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Classification of Crystallographic Groups Associated with the Infinite Dihedral Group*

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Abstract Let V be a 2-dimensional vector space over the real field $\mathbf{R}$ with an affine or indefinite symmetric bilinear form. The infinite dihedral group W can be viewed as a subgroup of GL (V). In the present paper we will classify all crystallographic groups associated with W up to conjugation in the affine group $\mathbf{A}(V)$.

Keywords crystallographic groups, infinite dihedral group.
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1. Introduction

Let V be an n-dimensional vector space over the real field $\mathbf{R}$. We endow V with a symmetric bilinear form $(,)$ such that V has a basis $a_1, a_2, \cdots, a_n$ satisfying the following conditions:

i). $(a_i, a_i) \neq 0$ for $i = 1, 2, \cdots, n$;

ii). the matrix $B = (b_{ij})_{n\times n}$ with $b_{ij} = \langle a_j, a_i\rangle$ is an indecomposable generalized Cartan matrix [1], where we denote $2(a_i, a_j)/(a_j, a_j)$ by $\langle a_i, a_j\rangle$.

The vector space V endowed with such a symmetric bilinear form $(\ ,\ )$ is called Euclidean, affine or indefinite if the corresponding matrix B is offinite, affine or indefinite type. A subset Λ of V is called a lattice, provided there exists a basis $v_1, v_2, \cdots, v_n$ of V such that $\Lambda = \oplus_{i=1}^{n}\mathbf{Z}v_i$. For any subgroup W of GL (V), a lattice Λ is called W-invariant if $w(v) \in \Lambda$ for any $w \in W$ and $v \in \Lambda$. By $\mathbf{A}(V)$ we denote the affine group of V consisting of all elements (a, A), where $a \in V$ and $A \in$ GL (V), with multiplication $(a, A)(b, B) = (a + Ab, AB)$, and we identify the subgroup $T(V)$ of $\mathbf{A}(V)$ consisting of all elements (a, I), $a \in V$, with V itself. It is clear that there exists a homomorphism p: $\mathbf{A}(V) \to$ GL (V) given by $p(a, A) = A$. Let W be a subgroup of GL (V). A subgroup G of $\mathbf{A}(V)$ is called a crystallographic group associated with W if the homomorphic image $p(G)$ coincides with W and the kernel of p, $\Lambda = G \cap T(V)$ is a lattice of V. We also call G the extension of W by Λ. It is an interesting problem to classify all crystallographic groups in the affine group $\mathbf{A}(V)$. When V is an n-dimensional Euclidean space and W is the

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Weyl group of an irreducible root system, the classification of crystallographic groups associated with W is well known[2], [3]. When V is an n-dimensional indefinite space, W is an infinite Coxeter group and the Coxeter graph is simple-laced, the results on this subject have been given by Li.[4] In the present paper we are going to classify all crystallographic groups associated with the infinite dihedral group W in the affine group $\mathbf{A}(V)$, where V is an affine or indefinite 2-dimensional space and the Coxeter graph is not simple-laced. The group W with s_1 and s_2 as a set of generators, subject only to relations $s_1^2 = s_2^2 = 1$, is called the infinite dihedral group. Let V be a 2-dimensional vector space over the real field $\mathbf{R}$ with a symmetric bilineal form $(,)$ such that V has a basis a_1 and a_2 satisfying $(a_i, a_i) \neq 0$ for $i = 1, 2$ and the corresponding Cartan matrix

$$\begin{pmatrix} 2 & \langle a_2, a_1\rangle \\ \langle a_1, a_2\rangle & 2 \end{pmatrix} = \begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix},$$

where m and n are positive integers. It is clear that B is of affine or indefinite type if and only if $mn = 4$ or $mn > 4$. For $i = 1, 2$ we can define reflections σ_{g_i} by the rule

$$\sigma_i(\lambda) = \lambda - \langle\lambda, a_i\rangle a_i,$$

where we denote $2(\lambda, a_i)/(a_i, a_i)$ by $\langle\lambda, a_i\rangle$. An easy calculation shows that σ_{g_i} preserves the form $(,)$.

