,U^*,v) - J(U,U,v), \\ &E_3(v) = \delta\tau J(U_t^*,U^*,v), && E_4(v) = (\nabla(P^*-P),v), \\ &E_5(v) = \theta\tau(\nabla P_t^*, v), && E_6(v) = \nu\sigma\tau(\nabla U_t, \nabla v), \\ &E_7(v) = \beta(\nabla(P+\theta\tau P_t),\nabla v), && E_8(v) = (\nabla\cdot(U^*-U),v), \\ &E_9(v) = \theta\tau(\nabla\cdot U_t^*, v). \end{aligned}$$

Let the pair (u,p) be the solution of $(2.3)_1$—$(2.3)_2$. Define

$$\tilde{U} = u - U^*, \quad \tilde{P} = p - P^*.$$

By subtracting (5.1) from $(2.3)_1$—$(2.3)_2$, we have

$$\begin{cases} (\tilde{U}_t, v) + J(U^* + \delta\tau U_t^*, \tilde{U}, v) + J(\tilde{U} + \delta\tau\tilde{U}_t, U^* + \tilde{U}, v) + \\ \quad \nu(\nabla(\tilde{U} + \sigma\tau\tilde{U}_t), \nabla v) + (\nabla(\tilde{P} + \theta\tau\tilde{P}_t), v) = -\sum_{l=1}^{6} E_l(v), \qquad \forall v \in (V_N)^n, \\ \beta(\nabla(\tilde{P} + \theta\tau\tilde{P}_t), \nabla v) + (\nabla \cdot (\tilde{U} + \theta\tau\tilde{U}_t), v) = -\sum_{l=7}^{9} E_l(v), \qquad \forall v \in W_N, \\ \tilde{U}(0) = 0. \end{cases}$$

We estimate $|E_l(v)|$, $l = 1, 2, \cdots, 9$. It is easily seen that

$$|E_1(2\tilde{U} + 2\theta\tau\tilde{U}_t)| \le \|\tilde{U}\|^2 + \varepsilon\tau\|\tilde{U}_t\|^2 + (1 + \frac{\tau\theta^2}{\varepsilon})\|U_t^* - \frac{\partial U}{\partial t}\|^2.$$

By using Lemma 1, Lemma 4 and Lemma 5, we have

$$\begin{aligned} |E_2(2\tilde{U} + 2\theta\tau\tilde{U}_t)| &\le |J(U, U^* - U, 2\tilde{U} + 2\theta\tau\tilde{U}_t)| + |J(U^* - U, U^*, 2\tilde{U} + 2\theta\tau\tilde{U}_t)| \\ &\le C(|U|_1 + |U^*|_1)|\tilde{U} + \theta\tau\tilde{U}_t|_1|U^* - U|_1 \\ &\le \varepsilon\nu|\tilde{U}|_1^2 + \varepsilon\nu\tau^2|\tilde{U}_t|_1^2 + \frac{C}{\varepsilon\nu}(1 + \theta^2)\|U\|_1^2|U^* - U|_1^2, \end{aligned}$$

and

$$\begin{aligned} \sum_{l=3}^{6} |E_l(2\tilde{U} + 2\theta\tau\tilde{U}_t)| &\le \varepsilon\nu|\tilde{U}|_1^2 + \varepsilon\nu\tau^2|\tilde{U}_t|_1^2 + \frac{C}{\varepsilon\nu}(1 + \theta^2)\{\delta^2\tau^2\|U_t\|_1^2\|U\|_1^2 + \\ &\quad \|P^* - P\|^2 + \theta^2\tau^2\|P_t\|_1^2 + \nu^2\sigma^2\tau^2\|U_t\|_1^2\}. \end{aligned}$$

