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On Totally Real Pseudo-Umbilical Submanifolds in a Complex Projective Space

ZHANG Liang
(College of Mathematics and Computer Science, Anhui Normal University, Anhui 241000, China)
(E-mail: zhliang43@sohu.com)

Abstract Let M^n be a totally real submanifold in a complex projective space $\mathbf{C}P^{n+p}$. In this paper, we study the position of the parallel umbilical normal vector field of M^n in the normal bundle. By choosing a suitable frame field, we obtain a pinching theorem, in the case $p>0$, for the square of the length of the second fundamental form of a totally real pseudo-umbilical submanifold with parallel mean curvature vector.

Keywords complex projective space; totally real submanifolds; pseudo-umbilical submanifolds; parallel mean curvature vector.

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1. Introduction

Let $\mathbf{C}P^{n+p}$ be a $2(n+p)$-dimensional complex projective space endowed with the Fubini-Study metric of constant holomorphic sectional curvature 4. Let M^n be an n-dimensional submanifold in $\mathbf{C}P^{n+p}$. M^n is called totally real if each tangent space of M^n is mapped into the normal space by the complex structure J of $\mathbf{C}P^{n+p}$. In [1,2], in the case $p=0$, a pinching theorem and the corresponding rigidity theorem for totally real minimal submanifolds were obtained. In [3], Du investigated the conditions under which totally real pseudo-umbilical submanifolds must be minimal for $p=0$. In this paper, we study the totally real pseudo-umbilical submanifolds with parallel mean curvature vector in $\mathbf{C}P^{n+p}$ for general complex codimension p. In order to choose a suitable frame field, one should consider the position of parallel umbilical normal vector field of totally real submanifold M^n in the normal bundle.

Let TM^n and $T^{\perp}M^n$ be the tangent bundle and the normal bundle of submanifold M^n. Write

$$V=\bigcup_{x\in M^n} J(T_xM^n).$$

Obviously, V is a rank n subbundle of $T^{\perp}M^n$, $J: T_xM^n\to V_x$ is an isomorphism. Denote by $V^{\perp}$ the orthogonal complement of V in $T^{\perp}M^n$. For parallel umbilical normal vector field, we

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first prove

Theorem 1 *Let M^n be an $n(\geq 2)$-dimensional totally real submanifold in $\mathbf{C}P^{n+p}$ and ξ be a parallel umbilical normal vector field of M^n. Then ξ must be in $V^\perp$, i.e., $\xi \in C^\infty(V^\perp)$.*

Remark In [3], it is proved that totally real pseudo-umbilical submanifolds with parallel mean curvature vector in $\mathbf{C}P^n$ must be minimal. This result is a direct corollary of Theorem 1.

Furthermore, according to Theorem 1, by choosing a suitable moving frame, we get the following pinching theorem for the square of the length of the second fundamental form.

Theorem 2 *Let M^n be an n-dimensional compact totally real pseudo-umbilical submanifold with parallel mean curvature in $\mathbf{C}P^{n+p}$, $n \geq 2$, $p \geq 1$. If the square of the length of the second fundamental form*

$$S \leq \frac{n}{3}(2 + 5H^2),$$

where H is the mean curvature of M, then either $S = nH^2$, M^n is totally umbilical, or $S = \frac{n}{3}(2 + 5H^2)$. In the latter case, $n = 2$ and M^2 is locally isometric to the sphere $S^2(\sqrt{\frac{3}{1+H^2}})$ of radius $\sqrt{\frac{3}{1+H^2}}$.

2. Basic formulas

Let M^n be an n-dimensional totally real submanifold in $\mathbf{C}P^{n+p}$. Choose a local field of orthonormal frames

$$\begin{aligned} e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+p}, e_{1^*} = Je_1, \ldots, e_{n^*} = Je_n, e_{(n+1)^*} = Je_{n+1}, \ldots, e_{(n+p)^*} \\ = Je_{n+p} \end{aligned} \tag{2.1}$$

in $\mathbf{C}P^{n+p}$, in such a way that, restricted to M^n, $\{e_1, \ldots, e_n\}$ are tangent to M^n. For convenience, we use the following convention on the range of indices:

$$\begin{aligned} A, B, C, \ldots &= 1, \ldots, n+p,\ 1^*, \ldots, (n+p)^*; \\ i, j, k, \ldots &= 1, \ldots, n; \\ \alpha, \beta, \gamma, \ldots &= n+1, \ldots, n+p,\ 1^*, \ldots, (n+p)^*; \\ \lambda, \mu, \ldots &= n+1, \ldots, n+p. \end{aligned}$$