Proposition 1.1 *There exists a unique homomorphism $\sigma: W \to GL(V)$, sending s_i to σ_i. Moreover, σ is faithful if and only if B is of affine or indefinite type.*

Proof It is clear that σ is faithful if and only if $\sigma_1\sigma_2$ is not of finite order. With respect to the basis $\{a_1, a_2\}$ $\sigma_1\sigma_2$ can be expressed as the matrix

$$\begin{pmatrix} -1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix} = \begin{pmatrix} mn-1 & -m \\ n & -1 \end{pmatrix},$$

whose characteristic polynomial is $\lambda^2 - (mn-2)\lambda + 1$. It is easy to see that $\sigma_1\sigma_2$ is of finite order if and only if $\sigma_1\sigma_2$ has two different complex eigenvalues, which is equivalent to the fact that $mn < 4$. So W acts on V faithfully if and only if B is of affine or indefinite type.

Proposition 1.1 shows that the infinite dihedral group W can be viewed as a subgroup of $GL(V)$ when V is an affine or indefinite 2-dimensional space. In order to complete the classification of crystallographic groups associated with the infinite dihedral group W in $GL(V)$, where V is an affine or indefinite 2-dimensional space, we should proceed in the following way: First of all, we will determine all lattices which are W-invariant; then, we will calculate the first cohomology group $H^1(W, V/\Lambda)$ for every W-invariant lattice Λ; Finally, we will find the important normalizer of W in $GL(V)$. For the outline of the theory of crystallographic groups and all notationand terminology, we refer the readers to [5].

2 W-Invariant lattices Λ

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Let W be the infinite dihedral group and let W act on a 2-dimensional affine or indefinite real vector space V in the way indicated in Proposition 1.1. We assume that V has the basis $\{a_1, a_2\}$ such that the Cartan matrix

$$\begin{pmatrix} \langle a_1, a_1\rangle & \langle a_2, a_1\rangle \\ \langle a_1, a_2\rangle & \langle a_2, a_2\rangle \end{pmatrix} = \begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix},$$

where m, n are positive integers and $mn \geq 4$.

Lemma 2.1 *Assume that Λ is a W-invariant lattice in V. For $i = 1, 2$ there exists a unique positive real b_i such that*

$$\Lambda \cap \mathbf{R}a_i = \mathbf{Z}b_i a_i.$$

Proof See [4], Proposition 2.3.

Let $\beta_i = b_i a_i$ for $i = 1, 2$. The set $B = \{\beta_1, \beta_2\}$ of positive multiples of the a_1 and a_2 is called a basic system of W if there exists a W-invariant lattice Λ such that

$$\Lambda \cap \mathbf{R}a_i = \mathbf{Z}\beta_i.$$

Definition 2.2 *Two lattices Λ_1 and Λ_2 are called equivalent if there exists some $c \neq 0$ such that $c\Lambda_1 = \Lambda_2$.*

When the space V is indefinite, all W-invariant lattices in V have been determined easily. We just recall the results due to Li.[4]

Theorem 2.3 *Assume that V is a 2-dimensional indefinite real vector space. For every W-invariant lattice Λ in V, denote by Λ^* the following lattice:*

$$\Lambda^* = \{v \in V \mid s_i(v) - v \in \Lambda \text{ for } i = 1, 2\}.$$

i) Every W-invariavt lattice Λ in V is such that $Q(B) \subseteq \Lambda \subseteq P(B)$ for some basic system $B = \{\beta_1, \beta_2\}$ of W, where $Q(B) = \mathbf{Z}\beta_1 \oplus \mathbf{Z}\beta_2$ and $P(B) = Q(B)^*$. Conversely, every lattice of this form is W-invariant.

ii) The number of equivalent classes of W-invariant lattices in V is finite.

We focus now on the classification of W-invariant lattices in an affine space V. It is clear that V is affine if and only if $mn = 4$. We can replace a_1 and a_2 by their positive multiples without altering the action of s_1 and s_2. therefore we may assume, without losing generality, that V has the basis $\{a_1, a_2\}$ such that the Cartan matrix

$$\begin{pmatrix} \langle a_1, a_1\rangle & \langle a_2, a_1\rangle \\ \langle a_1, a_2\rangle & \langle a_2, a_2\rangle \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Let $\delta = a_1 + a_2$. By direct calculation we have

$$s_1 s_2(\lambda) = \lambda - \langle \lambda, a_2\rangle \delta$$

for any $\lambda \in V$. Now we may give an explicit description of inequivalent W-invariant lattices in the affine space V. At this point the results are different from those given in the indefinite space V. Inequivalent W-invariant lattices in the affine space V depend upon a parameter k and the number of inequivalent lattices is infinite.