Similarly

$$\begin{aligned} |E_7(2\tilde{P} + 2\theta\tau\tilde{P}_t)| &\le \frac{\beta}{3}|\tilde{P} + \theta\tau\tilde{P}_t|_1^2 + 6\beta(1 + \theta^2\tau^2)(|P|_1^2 + |P_t|_1^2) \\ |E_8(2\tilde{P} + 2\theta\tau\tilde{P}_t)| &\le \frac{\beta}{3}|\tilde{P} + \theta\tau\tilde{P}_t|_1^2 + \frac{3}{\beta}\|U^* - U\|^2, \\ |E_9(2\tilde{P} + 2\theta\tau\tilde{P}_t)| &\le \frac{\beta}{3}|\tilde{P} + \theta\tau\tilde{P}_t|_1^2 + \frac{3\theta^2\tau^2}{\beta}\|U_t\|_1^2. \end{aligned}$$

So far, we can obtain a conclusion similar to Theorem 1, but with

$$\tilde{\rho}(t) = \|\tilde{U}(0)\|^2 + \nu\tau(\sigma + \theta)|\tilde{U}(0)|_1^2 + \tau \sum_{t' \in S_\tau, t' < t} G_2(t'),$$

where

$$\begin{aligned} G_2(t) = (1 + \frac{\tau\theta^2}{\varepsilon})\|U_t^* - \frac{\partial U}{\partial t}\|^2 + \frac{C}{\varepsilon\nu}(1 + \theta^2)\{\|U\|_1^2|U^* - U|_1^2 + \\ \delta^2\tau^2\|U_t\|_1^2\|U\|_1^2 + \|P^* - P\|^2 + \theta^2\tau^2\|P_t\|_1^2 + \nu^2\sigma^2\tau^2\|U_t\|_1^2\} + \\ 6\beta(1 + \theta^2\tau^2)(|P|_1^2 + |P_t|_1^2) + \frac{3}{\beta}(\|U^* - U\|^2 + \theta^2\tau^2\|U_t\|_1^2). \end{aligned}$$

It means that if

$$\tilde{\rho}(T) = O(N^{-2\alpha_n}), \tag{5.2}$$

then we have for all $t \in S_\tau$

$$E(\tilde{U}, \tilde{P}, t) = O(\tilde{\rho}(t)).$$

In order to obtain the convergence, we only need to estimate the order of $\tilde{\rho}(t)$ and verify (5.2). Let $r \geq 1$ and $k \geq \max\{1, r-1\}$. Since

$$\frac{\partial U(t)}{\partial t} - U_t(t) = -\frac{1}{\tau}\int_t^{t+\tau}(t+\tau-\xi)\frac{\partial^2 U(\xi)}{\partial \xi^2}\,\mathrm{d}\xi.$$

Then

$$\begin{aligned}\|U_t^*(t) - \frac{\partial U(t)}{\partial t}\| &\leq \|U_t(t) - U_t^*(t)\| + \|U_t(t) - \frac{\partial U(t)}{\partial t}\| \\ &\leq C\tau^{-\frac{1}{2}}N^{-r}\left\{\int_t^{t+\tau}\|\frac{\partial U(t')}{\partial t}\|_r^2\,\mathrm{d}t'\right\}^{\frac{1}{2}} + C\tau^{\frac{1}{2}}\left\{\int_t^{t+\tau}\|\frac{\partial^2 U(t')}{\partial t^2}\|^2\,\mathrm{d}t'\right\}^{\frac{1}{2}}.\end{aligned}$$

By Lemma 1 and Lemma 5, we have

$$\|U^* - U\| \leq CN^{-r}\|v\|_r, \qquad |U^* - U|_1 \leq CN^{1-r}\|U\|_r,$$

$$\|P^* - P\| \leq CN^{-k}\|P\|_k.$$

Furthermore

$$\|U_t\|_1 = \frac{1}{\tau}\left\|\int_t^{t+\tau}\frac{\partial U(t')}{\partial t}\,\mathrm{d}t\right\|_1 \leq \tau^{-\frac{1}{2}}\left(\int_t^{t+\tau}\left\|\frac{\partial U(t')}{\partial t}\right\|_1^2\,\mathrm{d}t'\right)^{\frac{1}{2}}.$$

$$\|P_t\|_1 \leq \tau^{-\frac{1}{2}}\left(\int_t^{t+\tau}\left\|\frac{\partial P(t')}{\partial t}\right\|_1^2\,\mathrm{d}t'\right)^{\frac{1}{2}}, \qquad \tilde{U}(0) = 0.$$

Thus we have from the above estimates that

$$\tilde{\rho}(t) \leq M_2\{\beta^{-1}(\tau^2 + N^{-2r}) + N^{2-2r} + \beta\},$$

where M_2 is a positive constant depending only on ν and the norms of U and P in the spaces mentioned in the above. Finally by an argument similar to the proof of Theorem 1, we have the following result.