Let $\{\omega^A\}$ be the dual frames of $\{e_A\}$. Then the structure equations of $\mathbf{C}P^{n+p}$ are given by

$$\mathrm{d}\omega^A = \sum_B \omega^B \wedge \omega_B^A, \tag{2.2}$$

$$\mathrm{d}\omega_B^A = \sum_C \omega_B^C \wedge \omega_C^A + \frac{1}{2} \sum_{C,D} K_{ABCD}\, \omega^C \wedge \omega^D, \tag{2.3}$$

where[1]

$$\begin{aligned} \omega_i^j = \omega_{i^*}^{j^*},\ \omega_i^\lambda = \omega_{i^*}^{\lambda^*},\ \omega_i^{j^*} = \omega_j^{i^*}, \\ \omega_i^{\lambda^*} = \omega_\lambda^{i^*},\ \omega_\lambda^\mu = \omega_{\lambda^*}^{\mu^*},\ \omega_\lambda^{\mu^*} = \omega_\mu^{\lambda^*}; \end{aligned} \tag{2.4}$$

$$K_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{CA}J_{DB} - J_{DA}J_{CB} + 2J_{BA}J_{DC}. \tag{2.5}$$

In (2.5), J_{AB} is the component of the linear transformation J, i.e.,

$$(J_{AB}) = \begin{array}{cc} \left(\begin{array}{c|c} \mathbf{0} & I_{n+p} \\ \hline -I_{n+p} & \mathbf{0} \end{array}\right) & \begin{array}{l} \}\, i \\ \}\, \lambda \\ \}\, i^* \\ \}\, \lambda^* \end{array} \\ \underbrace{\quad}_{i}\,\underbrace{\quad}_{\lambda}\,\underbrace{\quad}_{i^*}\,\underbrace{\quad}_{\lambda^*} & \end{array} \tag{2.6}$$

where I_{n+p} denotes the identity matrix of degree $n+p$.

Restricting these forms to M^n, we have

$$\omega^\alpha = 0, \tag{2.7}$$

$$\omega_i^\alpha = \sum_j h_{ij}^\alpha\, \omega^j, \quad h = \sum_{\alpha,i,j} h_{ij}^\alpha\, \omega^i \otimes \omega^j \otimes e_\alpha,$$

$$h_{jk}^{i^*} = h_{ik}^{j^*} = h_{ij}^{k^*}, \tag{2.8}$$

$$\begin{cases} \mathrm{d}\omega^{\mathrm{i}} = \sum_{\mathrm{j}} \omega^{\mathrm{j}} \wedge \omega_{\mathrm{j}}^{\mathrm{i}}, \\ \mathrm{d}\omega_j^i = \sum_k \omega_j^k \wedge \omega_k^i + \frac{1}{2}\sum_{k,l} R_{ijkl}\, \omega^k \wedge \omega^l, \end{cases} \tag{2.9}$$

$$R_{ijkl} = K_{ijkl} + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{2.10}$$

$$\mathrm{d}\omega_\beta^\alpha = \sum_\gamma \omega_\beta^\gamma \wedge \omega_\gamma^\alpha + \frac{1}{2}\sum_{k,l} R_{\alpha\beta kl}\, \omega^k \wedge \omega^l, \tag{2.11}$$

$$R_{\alpha\beta ij} = K_{\alpha\beta ij} + \sum_k (h_{ik}^\alpha h_{kj}^\beta - h_{jk}^\alpha h_{ki}^\beta), \tag{2.12}$$

where h is the second fundamental form of M^n, and R_{ijkl}, $R_{\alpha\beta ij}$ are the components of the Riemannian curvature tensor R and the normal curvature tensor $R^\perp$ with respect to $\{e_A\}$. The mean curvature vector of M^n is defined as follows