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Theorem 2.4 *Assume that V is a 2-dimensional real affine space, having a basis $\{a_1, a_2\}$ such that the Cartan integers $\langle a_1, a_2\rangle = \langle a_2, a_1\rangle = -2$. Let k denote any positive integer. The representative elements Λ of inequivalent W-invariant lattices in V are explicitly listed as follows:*

$$\Lambda = \mathbf{Z}\beta_1 \oplus \mathbf{Z}\left(\frac{\beta_1}{2} + \frac{\delta}{2}\right),$$

where $\beta_1 = ka_1$, or

$$\Lambda = \mathbf{Z}\beta_1 \oplus \mathbf{Z}\delta,$$

where $\beta_1 = \frac{k}{2}a_1$.

Proof Since δ is fixed by the action of W and $s_1(\beta_1) = -\beta_1$,

$$s_1 s_2(\beta_1) = \beta_1 - \langle \beta_1, a_2\rangle\delta,$$

it is easy to see that the above lattices are all W-invariant. Conversely, suppose that Λ is a W-invariiant lattice. We have

$$v - s_1 s_2(v) = \langle v, a_2\rangle\delta \in \Lambda,$$

for any $v \in V$. Since the vector a_2 is nonisotropic, $\langle v, a_2\rangle \neq 0$ for some $v \in V$. It follows that there exists some real $b > 0$ such that $\Lambda \cap \mathbf{R}\delta = \mathbf{Z}b\delta$. Replacing Λ by $b^{-1}\Lambda$ without altering the equivalent class of Λ, we may assume that $\Lambda \cap \mathbf{R}\delta = \mathbf{Z}\delta$. Similarly, we have $\Lambda \cap \mathbf{R}a_1 = \mathbf{Z}b_1 a_1$ for some $b_1 > 0$. Let $\beta_1 = b_1 a_1$. Since

$$\beta_1 - s_1 s_2(\beta_1) = \langle \beta_1, a_2\rangle\delta = -2b_1\delta \in \Lambda,$$

it follows that $2b_1 \in \mathbf{Z}$ and $b_1 = \frac{k}{2}$ for some positive integer k. Every element $v \in \Lambda$ can be written uniquely as $v = x\beta_1 + y\delta$ for some $x, y \in \mathbf{R}$. For any $v = x\beta_1 + y\delta \in \Lambda$, we have

$$s_1(v) = -x\beta_1 + y\delta \in \Lambda,$$

which means that $2x \in \mathbf{Z}$ and $2y \in \mathbf{Z}$. It is clear that x and y must take integers or half integers simultaneously, otherwise we would have $\frac{1}{2}\beta_1 \in \Lambda$ or $\frac{1}{2}\delta \in \Lambda$, which is a contradiction. Therefore, if $\frac{1}{2}\beta_1 + \frac{1}{2}\delta \notin$, $x, y \in \mathbf{Z}$ for any $x\beta_1 + y\delta \in \Lambda$, and we have $\Lambda = \mathbf{Z}\beta_1 \oplus \mathbf{Z}\delta$. If $\frac{1}{2}\beta_1 + \frac{1}{2}\delta \in \Lambda$, then $\Lambda = \mathbf{Z}\beta_1 \oplus \mathbf{Z}\left(\frac{\beta_1}{2} + \frac{\delta}{2}\right)$, and $b_1 \in \mathbf{Z}$ since

$$s_1 s_2\left(\frac{\beta_1}{2} + \frac{\delta}{2}\right) - \left(\frac{\beta_1}{2} + \frac{\delta}{2}\right) = -\left\langle \frac{\beta_1}{2} + \frac{\delta}{2}, a_2\right\rangle\delta = b_1\delta \in \Lambda$$

3. The first cohomology group $H^1(W, V/\Lambda)$

Let Λ be a W-invariant lattice in V. The connection between classifing all crystallographic groups G, the extension of W by Λ, and calculating the first cohomology group $H^1(W, V/\Lambda)$ has been emphasized by Maxswell[5]. Since W is generated by s_1 and s_2, subject only to relations $s_1^2 = s_2^2 = 1$, a mapping $c: W \to V/\Lambda$ extends to a cocycle of W if and only if $c(s_i) + s_i c(s_i) = 0$ for $i = 1, 2$. In calculating the cohomology group $H^1(W, V/\Lambda)$ we must distinguish two different cases,

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in which the space V is indefinite or affine.

