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一个组合恒等式的推广*

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摘 要：本文推广了文[1]中的一个组合恒等式.

关键词：组合恒等式；母函数；分部求和公式.
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文[1]中给出了由常庚哲、单墫首先提出并证明了的如下组合恒等式

$$\sum_{k=0}^{n-1}\left\{\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{k}\right\}\left\{\binom{n}{k+1}+\binom{n}{k+2}+\cdots+\binom{n}{n}\right\}=\frac{n}{2}\binom{2n}{n},\qquad(1)$$

而后，文[2]又给出了(1)式的几个证明.

恒等式(1)外形十分精美. 将组合数$\binom{n}{0},\binom{n}{1},\cdots,\binom{n}{n-1},\binom{n}{n}$截成两段，分别求和，再作这两个和的乘积，(1)的左边是所有可能的乘积之和，而(1)的右边竟是那么一个简单的表达式！

在这篇短文中，我们将这个有趣的组合恒等式推广为如下的

定理 设α是实数，且$\alpha\neq 0$，则

$$\sum_{k=0}^{n-1}\left\{\binom{n}{0}+\binom{n}{1}\alpha+\cdots+\binom{n}{k}\alpha^k\right\}\left\{\binom{n}{k+1}\frac{1}{\alpha^{k+1}}+\binom{n}{k+2}\frac{1}{\alpha^{k+2}}+\cdots+\binom{n}{n}\frac{1}{\alpha^n}\right\}$$
$$=n\cdot\frac{1-\alpha}{\alpha^n}\sum_{k=0}^{n-1}\alpha^k\binom{2n-1}{k}+n\binom{2n-1}{n-1}.\qquad(2)$$

证明 记(2)式的左端为a_{n-1}，再记$b_k=\sum_{l=0}^{k}\binom{n}{l}\alpha^l$，则有

$$\sum_{l=k+1}^{n}\binom{n}{l}\frac{1}{\alpha^l}=\sum_{l=k+1}^{n}\binom{n}{n-l}\frac{1}{\alpha^l}=\frac{1}{\alpha^n}\sum_{j=0}^{n-k-1}\binom{n}{j}\alpha^j=\frac{1}{\alpha^n}b_{n-k-1},$$

于是

$$a_{n-1}=\frac{1}{\alpha^n}\sum_{k=0}^{n-1}b_k\,b_{n-k-1},$$

即

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$$\alpha^n a_{n-1} = \sum_{k=0}^{n-1} b_k b_{n-k-1}.$$

因此，如果$\{b_k\}$的母函数是$f(x)$，即$f(x) = \sum_{k=0}^{\infty} b_k x^k$，那么根据形式幂级数的乘法规则知

$$f^2(x) = \sum_{l=0}^{\infty} \left(\sum_{k=0}^{l} b_k b_{l-k} \right) x^l,$$

由此可见$\alpha^n a_{n-1}$是$f^2(x)$的展开式中x^{n-1}的系数。已知$\left\{ \binom{n}{l} \alpha^l \right\}$的母函数是$(1+\alpha x)^n$，故$\{b_k\}$的母函数$f(x) = \dfrac{(1+\alpha x)^n}{1-x}$，于是$f^2(x) = \dfrac{(1+\alpha x)^{2n}}{(1-x)^2}$。下面把$f^2(x)$展开成幂级数。

由于

$$(1+\alpha x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} \alpha^k x^k, \quad \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1) x^k,$$

从而

$$\frac{(1+\alpha x)^{2n}}{(1-x)^2} = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} \binom{2n}{l} \alpha^l (k-l+1) \right) x^k.$$

这样我们就得到

$$\begin{aligned}
\alpha^n a_{n-1} &= \sum_{l=0}^{n-1} \binom{2n}{l} \alpha^l (n-l) = n \sum_{l=0}^{n-1} \binom{2n}{l} \alpha^l - \sum_{l=0}^{n-1} \binom{2n}{l} l \, \alpha^l \\
&= n + n \sum_{l=1}^{n-1} \binom{2n}{l} \alpha^l - \sum_{l=1}^{n-1} \binom{2n}{l} l \, \alpha^l \\
&= n + n \sum_{l=1}^{n-1} \left(\binom{2n-1}{l} + \binom{2n-1}{l-1} \right) \alpha^l - 2n \sum_{l=1}^{n-1} \binom{2n-1}{l-1} \alpha^l \\
&= n + n \sum_{l=1}^{n-1} \left(\binom{2n-1}{l} - \binom{2n-1}{l-1} \right) \alpha^l \\
&= n - n\alpha \sum_{l=1}^{n-1} \left(\binom{2n-1}{l-1} - \binom{2n-1}{l} \right) \alpha^{l-1} \\
&= n - n\alpha \sum_{i=0}^{n-2} \left(\binom{2n-1}{i} - \binom{2n-1}{i+1} \right) \alpha^i \\
&= n - 2n\alpha + 2n^2 \alpha - n\alpha \sum_{i=1}^{n-2} \left(\binom{2n-1}{i} - \binom{2n-1}{i+1} \right) \alpha^i.
\end{aligned} \tag{3}$$