$$\zeta = \frac{1}{n}\sum_\alpha \mathrm{tr}H_\alpha \cdot e_\alpha\,,$$

where $\mathrm{tr}H_\alpha$ is the trace of the matrix $H_\alpha = (h_{ij}^\alpha)$. We call $H = |\zeta|$ the mean curvature of M^n. Write

$$S = \sum_\alpha \mathrm{tr}H_\alpha^2.$$

Let ρ be the scalar curvature of M^n. Then we have[1]

$$\rho = n(n-1) + n^2H^2 - S. \tag{2.13}$$

Define the covariant derivative of h_{ij}^α as follows

$$\sum_k h_{ijk}^\alpha\, \omega^k = dh_{ij}^\alpha + \sum_\beta h_{ij}^\beta\, \omega_\beta^\alpha - \sum_l h_{lj}^\alpha\, \omega_i^l - \sum_l h_{il}^\alpha\, \omega_j^l. \tag{2.14}$$

Then[4]

$$h_{ijk}^\alpha - h_{ikj}^\alpha = -K_{\alpha ijk}, \tag{2.15}$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m (h^\alpha_{mi} R_{mjkl} + h^\alpha_{mj} R_{mikl}) - \sum_\beta h^\beta_{ij} R_{\alpha\beta kl}. \tag{2.16}$$

Note that M^n is an invariant submanifold[1] in $\mathbf{C}P^{n+p}$, i.e., $K_{\alpha ijk} = 0$, so

$$h^\alpha_{ijk} = h^\alpha_{ikj}. \tag{2.17}$$

From (2.16), (2.17), the Laplacian of h^α_{ij} is

$$\begin{aligned}\Delta h^\alpha_{ij} &= \sum_k h^\alpha_{ijkk} \\ &= \sum_k h^\alpha_{kkij} + \sum_{k,m} (h^\alpha_{im} R_{mkjk} + h^\alpha_{km} R_{mijk}) - \sum_{\beta,k} h^\beta_{ki} R_{\alpha\beta jk}. \end{aligned} \tag{2.18}$$

For the frame field chosen above (see (2.1)), using (2.5), it is not difficult to obtain the following lemma.

Lemma 2.1 *Let M^n be an n-dimensional totally real submanifold in $\mathbf{C}P^{n+p}$. Then*

(1) $K_{i^* j^* kl} = K_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$;

(2) $K_{\lambda A ij} = 0$, $K_{\lambda^* A ij} = 0$;

(3) $K_{\alpha ijk} = 0$, $K_{\alpha\lambda jk} = 0$.

3. Position of parallel umbilical normal vector field in the normal bundle

Let M^n be a totally real submanifold in $\mathbf{C}P^{n+p}$. $D^\perp$ denotes the normal connection of $T^\perp M^n$. We can split $T^\perp M^n$ into the direct sum

$$T^\perp M^n = V \oplus V^\perp,$$

where V and $V^\perp$ are two subbundles of $T^\perp M^n$ as we state in the introduction.

Proof of Theorem 1 Suppose the parallel umbilical normal vector field $\xi = \xi_1 + \xi_2$, where $\xi_1 \in C^\infty(V)$, $\xi_2 \in C^\infty(V^\perp)$. It suffices to prove that $\xi_1 = 0$. Choose local field of orthonormal frames (2.1) such that

$$\xi_1 = |\xi_1| e_{1^*}, \quad \xi_2 = |\xi_2| e_{n+1}.$$

From that ξ is umbilical, we have

$$|\xi_1| h^{1^*}_{ij} + |\xi_2| h^{n+1}_{ij} = B\delta_{ij},$$

where $B = \langle \xi, \zeta \rangle$. It is clear that the above formula is equivalent to

$$|\xi_1| \omega^{1^*}_i + |\xi_2| \omega^{n+1}_i = B\omega^i.$$

On the other hand, ξ is parallel, so

$$\begin{aligned} 0 =& D^\perp \xi = D^\perp (|\xi_1| e_{1^*} + |\xi_2| e_{n+1}) \\ =& (\mathrm{d}|\xi_1| + |\xi_2| \omega^{1^*}_{n+1}) \otimes e_{1^*} + (\mathrm{d}|\xi_2| + |\xi_1| \omega^{n+1}_{1^*}) \otimes e_{n+1} + \\ & \sum_{\alpha \neq n+1, 1^*} (|\xi_1| \omega^\alpha_{1^*} + |\xi_2| \omega^\alpha_{n+1}) \otimes e_\alpha. \end{aligned}$$

