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Note on Existence of Periodic Solutions of One Class Differential Equations *

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Abstract In the present paper we investigate the number of periodic solutions of the following differential equation

$$\frac{dy}{dt} = \frac{A_1(t)y + A_2(t)y^2 + A_3(t)y^3}{a_0(t) + a_1(t)y + a_2(t)y^2} \qquad (**)$$

which was discussed in paper [1,2], and obtain the theorem by the method of cross - ratio of the solutions of $(**)$ without the traditional condition assumption that the functions $A_i(t)$, $a_j(t)$ $(i = 1,2,3;\ j = 0,1,2)$ are differential.

Keywords differential equation, periodic solution, existence, transformation.

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1. Introduction

It's well - known that the question on the existence and the number of periodic solutions about differential equation plays an important role in the studies of nonlinear oscillation and the qualtative theory of ordinary differential equations. We see easily that the results in this area is useful to the existence and uniqueness of limit cycle of differential systems and applied widely in Physics and Engineering.

Condsider the following differential equation

$$\frac{dy}{dt} = f(t,y) = \frac{A_1(t)y + A_2(t)y^2 + A_3(t)y^3}{a_0(t) + a_1(t)y + a_2(t)y^2}. \qquad (1.1)$$

Constructing one strip domain such that the solution of (1.1) $y(t)$ in the boundary of this domain will not leave it, and then by using Brouwer fixed point theory, it is easy to get the theorems for the existence of periodic solutions of (1.1) [1,2]. Comrade Lilin once obtained the theorem for the number of periodic solutions of (1.1) by the Lloyd's method in [2] in the above theorem, however, it is needed

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that the condition at the functions $A_i(t)$, $a_j(t)$ $(i=1,2,3;\ j=0,1,2)$ is not only continuous T-periodic, but also differential (see Theorem 2.1 in [2]).

Out results in this paper, obtained by the method of cross-ratio of solutions of (1.1), gives the sufficient conditions under which equation (1.1) has at most two non-trivial periodic solutions without the traditional assumption that $A_i(t)$, $a_j(t)$ $(i=1,2,3;\ j=0,1,2)$ are differential functions, as is well known such an assumption plays an important roal in the proofs of many results [1,2,etc].

2. Results in [2]

Change equation (1.1) to following form

$$\frac{dz}{dt}=P_0(t)+P_1(t)z+\cdots+P_n(t)z^n, \tag{2.1}$$

where $P_i(t)$ $(i=0,1,2,\cdots,n)$ are T-periodic functions, n may be inifinity, the number of T-periodic solutions of (2.1) has been discussed in paper[3,4], thus we can use these results for ours. Firstly, we consider the problem to transform (1.1) into (2.1).

Generally, $a_0(t)+a_1(t)y+a_2(t)y^2$ may be decomposed into $(b_0(t)+b_1(t)y)(c_0(t)+c_1(t)y)$, where $b_0(t)$, $c_0(t)$, $b_1(t)$, $c_1(t)$ are real functions, and $b_0(t)c_0(t)=a_0(t)$, $b_1(t)c_1(t)=a_2(t)$, $b_0(t)c_1(t)+c_0(t)b_1(t)=a_1(t)$. Here, we assume $a_1^2(t)-4a_0(t)a_2(t)>0$, then this decomposition is suitable. Let

$$r=\frac{y}{b_0(t)+b_1(t)y}\quad\text{then}\quad y=\frac{b_0(t)r}{1-b_2(t)r}. \tag{2.2}$$

Transformation (2.2) is topological in the neighbourhood of $y=0$. By the way, we always assume that the results of this section only hold in the neighbourhood of $y=0$. Using (2.2), (1.1) becomes

$$\frac{dr}{dt}=\frac{(B_1r+B_2^2r+B_3^3r+B_4^4r)-\left(\frac{\dot b_0r}{b_0}+\frac{(b_0\dot b_1-b_1\dot b_0)r^2}{b_0}\right)}{(c_0+er)}, \tag{2.3}$$

where the “ · ”denote the derivative with respect to t, and

$$\begin{aligned}
e&=(a_1^2-4a_0a_2)^{\frac{1}{2}},\\
B_1&=\frac{A_1}{b_0},\\
B_2&=A_2-\frac{3A_1b_1}{b_0},\\
B_3&=A_3b_0-2A_2b_1+\frac{3A_1b_1^2}{b_0},\\
B_4&=A_2b_1^2-A_3b_0b_1-\frac{A_1b_1^3}{b_0}.
\end{aligned} \tag{2.4}$$