记$c_i = \alpha^i - \alpha^{i-1}$，$d_i = \binom{2n-1}{i}$，再记$S_i = c_1 + c_2 + \cdots + c_i = \alpha^i - 1$，代入分部求和公式

$$\sum_{i=1}^{n-1} c_i d_i = \sum_{i=1}^{n-2} S_i (d_i - d_{i+1}) + S_{n-1} d_{n-1},$$

得

$$\begin{aligned}
&\sum_{i=1}^{n-2} (\alpha^i - 1) \left(\binom{2n-1}{i} - \binom{2n-1}{i+1} \right) \\
&\quad = \sum_{i=1}^{n-1} (\alpha^i - \alpha^{i-1}) \binom{2n-1}{i} - (\alpha^{n-1} - 1) \binom{2n-1}{n-1},
\end{aligned}$$

移项

$$\sum_{i=1}^{n-2}\left(\binom{2n-1}{i}-\binom{2n-1}{i+1}\right)\alpha^i$$

$$=\sum_{i=1}^{n-2}\left(\binom{2n-1}{i}-\binom{2n-1}{i+1}\right)+\sum_{i=1}^{n-1}(\alpha^i-\alpha^{i-1})\binom{2n-1}{i}-(\alpha^{n-1}-1)\binom{2n-1}{n-1}$$

$$=\binom{2n-1}{1}-\binom{2n-1}{n-1}+(\alpha-1)\sum_{i=1}^{n-1}\alpha^{i-1}\binom{2n-1}{i}-\alpha^{n-1}\binom{2n-1}{n-1}+\binom{2n-1}{n-1}$$

$$=2n-1+(\alpha-1)\sum_{i=1}^{n-1}\alpha^{i-1}\binom{2n-1}{i}-\alpha^{n-1}\binom{2n-1}{n-1},\qquad(4)$$

将(4)代入(3),并整理得

$$\alpha^n a_{n-1}=n(1-\alpha)\left(1+\sum_{i=1}^{n-1}\alpha^i\binom{2n-1}{i}\right)+n\alpha^n\binom{2n-1}{n-1}$$

$$=n(1-\alpha)\sum_{i=0}^{n-1}\alpha^i\binom{2n-1}{i}+n\alpha^n\binom{2n-1}{n-1},$$

即

$$a_{n-1}=n\cdot\frac{1-\alpha}{\alpha^n}\sum_{i=0}^{n-1}\alpha^i\binom{2n-1}{i}+n\binom{2n-1}{n-1}.$$ □

如果在等式(2)中取 $\alpha=1$,我们就得到

$$a_{n-1}=n\binom{2n-1}{n-1}=\frac{n}{2}\binom{2n}{n}.$$

这刚好是(1)式,由此可见,(2)式为(1)式的推广.

如果在(2)中取 $\alpha=-1$,我们又得到

$$a_{n-1}=n\cdot\frac{2}{(-1)^n}\sum_{k=0}^{n-1}(-1)^k\binom{2n-1}{k}+n\binom{2n-1}{n-1}.$$

应用下面的组合恒等式[3]

$$\sum_{k=0}^{m}(-1)^k\binom{n}{k}=\begin{cases}0, & m=n;\\ (-1)^m\binom{n-1}{m}, & m<n.\end{cases}$$

可得

$$a_{n-1}=n\cdot\frac{2}{(-1)^n}\cdot(-1)^{n-1}\binom{2n-2}{n-1}+n\binom{2n-1}{n-1}$$

$$=-2n\binom{2n-2}{n-1}+n\binom{2n-1}{n}$$

$$=-2n\binom{2n-2}{n-1}+(2n-1)\binom{2n-2}{n-1}=-\binom{2n-2}{n-1}.$$

这样我们又得到一个新的有趣的组合恒等式:

$$\sum_{k=0}^{n-1}\left\{\binom{n}{0}-\binom{n}{1}+\cdots+(-1)^k\binom{n}{k}\right\}\left\{(-1)^{k+1}\binom{n}{k+1}+\cdots+(-1)^n\binom{n}{n}\right\}$$

$$=-\binom{2n-2}{n-1}.$$

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Generalization of a Combinatorial Identity

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Abstract: In this note, we obtain a new combinatorial identity, which generalizes an identity proved by Chang Geng-zhe and Shan Zun in 1983.

Key words: combinatorial identity; generating function; partial summation formula.

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