Therefore

$$\begin{cases} \mathrm{d}|\xi_1| + |\xi_2|\omega_{n+1}^{1^*} = 0, & (3.1)\\ \mathrm{d}|\xi_2| + |\xi_1|\omega_{1^*}^{n+1} = 0, & (3.2)\\ |\xi_1|\omega_{1^*}^{\alpha} + |\xi_2|\omega_{n+1}^{\alpha} = 0,\ \alpha \neq n+1\,,1^* & (3.3)\end{cases}$$

Taking exterior derivative of (3.3), we obtain

$$\begin{aligned} 0 =&(\mathrm{d}|\xi_1| + |\xi_2|\omega_{n+1}^{1^*}) \wedge \omega_{1^*}^{\alpha} + (\mathrm{d}|\xi_2| + |\xi_1|\omega_{1^*}^{n+1}) \wedge \omega_{n+1}^{\alpha} + \\ &\sum_{\beta\neq n+1,1^*} (|\xi_1|\omega_{1^*}^{\beta} + |\xi_2|\omega_{n+1}^{\beta}) \wedge \omega_{\beta}^{\alpha} + \\ &\sum_j (|\xi_1|\omega_{1^*}^{j} + |\xi_2|\omega_{n+1}^{j}) \wedge \omega_{j}^{\alpha} + \\ &\frac{|\xi_1|}{2}\sum_{i,j} K_{\alpha 1^* ij}\,\omega^i \wedge \omega^j + \frac{|\xi_2|}{2}\sum_{i,j} K_{\alpha n+1 ij}\,\omega^i \wedge \omega^j \\ =& -B\sum_j \omega^j \wedge \omega_j^{\alpha} + \frac{|\xi_1|}{2}\sum_{i,j} K_{\alpha 1^* ij}\,\omega^i \wedge \omega^j + \frac{|\xi_2|}{2}\sum_{i,j} K_{\alpha n+1 ij}\,\omega^i \wedge \omega^j. \qquad (3.4)\end{aligned}$$

From Lemma 2.1, we know that $K_{\alpha n+1 ij} = 0$. And we also have

$$0 = \mathrm{d}\omega^{\alpha} = \sum_j \omega^j \wedge \omega_j^{\alpha},$$

so (3.4) reduces to

$$\frac{|\xi_1|}{2}\sum_{i,j} K_{\alpha 1^* ij}\omega^i \wedge \omega^j = 0,$$

i.e.

$$|\xi_1| K_{\alpha 1^* ij} = 0.$$

Setting $\alpha = 2^*$ and using Lemma 2.1 gives

$$0 = |\xi_1|(\delta_{2i}\delta_{1j} - \delta_{2j}\delta_{1i}).$$

Setting $i = 2$ and $j = 1$, we have $|\xi_1| = 0$. This completes the proof. □

4. Pinching for the square of the length of the second fundamental form

According to Theorem 1, we can choose a local field of orthonormal frames

$$e_1, \dots, e_n, e_{n+1}, \dots, e_{n+p},$$

$$e_{1^*} = Je_1, \dots, e_{n^*} = Je_n, e_{(n+1)^*} = Je_{n+1}, \dots, e_{(n+p)^*} = Je_{n+p},$$

in $\mathbf{C}P^{n+p}$, in such a way that, restricted to M^n, $\{e_1, \dots, e_n\}$ are tangent to M^n, and $\zeta = He_{n+1}$. It is easy to see that

$$\mathrm{tr}H_{\alpha} = \begin{cases} nH, & \alpha = n+1 \\ 0, & \alpha \neq n+1. \end{cases} \qquad (4.1)$$

Set

$$\tau = S - nH^2.$$

From (2.18), noting that M^n has parallel mean curvature vector and $\sum_k h^\alpha_{kkij}=0$, one gets

$$\frac{1}{2}\Delta\tau=\sum_{\alpha\neq n+1}\sum_{i,j,k}(h^\alpha_{ijk})^2+\sum_{\alpha\neq n+1}\sum_{i,j,k,m}h^\alpha_{ij}(h^\alpha_{im}R_{mkjk}+h^\alpha_{km}R_{mijk})-\sum_{\alpha\neq n+1}\sum_{\beta,k}h^\alpha_{ij}h^\beta_{ki}R_{\alpha\beta jk}. \tag{4.2}$$