Let $z=\dfrac{r}{c_0+er}$, then

$$r=\frac{c_0z}{1-ez}, \tag{2.5}$$

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We can think transformation (2.5) is topological. By (2.5), (2.3) becomes

$$\frac{dz}{dt} = P_1 z + P_2 z^2 + P_3 z^3 + \frac{P_4 z^4}{1 - ez} \tag{2.6}$$

or

$$\frac{dz}{dt} = \sum_{n=1}^{\infty} P_n z^n, \tag{2.7}$$

where

$$\begin{aligned} P_1 &= \frac{(B_1 - \dot{c}_0)}{c_0} - \frac{\dot{b}_0}{b_0}, \\ P_2 &= \frac{\dot{c}_0 e - c_0 \dot{e} - 2B_1 c_1}{c_0} + B_2 + \frac{\dot{b}_0 e - b_0 \dot{b}_1 + b_1 \dot{b}_0}{b_0}, \\ P_3 &= B_3 c_0 - B_2 c_1 + \frac{B_1 c_1^2}{c_0}, \\ P_4 &= B_4 c_0^2, \\ P_5 &= B_4 c_0^2 e. \end{aligned} \tag{2.8}$$

Now, we assume $B_4 = 0$, that is

$$A_3 y^2 + A_2 y + A_1 = 0, \tag{2.9}$$

where $y = \frac{-b_0(t)}{b_1(t)}$, then $P_k (k > 3)$ of (2.7) is zero. By the results of [3], we give the following

Theorem *If $a_i(t), A_j(t)$ $(i = 0, 1, 2; j = 1, 2, 3)$ are continuous differential T-periodic functions, (2.9) holds. Then when*

$$B_1 + B_2 u + b_3 u^2 > 0, \tag{2.10}$$

where $u = \frac{-c_0(t)}{c_1(t)}$. equation (1.1) has at most two nontrivial T-periodic solutions.

Proof Since (2.9) holds, (2.7) becomes

$$\frac{dz}{dt} = P_1 z + P_2 z^2 + P_3 z^3. \tag{2.11}$$

The results of paper [3] guarantee that (2.11) has at most three T-periodic solutions, when (2.10) holds, that is the P_3 of (2.11) is more than zero. Because $z = 0$ is one of T-periodic solutions of (2.11), and transformation (2.2), (2.5) are topological, then (1.1) and (2.7) has the same number of T-periodic solutions, that is, three. But $z = 0$, that is, $y = 0$ is trivial. Then the conclusion of Theorem 2.1 is true.

3. Our Main Results

Firstly, we introduce the following lemma which is needed in the proof of the Theorem.

Lemma[5] *If I is some open interval, $y \in I$, $g''(y)$ exists and $g''(y) \geq 0$, and let $a, b, c, d \in I$ such that $a < b < c < d$, then*

$$\frac{1}{d + c - b - a}\left[\frac{d - b}{b - a}\int_a^b g(y)\,dy + \frac{c - a}{d - c}\int_c^d g(y)\,dy\right] \geq \frac{1}{c - b}\int_b^c g(y)\,dy. \tag{3.1}$$

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Which also can be written

$$\frac{d-c}{(c-a)(d-a)}\int_a^b g(y)\,dy+\frac{b-a}{(d-b)(d-a)}\int_c^d g(y)\,dy$$

$$\geq\frac{(d-c)(b-a)(d+c-b-a)}{(c-a)(d-a)(d-b)(c-b)}\int_b^c g(y)\,dy. \tag{3.2}$$

The equality sign holds if and only if $g(y)$ is linear for $y\in[a,d]$. If $g''(y)$ exists and $g''(y)\leq 0$, then the inequality sign of (3.1) and (3.2) is "$\leq$" and the equality sign holds if and only if $g(y)$ is linear for $y\in[a,d]$.