Case 1 *Assume that V is indefinite. Let Λ is a W-invariant lattice in V, having the basic system $\{\beta_1, \beta_2\}$ such that the Cartan integers $\langle\beta_1, \beta_2\rangle = -n$, $\langle\beta_2, \beta_1\rangle = -m$, where $m, n > 0, mn > 4$.*

Proposition 3.1 *A mapping $b: W \to V/\Lambda$ extends to a coboundary of W if and only if there exist $x_i \in \mathbf{R}$ such that $b(s_i) = x_i\beta_i \pmod{\Lambda}$ for $i = 1, 2$.*

Proof As in Proposition 3.4 of Maxswell[5]

For $v_1, v_2 \in V$, it is clear to see that $c(s_i) = v_i \pmod{\Lambda}$ extends to a cocycle of W if and only if

$$2v_i = \langle v_i, \beta_i\rangle\beta_i \pmod{\Lambda}$$

for $i = 1, 2$. Write $v = x\beta_1 + y\beta_2$, $2v = \langle v, \beta_1\rangle\beta_1 \pmod{\Lambda}$ if and only if

$$2y\beta_2 = -my\beta_1 \pmod{\Lambda}.$$

This yields that $2y \in \mathbf{Z}$ and $my \in \mathbf{Z}$. Similarly, $2v = \langle v, \beta_2\rangle\beta_2 \pmod{\Lambda}$ if and only if $2x \in \mathbf{Z}$ and $nx \in \mathbf{Z}$. Then,

$$c(s_1) = x_1\beta_1 + x_2\beta_2 \pmod{\Lambda}, \qquad c(s_2) = y_1\beta_1 + y_2\beta_2 \pmod{\Lambda}$$

extends to a cocycle of W if and only if the following conditions are satisfied: $2x_2 \in \mathbf{Z}$, $mx_2 \in \mathbf{Z}$, $2y_1 \in \mathbf{Z}$ and $ny_1 \in \mathbf{Z}$. Taking into account the form of coboundaries of W given by Proposition 3.1, we have the central result.

Theorem 3.2 *For the first cohomology group $H^1(W, V/\Lambda)$ we have*

i) $H^1(W, V/\Lambda) = 0$ *if m and n are both odd;*

ii) $H^1(W, V/\Lambda) \cong F_2$, *more precisely, the cocycle, as a generator of $H^1(W, V/\Lambda)$, is defined by*

$$c(s_1) = \frac{1}{2}\beta_2 \pmod{\Lambda}, \qquad c(s_2) = 0 \pmod{\Lambda}$$

if m is even and n is odd. Similarly, if m is odd and n is even, $H^1(W, V/\Lambda)$ is generated by the cocycle defined by

$$c(s_1) = 0 \pmod{\Lambda}, \qquad c(s_2) = \frac{1}{2}\beta_1 \pmod{\Lambda}.$$

iii) $H^1(W, V/\Lambda) \cong F_2 \times F_2$, *more precisely, two cocycles, as generators of $H^1(W, V/\Lambda)$, are defined by*

$$c(s_1) = \frac{1}{2}\beta_2 \pmod{\Lambda}, \qquad c(s_2) = 0 \pmod{\Lambda}$$

and

$$c'(s_1) = 0 \pmod{\Lambda}, \qquad c'(s_2) = \frac{1}{2}\beta_1 \pmod{\Lambda},$$

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if m and n are both even.

Case 2 *Assume that V is affine, having the basis $\{a_1, a_2\}$ such that the Cartan integers $\langle a_1, a_2\rangle = \langle a_2, a_1\rangle = -2$. Let Λ be a W-invariant lattice with $\Lambda \cap \mathbf{R}\delta = \mathbf{Z}\delta$.*

Proposition 3.3 *A mapping $b: W \to V/\Lambda$ is a coboundary if and only if there exists some $x \in \mathbf{R}$ such that*

$$b(s_1) = xa_1 \pmod{\Lambda}, \qquad b(s_2) = -xa_2 \pmod{\Lambda}.$$

Proof It is clear that any coboundary of W has the indicated form since

$$\langle v, a_1\rangle = -\langle v, a_2\rangle$$

for any $v \in V$. To conclude, we notice that for any $x \in \mathbf{R}$ there exists $v = \frac{1}{2}xa_1 \in V$ such that $\langle v, a_1\rangle a_1 = xa_1$ and $\langle v, a_2\rangle a_2 = -xa_2$.

In indefinite case the first cohomology group $H^1(W, V/\Lambda)$ is finite, as we see in Theorem 3.2, but in affine case the group is infinite.