By using (2.5), (2.10), (2.12), (4.1) and the fact that M^n is pseudo-umbilical, we can get

$$\begin{aligned}&\sum_{\alpha\neq n+1}\sum_{i,j,k,m}h^\alpha_{ij}(h^\alpha_{im}R_{mkjk}+h^\alpha_{km}R_{mijk})\\&=n(1+H^2)\tau+\frac{1}{2}\sum_{\alpha,\beta\neq n+1}\mathrm{tr}(H_\alpha H_\beta-H_\beta H_\alpha)^2-\sum_{\alpha,\beta\neq n+1}\big(\mathrm{tr}(H_\alpha H_\beta)\big)^2,\end{aligned} \tag{4.3}$$

$$\begin{aligned}&\sum_{\alpha\neq n+1}\sum_{\beta,k}h^\alpha_{ij}h^\beta_{ki}R_{\alpha\beta jk}\\&=-\sum_i\mathrm{tr}H^2_{i^*}-\frac{1}{2}\sum_{\alpha,\beta\neq n+1}\mathrm{tr}\left(H_\alpha H_\beta-H_\beta H_\alpha\right)^2.\end{aligned} \tag{4.4}$$

From (4.2), (4.3), (4.4), we have

Lemma 4.1 *For any real number a, the Laplacian of τ is*

$$\begin{aligned}\frac{1}{2}\Delta\tau=&\sum_{\alpha\neq n+1}\sum_{i,j,k}\left(h^\alpha_{ijk}\right)^2+\sum_i\mathrm{tr}H^2_{i^*}-an(1+H^2)\tau+\\&(1+a)\sum_{\alpha\neq n+1}\sum_{i,j,k,m}h^\alpha_{ij}\left(h^\alpha_{im}R_{mkjk}+h^\alpha_{km}R_{mijk}\right)+\\&\frac{1-a}{2}\sum_{\alpha,\beta\neq n+1}\mathrm{tr}\left(H_\alpha H_\beta-H_\beta H_\alpha\right)^2+a\sum_{\alpha,\beta\neq n+1}\big(\mathrm{tr}(H_\alpha H_\beta)\big)^2.\end{aligned}$$

Lemma 4.2[5] *Let $A_1,A_2,\ldots,A_m$ be symmetric $(n\times n)$-matrices, $m\geq 2$. Then*

$$\sum_{\alpha,\beta=1}^m\mathrm{tr}(A_\alpha A_\beta-A_\beta A_\alpha)^2-\sum_{\alpha,\beta=1}^m\big(\mathrm{tr}(A_\alpha A_\beta)\big)^2\geq-\frac{3}{2}\Big(\sum_{\alpha=1}^m\mathrm{tr}(A^2_\alpha)\Big)^2,$$

and the equality holds if and only if one of the following cases occurs:

(1) $A_1=\cdots=A_m=0$;

(2) Only two of the matrices $A_\alpha(\alpha=1,\ldots,m)$ cannot be zero. Moreover, if $A_1\neq 0,A_2\neq 0,A_3=\cdots=A_m=0$, then $\mathrm{tr}A_1^2=\mathrm{tr}A_2^2$, and there exists an orthogonal matrix T, such that

$$TA_1T^{-1}=\frac{1}{2}\sqrt{\sum_{\alpha=1}^m\mathrm{tr}A^2_\alpha}\left(\begin{array}{cc|c}0&1&\\1&0&\mathbf{0}\\\hline\multicolumn{2}{c|}{\mathbf{0}}&\mathbf{0}\end{array}\right),TA_2T^{-1}=\frac{1}{2}\sqrt{\sum_{\alpha=1}^m\mathrm{tr}A^2_\alpha}\left(\begin{array}{cc|c}1&0&\\0&-1&\mathbf{0}\\\hline\multicolumn{2}{c|}{\mathbf{0}}&\mathbf{0}\end{array}\right).$$

Proof of Theorem 2 Setting $a=-1$ in Lemma 4.1, from Lemma 4.2, one can get

$$\frac{1}{2}\Delta\tau\geq\sum_{\alpha\neq n+1}\sum_{i,j,k}(h^\alpha_{ijk})^2+\sum_i\mathrm{tr}H^2_{i^*}+\tau\Big(n(1+H^2)-\frac{3}{2}\tau\Big). \tag{4.5}$$

If $S\leq\frac{n}{3}(2+5H^2)$, i.e., $\tau\leq\frac{2}{3}n(1+H^2)$, then the right hand of (4.5) is nonnegative and τ is a subharmonic function on M^n. By Hopf's maximum principle, we know that τ is a constant.