We consider the differential equation (1.1). Where $A_i(t)$, $a_j(t)$ ($i=1,2,3$; $j=0,1,2$) are continuous T-periodic functions, may not be differential. Our main idea is similar to that of Theorem 1 in paper [5], that is, if $\varnothing_i(t)$ ($i=1,2,3,4$) are four different T-periodic solutions on the same subinterval J of I, and $f'(t,y)$ is concave (convex) in y for any fixed $t\in I$, by applying the lemma, the cross-ratio (with the solutions properly number)

$$R(t)=\frac{\varnothing_3(t)-\varnothing_1(t)}{\varnothing_4(t)-\varnothing_1(t)}:\frac{\varnothing_3(t)-\varnothing_2(t)}{\varnothing_4(t)-\varnothing_2(t)},\quad t\in J$$

is monotone increasing (monotone decresing), which contradicts with the hypothsis that $R(t)$ is T-periodic.

Here, we also assume that $a_1(t)^2-4a_0(t)a_2(t)>0$, then $a_0(t)+a_1(t)y+a_2(t)y^2$ may be decomposed into $(b_0(t)+b_1(t)y)(c_0(t)+c_1(t)y)$, where $b_i(t)$, $c_i(t)$, $i=0,1$; are real functions, and $b_0(t)c_0(t)=a_0(t)$, $b_1(t)c_0(t)=a_2(t)$, $b_0(t)c_1(t)+c_0(t)b_1(t)=a_1(t)$, thus, we have

$$\begin{aligned} f(t,y) &= \frac{A_1(y)+A_2(t)y^2+A_3(t)y^3}{a_0(t)+a_1(t)y+a_2(t)y^2}\\ &= \frac{A_1(t)y+A_2(t)y^2+A_3(t)y^3}{(b_0(t)+b_1(t)y)(c_0(t)+c_1(t)y)}\\ &= \frac{d_0(t)}{b_0(t)+b_1(t)y}+\frac{d_1(t)}{c_0(t)+c_1(t)y}+d_2(t)y+d_3(t). \end{aligned}$$

Which $d_i(t)$ ($i=0,1,2,3$) satisfy that $d_0c_0+d_1b_0+d_0c_1y+d_1b_1y+(d_2y+d_3)(a_0+a_1y+a_2y^2)=A_1y+A_2y^2=A_3y^3$, that is

$$\left.\begin{aligned} d_0c_0+d_1b_0+d_3a_0 &= 0\\ d_0c_1+d_1b_1+a_0d_2+d_3a_1 &= A_1\\ d_2a_1+d_3a_2 &= A_2\\ d_2a_2 &= A_3 \end{aligned}\right\}\text{则} \tag{3.4}$$

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We define that Ω_1 denotes the domain $\{ y : b_0(t) + b_1(t) y > 0$ and $c_0(t) + c_1(t) y > 0\}$, Ω_2 denotes the domain $\{ y : b_0(t) + b_1(t) y > 0$ and $c_0(t) + c_1(t) y < 0\} \cup \{ y : b_0(t) + b_1(t) y < 0$ and $c_0(t) + c_1(t) y > 0\}$ and Ω_3 denotes the domain $\{ b_0(t) + b_1(t) y < 0$ and $c_0(t) + c_1(t) y < 0\}$.