Theorem 2 *Assume that*

(i) condition (ii) or (iii) of Theorem 1 holds;

(ii) For $r > 1$, corresponding to $n = 2$, or $r \geq 2$, corresponding to $n = 3$, and $k \geq \max\{1, r-1\}$,

$$\begin{aligned} U &\in C(0,T;(H_0^1(\Omega))^n \bigcap (H^r(\Omega))^n), \quad \frac{\partial U}{\partial t} \in L^2(0,T;(H^r(\Omega))^n), \\ &\quad \frac{\partial^2 U}{\partial t^2} \in L^2(0,T;(L^2(\Omega))^n), \\ P &\in C(0,T;H^k(\Omega)), \quad \frac{\partial P}{\partial t} \in L^2(0,T;H^1(\Omega)); \end{aligned}$$

(iii) *For certain positive constant* M_3,

$$M_2 e^{M_3 T}\{\beta^{-1}(\tau^2+N^{-2r})+N^{2-2r}+\beta\} \le \frac{\varepsilon\nu^2}{C(\theta-\delta)^2 N^{2\alpha_n}}.$$

Then for all $t \in S_\tau$,

$$||U(t)-u(t)||^2 \le M_4\{\beta^{-1}(\tau^2+N^{-2r})+N^{2-2r}+\beta\},$$

where M_3 *and* M_4 *are two positive constants depending only on* ν *and the norms of* U *and* P *in the spaces as mentioned above.*

Remark 5.1 *Indeed, condition (iii) is not too restrictive. We consider for instance the case* $n=3$, *corresponding* $\alpha_n = 1$. *Then on the assumption that* $r=2$, $k=1$, $\tau = O(N^{-2})$ *and* $C_1 \le \beta N^2 \le C_2$ *for some positive constants* C_1 *and* C_2, *we have*

$$\beta^{-1}(\tau^2+N^{-2r})+N^{2-2r}+\beta = O(N^{-2}) = O(N^{-2\alpha_n}).$$

Remark 5.2 *If* $\theta=\delta$, *then condition (iii) can be eliminated.*

References

1 Temam R. *Navier-Stokes Equations.* North-Holland, Amsterdam, 1977.

2 Guo Benyu. *Spectral method for solving Navier-Stokes equation.* Scientia Sinica. Series A, 1985, **28**: 1139~1153

3 Brezzi F and Pitkäranta J. *On the stabilization of finite element approximation of the Stokes problem.* in Efficient Solutions of Elliptic Systems, Notes on Numerical Fluid Mechanics, Vol. 10, ed. by Hackbusch, W., Viewig, 1984, 11~19

4 Shen Jie. *On pressure stabilization method and projection method for unsteady Navier-Stokes equations.* in Advances in Computer Methods for Partial Differential Equations VII. ed. by Vichnevetsky, R., Knight, D. and Richter, G., IMACS, 1992, 658~662

5 Canuto C, Hussaini M Y, Quarteroni A and Zang T A. *Spectral Method in Fluid Dynamics.* Springer-Verlag, Berlin, 1988.

6 Guo Benyu. *Difference Methods for Partial Differential Equations.* Science Press, Beijing, 1988.

发展型 Navier-Stokes 方程基于压力稳定化方法的谱逼近格式

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摘　要

本文采用压力稳定化方法近似模拟不可压缩条件，进而构造了发展型非周期 Navier-Stokes 方程的全离散 Legendre 谱逼近计算格式，严格分析了格式的广义稳定性与收敛性．本文建立的逼近结果也适用其它非周期问题．