Hence

$$h^{\alpha}_{ijk} = 0, \quad \forall i, j, k, \alpha \tag{4.6}$$

$$H_{i^*} = 0, \quad \forall i \tag{4.7}$$

$$\tau\big(n(1+H^2) - \frac{3}{2}\tau\big) = 0. \tag{4.8}$$

(4.8) implies: if $S < \frac{n}{3}(2+5H^2)$, i.e., $\tau < \frac{2}{3}n(1+H^2)$, then $\tau = 0$, M^n is totally umbilical, and if $S = \frac{n}{3}(2+5H^2)$, i.e., $\tau = \frac{2}{3}n(1+H^2)$, then

$$\sum_{\alpha,\beta\neq n+1} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha,\beta\neq n+1} (\mathrm{tr}(H_\alpha H_\beta))^2 = -\frac{3}{2}\tau^2.$$

From Lemma 4.2 and (4.7), there exist $\alpha_1, \alpha_2 \in \{n+2, \dots, n+p, (n+1)^*, \dots, (n+p)^*\}$, such that

$$H_{\alpha_1} = \frac{\sqrt{\tau}}{2}\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} \end{array}\right), \quad H_{\alpha_2} = \frac{\sqrt{\tau}}{2}\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} \end{array}\right), \tag{4.9}$$

$$H_\alpha = 0, \quad \alpha \neq n+1, \alpha_1, \alpha_2. \tag{4.10}$$

Using (4.6), (4.10) and noting that M^n has parallel mean curvature vector, $\omega^{\alpha}_{n+1} = 0$, we have

$$\begin{aligned} 0 &= \mathrm{d}h^{\alpha}_{ij} + \sum_\beta h^{\beta}_{ij}\omega^{\alpha}_{\beta} - \sum_l h^{\alpha}_{lj}\omega^{l}_{i} - \sum_l h^{\alpha}_{il}\omega^{l}_{j} \\ &= \mathrm{d}h^{\alpha}_{ij} + h^{\alpha_1}_{ij}\omega^{\alpha}_{\alpha_1} + h^{\alpha_2}_{ij}\omega^{\alpha}_{\alpha_2} - \sum_l h^{\alpha}_{lj}\omega^{l}_{i} - \sum_l h^{\alpha}_{il}\omega^{l}_{j}. \end{aligned}$$

Setting $\alpha = \alpha_1$, $i = 1$, $j \geq 3$ and $\alpha = \alpha_2, i = 2, j \geq 3$, we see that

$$\omega^2_j = 0, \quad j \geq 3, \tag{4.11}$$

$$\omega^1_j = 0, \quad j \geq 3. \tag{4.12}$$

Taking exterior derivative of (4.11) and using (4.9)–(4.12) and Lemma 2.1 gives

$$0 = \omega^2 \wedge \omega^j, \quad j \geq 3.$$

This shows that $n = 2$. By Gauss equation, we compute the sectional curvature of M^2

$$\begin{aligned} R_{1212} &= K_{1212} + \sum_\alpha [h^{\alpha}_{11}h^{\alpha}_{22} - (h^{\alpha}_{12})^2] = 1 + H^2 - \big(\frac{\sqrt{\tau}}{2}\big)^2 - \big(\frac{\sqrt{\tau}}{2}\big)^2 \\ &= \frac{1}{3}(1+H^2). \end{aligned}$$

Since H is a constant, M^2 is locally isometric to the sphere $S^2\big(\sqrt{\frac{3}{1+H^2}}\big)$. This completes the proof. $\square$

Remark From (2.13), Theorem 2 also gives a pinching constant $n(n-\frac{5}{3})(1+H^2)$ for the scalar curvature.

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