Derivate $f(t, y)$ in y, we have

$$\begin{aligned} f'(t, y) &= d_2 - \frac{d_0 b_1}{(b_0 + b_1 y)^2} - \frac{d_1 c_1}{(c_0 + c_1 y)^2}, \\ f''(t, y) &= \frac{2 d_0 b_1^2}{(b_0 + b_1 y)^3} + \frac{2 d_1 c_1^2}{(c_0 + c_1 y)^3}, \\ f'''(t, y) &= - \frac{6 d_0 b_1^3}{(b_0 + b_1 y)^4} - \frac{6 d_1 c_1^3}{(c_0 + c_1 y)^4}. \end{aligned}$$

Thus, we obtain the following

Theorem 1 *If $A_i(t)$, $a_j(t)$ $(i = 1, 2, 3; j = 0, 1, 2)$ are continuous T-periodic functions, $a_1^2 - 4 a_0 a_2 > 0$, and $-d_0 b_1 \geq 0$, $-d_1 c_1 \geq 0$, (or $-d_0 b_1 \leq 0$, $-d_1 c_1 \leq 0$). Then equation (1.1) has at most two non-trivial T-periodic solutions at the domain Ω_1 or Ω_2 or Ω_3.*

Proof We discuss the case at the domain Ω_1 the other cases is treated analogously.

Recall that through any point $(t_0, \zeta) \in I \times R$, there passes a unique maximal solution $\varphi(t, t_0, \zeta)$ $(t \in I(t_0, \zeta)$ of (1.1)), in the following $t_0 \in [0, T]$ is arbitrary, but fixed.

Next, we assume $\varphi_i(t) = \varphi(t, t_0, \zeta_i)$ $(i = 1, 2, 3, 4; \zeta_1 < \zeta_2 < \zeta_3 < \zeta_4)$ are four different T-periodic solutions of (1.1) on the same interval $J \subset I$, and such that $\varphi_1 < \varphi_2 < \varphi_3 < \varphi_4$.

From (1.1), we get

$$\frac{\dot{\varphi}_2(t) - \dot{\varphi}_1(t)}{\varphi_2(t) - \varphi_1(t)} = \int_{\varphi_1(t)}^{\varphi_2(t)} \frac{f'(t, y)}{\varphi_2(t) - \varphi_1(t)} dy, \quad t \in J. \tag{3.5}$$

And by intergration

$$\ln \frac{\varphi_2(t) - \varphi_1(t)}{\zeta_2 - \zeta_1} = \iint_{a(t)} \frac{f'(\tau, y)}{\varphi_2(\tau) - \varphi_1(\tau)} d\tau dy, \tag{3.6}$$

where $a(t)$ denotes the domain $t_0 \leq \tau \leq t$, $\varphi_1(t) \leq y \leq \varphi_2(t)$ (see Fig. 1). From (3.1) and (3.6), we get

$$\begin{aligned} \ln \frac{R(t)}{R(t_0)} = & \iint_{a(t) \cup b(t)} \frac{f'(\tau, y)}{\varphi_3(\tau) - \varphi_1(\tau)} d\tau dy - \iint_{a(t) \cup b(t) \cup c(t)} \frac{f'(\tau, y)}{\varphi_4(\tau) - \varphi_1(\tau)} d\tau dy - \\ & \iint_{b(t)} \frac{f'(\tau, y)}{\varphi_3(\tau) - \varphi_2(\tau)} d\tau dy + \iint_{b(t) \cup c(t)} \frac{f'(\tau, y)}{\varphi_4(\tau) - \varphi_2(\tau)} d\tau dy \\ = & \iint_{a(t)} \frac{\varphi_4 - \varphi_3}{(\varphi_3 - \varphi_1)(\varphi_4 - \varphi_1)} f'(\tau, y) d\tau dy + \\ & \iint_{c(t)} \frac{\varphi_2 - \varphi_1}{(\varphi_4 - \varphi_2)(\varphi_4 - \varphi_1)} f'(\tau, y) d\tau dy - \\ & \iint_{b(t)} \frac{(\varphi_4 - \varphi_3)(\varphi_2 - \varphi_1)(\varphi_4 + \varphi_3 - \varphi_2 - \varphi_1)}{(\varphi_3 - \varphi_1)(\varphi_4 - \varphi_1)(\varphi_4 - \varphi_2)(\varphi_3 - \varphi_2)} f'(\tau, y) d\tau dy, \end{aligned} \tag{3.7}$$

where $b(t)$ denotes the domain $t_0 \leq \tau \leq t$, $\varphi_2(t) \leq y \leq \varphi_3(t)$, and $c(t)$ denotes the domain