Theorem 3.4 *The first cohomology group $H^1(W, V/\Lambda)$ is generated by the following cocycles, which are defined by*

i) $c(s_1) = xa_1 \pmod{\Lambda}$, $c(s_2) = 0 \pmod{\Lambda}$ *for any* $x \in \mathbf{R}$;

ii) $c(s_1) = 0 \pmod{\Lambda}$, $c(s_2) = ya_2 \pmod{\Lambda}$ *for any* $y \in \mathbf{R}$;

iii) $c(s_1) = \frac{1}{2}\delta \pmod{\Lambda}$, $c(s_2) = 0 \pmod{\Lambda}$;

iv) $c(s_1) = 0 \pmod{\Lambda}$, $c(s_2) = \frac{1}{2}\delta \pmod{\Lambda}$.

Proof It is easy to see that

$$c(s_1) = x_1a_1 + x_2\delta \pmod{\Lambda}, \qquad c(s_2) = y_1\delta + y_2a_2 \pmod{\Lambda}$$

extends to a cocycle of W if and only if $2x_2 \in \mathbf{Z}$ and $2y_1 \in \mathbf{Z}$. Since $\mathbf{Z}\delta \subseteq \Lambda$, we have the required result.

4 The normalizer of W in $GL(V)$

To solve the classification problem, it is necessary to find the nomalizer of W in $GL(V)$, as indicated by Martinais[3].

Lemma 4.1 *Let W be the infinite dihedral group with generators s_1 and s_2, subject only to $s_1^2 = s_2^2 = 1$. The outer automorphism group $Out(W)$ is of order 2 and $Out(W)$ is generated by the automorphism θ given by $\theta(s_1) = s_2$ and $\theta(s_2) = s_1$.*

Proof For any $w \in W$ denote by σ_w the inner automorphism of W induced by w, which means that

$$\sigma_w(s) = wsw^{-1}.$$

Assume that τ is any automorphism of W. Let A be the subgroup generated by s_1s_2. It is clear that A is an infinite cyclic group and A consists of all elements of order infinity. Then A is a characteristic subgroup of W, and $\tau(s_1s_2) = s_1s_2$ or $\tau(s_1s_2) = (s_1s_2)^{-1}$. If $\tau(s_1s_2) = (s_1s_2)^{-1}$, then

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$\sigma_{s_2}\tau(s_1s_2) = s_1s_2$. So without losing generality, we may assume that $\tau(s_1s_2) = s_1s_2$. Let $\tau(s_1) = s_1(s_1s_2)^m$. If m is even, there exists $w \in A$ such that $ws_1w^{-1} = s_1(s_1s_2)^m$. It follows that $\tau = \sigma_w$. If m is odd, there exists $w \in A$ such that $ws_1w^{-1} = s_1(s_1s_2)^{-(m-1)}$. Then $\sigma_w\tau(s_1) = s_2$ and $\sigma_w\tau(s_2) = \sigma_w\tau(s_1s_1s_2) = s_2s_1s_2$. It follows that $\sigma_w\tau = \sigma_{s_2}\theta$.

Now we can easily find the normalizer of W in $GL(V)$. Let V be a vector space, having the basis $\{a_1, a_2\}$ such that the Cartan matrix

$$\begin{pmatrix} \langle a_1, a_1\rangle & \langle a_2, a_1\rangle \\ \langle a_1, a_2\rangle & \langle a_2, a_2\rangle \end{pmatrix} = \begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}$$

where m, n are positive integers and $mn \geq 4$.

Theorem 4.2 *Denote by H the group of homotheties of V. Let $N(W)$ be the normalizer of W in $GL(V)$. Then $N(W)/HW$ is the group of order 2 generated by the element θ, where with respect to the basis $\{a_1, a_2\}$, $\theta = \begin{pmatrix} 0 & m \\ n & 0 \end{pmatrix}$.*

Proof With respect to the basis $\{a_1, a_2\}$ we have

$$s_1 = \begin{pmatrix} -1 & m \\ 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix}.$$

An easy calculation shows that $\theta s_1\theta^{-1} = s_2$ and $\theta s_2\theta^{-1} = s_1$. Then $\theta \in N(W)$. By Lemma 4.1 we only need to prove that any element τ of $N(W)$ is a homothety of V if the conjugation of τ leaves all elements of W fixed. But this is clear by Humphreys.[6]

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同无限二面体群关联的晶体群的分类

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摘　要

设 V 是实数域 $\mathbf{R}$} 上的一个2-维向量空间, V 带有一个仿射的或不定的对称双线性型. 无限二面体群 W 能够被看作 $GL(V)$ 的一个子群. 在本文中, 在仿射群 $A(V)$ 中共轭的意义下将分类同 W 关联的所有晶体群.

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