$$t_0 \leq \tau \leq t, \ \varphi_3(t) \leq y \leq \varphi_4(t).$$

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Suppressing the variable τ of intergration in the last three integrals, let $g(y) = f'(t, y)$, putting $a = \varnothing_1(\tau)$, $b = \varnothing_2(\tau)$, $c = \varnothing_3(\tau)$, and $d = \varnothing_4(\tau)$ in (3.2), clearly, $g''(y) \geq 0$ (or $g''(y) \leq 0$), $y \in R$, for any fixed $\tau \in [t_0, t]$, hence we may apply the lemma, we obtain

$$\frac{\varnothing_4 - \varnothing_3}{(\varnothing_3 - \varnothing_1)(\varnothing_4 - \varnothing_1)}\int_{\varnothing_1}^{\varnothing_2} f'(\tau, y)\, dy + \frac{\varnothing_2 - \varnothing_1}{(\varnothing_4 - \varnothing_2)(\varnothing_4 - \varnothing_1)}\int_{\varnothing_3}^{\varnothing_4} f'(\tau, y)\, dy - \frac{(\varnothing_4 - \varnothing_3)(\varnothing_2 - \varnothing_1)(\varnothing_4 + \varnothing_3 - \varnothing_2 - \varnothing_1)}{(\varnothing_3 - \varnothing_1)(\varnothing_4 - \varnothing_1)(\varnothing_4 - \varnothing_2)(\varnothing_3 - \varnothing_2)}\int_{\varnothing_2}^{\varnothing_3} f'(\tau, y)\, dy > 0. \tag{3.8}$$

Integrating the above inequality with respect to τ from t_0 to t, we get the last expression in (3.7) to be positive. Thus, we conclude that $R(t)$ is monotone incresing, that is, $R(0) < R(t)$, which yields a contradiction with the fact that $R(t)$ is a T - periodic function by the hypothsis. So (1.1) has at most three T - periodic solutions at the domain Ω_1 (Ω_2 or Ω_3), but $y = 0$ is trivial, then the proof of Theorem is completed.

If $a_1^2 - 4a_0 a_2 < 0$, we have

$$\frac{A_1 y + A_2 y^2 + A_3 y^3}{a_0 + a_1 y + a_2 y^2} = \frac{A_3}{a_2} y + \frac{A_2 a_2 - A_3 a_1}{a_2^2} + \frac{Ey + F}{a_0 + a_1 y + a_2 y^2},$$

where $E = A_1 - \dfrac{A_3 a_0}{a_2} - \dfrac{A_2 a_2 a_1 - A_3 a_1^2}{a_2^2}$, $F = \dfrac{A_3 a_1 a_0 - A_2 a_2 a_0}{a_2^2}$.

Similar to the proof of Theorem 1, we have

Theorem 2 *If $A_i(t)$, $a_i(t)$ ($i = 1, 2, 3$; $j = 0, 1, 2$) are continuous T - periodic functions, $a_1^2 - 4a_0 a_2 < 0$, and*

$$\left[\frac{Ey + F}{a_0 + a_1 y + a_2 y^2}\right]'' \geq 0 \quad (\text{or } \leq 0).$$

Then equation (1.1) has at most two non - trivial T - periodic solutions at the domain Ω_1 or Ω_2 or Ω_3.

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关于一类微分方程周期解的存在性的注记

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摘要

本文主要探讨下列周期系数微分方程

$$\frac{dy}{dt} = \frac{A_1(t)y + A_2(t)y^2 + A_3(t)y^3}{a_0(t) + a_1(t)y + a_2(t)y^2} \qquad (**)$$

的周期解个数问题,利用方程(＊＊)解的差率法得到了方程(＊＊)周期解的个数定理.本文仅在 $A_i(t), a_j(t)$ $(i = 1,2,3, j = 0,1,2)$ **是连续周期函数的条件下得到这一结论,从而减弱了文[2]中相应定理的条件,即** $A_i(t), a_j(t)$ **均是连续可微的周期函数.**

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