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A Note of Paper "Banach Spaces Failing the Almost Isometric Universal Extension Property"

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Abstract The definition of property $\mathcal{A}$ with constant α was introduced by D. M. Speegle, who proved that every infinite dimensional separable uniformly smooth Banach space has property $\mathcal{A}$ with constant $\alpha \in [0,1)$. In this paper, we give a sufficient condition for a Banach space to have property $\mathcal{A}$ with constant $\alpha \in [0,1)$, and some remarks on Speegle's paper [1].

Keywords property $\mathcal{A}$ with constant α; modulus of convexity; λ-EP; λ-UEP.

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The definition of property $\mathcal{A}$ with constant $\alpha \in [0,1)$ was introduced by Speegle in [1].

Definition 1 *We say a Banach space E has property $\mathcal{A}$ with constant α if there is a normalized weak* null sequence $\{e_n^*\}$, an $\alpha \in [0,1)$, and a normalized sequence $\{e_n\}$ in E such that (1) $e_n^*(e_n) \to 1$ and (2) $S(e_n,\alpha) \bigcap -S(e_m,\alpha) = \emptyset$ for all integers n and m, where $S(e,\alpha) = \{e^* \in B(E^*) : e^*(e) > \alpha\}$.*

In [1, Proposition 4], Speegle showed us that every infinite dimensional separable uniformly smooth Banach space has property $\mathcal{A}$ with constant $\alpha \in [0,1)$. Observing the proof of Proposition 4 more carefully, we can get the following theorem to give a somewhat weaker condition for an infinite dimensional Banach space to have property $\mathcal{A}$ with constant $\alpha \in [0,1)$.

In this paper, X, E will stand for separable Banach spaces with infinite dimensions.

Theorem 1 *Let X be an infinite-dimensional separable Banach space. If for any sequence $\{y_n\}$ in $S(X)$ we have $\inf_n \delta(y_n, 1/4) > 0$, where*

$$\delta(y_n, 1/4) = \inf\{1 - \frac{\|x + y_n\|}{2}; x \in S(X), \|x - y_n\| \geq 1/4\},$$

then X has property $\mathcal{A}$ with some $\alpha \in [0,1)$.

Proof We construct the necessary sequences $\{y_n^*\}$, $\{y_n\}$ and α. Let $\{x_n\}$ be a sequence which

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is dense in $B(X)$. Let y_1^* be in $S(X^*)$, and let y_1 be in $S(X)$ such that $y_1^*(y_1) \geq 1 - 1/2$. Let $S_1 = \{x^* \in S(X^*) : |x^*(y_1)| < 1/2^2, x^*(x_1) < 1/2\}$. Since X is infinite dimensional, by the Josefson-Nissenzweig's theorem, there is a normalized sequence which is w*-null. Hence we know that $S_1 \neq \emptyset$. So, let $y_2^* \in S_1$ and $y_2 \in S(X)$ such that $y_2^*(y_2) \geq 1 - 1/2^2$. Let $S_2 = \{x^* \in S(X^*) : |x^*(y_i)| < 3/2^3,\ x^*(x_i) < 1/3, 1 \leq i \leq 2\}$. Similarly, $S_2 \neq \emptyset$. So, let $y_3^* \in S_2$ and $y_3 \in S(X)$ such that $y_3^*(y_3) \geq 1 - 1/2^3$. Continuing in this fashion, we get

$$y_{n+1}^* \in S_n = \{x^* \in S(X^*) : |x^*(y_i)| < \frac{2^n - 1}{2^{n+1}},\ x^*(x_i) < \frac{1}{n+1},\ 1 \leq i \leq n\},$$

where $y_i \in S(X)$ satisfying $y_i^*(y_i) \geq 1 - 1/2^i$.

Obviously, the sequence $\{y_n^*(y_n)\}$ is convergent to 1. Since $\{x_n\}$ is dense in $B(X)$, we can show that the sequence $\{y_n^*\}$ is convergent to θ in the weak* topology. In fact, given $x \in B(X)$, $\varepsilon > 0$, there is $x_{m_0} \in \{x_n\}$ such that $\|x - x_{m_0}\| < \varepsilon$. For any large enough n with $n > \max\{m_0, 1/\varepsilon\}$, we have that $y_n^*(x_{m_0}) < 1/n$ for $y_n^* \in S_{n-1}$. Hence,

$$|y_n^*(x)| \leq |y_n^*(x) - y_n^*(x_{m_0})| + |y_n^*(x_{m_0})| \leq \|x - x_{m_0}\| + 1/n \leq 2\varepsilon.$$

Let $\alpha = 1 - \inf_n \delta(y_n, 1/4)$. From the condition of X, we have that $\alpha \in [0, 1)$. So, it remains to show that

$$S(y_i, \alpha) \bigcap -S(y_j, \alpha) = \emptyset \quad \text{for all} \quad i,\ j \in \mathbf{N}.$$

Obviously, it holds for $i = j$. For $i > j \geq 1$,

$$\|y_i + y_j\| \geq |y_i^*(y_i + y_j)| > 1 - \frac{1}{2^i} - \frac{2^{i-1} - 1}{2^i} = \frac{1}{2}.$$

Suppose that $x^* \in S(y_i, \alpha) \bigcap -S(y_j, \alpha)$. WLOG, $x^* \in S(X^*)$. Let $x \in S(X)$ such that $x^*(x) > \frac{1+\alpha}{2}$. We claim that $\|x - y_i\| < 1/4$ and $\|x + y_j\| < 1/4$. Conversely, if $\|x - y_i\| > 1/4$, then we have

$$\alpha = 1 - \inf_n \delta(y_n, 1/4) \geq 1 - \delta(y_i, 1/4) \geq \frac{\|x + y_i\|}{2} \geq \frac{x^*(x + y_i)}{2} > \frac{\frac{1+\alpha}{2} + \alpha}{2} = \frac{1 + 3\alpha}{4}.$$

It follows that $\alpha > 1$. A contradiction! It is similar for $\|x + y_j\| \geq 1/4$. Hence, $S(y_i, \alpha) \bigcap -S(y_j, \alpha) = \emptyset$. The proof is completed. □

Remark 1 Let $\delta_X(\varepsilon)$ be the modulus of convexity of the Banach space X. If $\delta_X(1/4) > 0$, then X has the property $\mathcal{A}$ with constant $\alpha = 1 - \delta_X(1/4)$.

Remark 2 In [1, Proposition 4], X needs to be reflexive and $\delta_{X^*}(1/4) > 0$. Although, we have no concrete nonreflexive Banach space which satisfies the conditions described in the Theorem 1, we conjecture that c_0 with the Day's norm is such a case. By the definition, $\delta_X(\varepsilon) \leq \inf_n \delta(y_n, \varepsilon)$ for any sequence $\{y_n\}$ in X. We should note that, even $\inf_n \delta(y_n, 1/4) > 0$ for any sequence $\{y_n\}$, we can't claim that $\delta_X(1/4) > 0$. Moreover, it is not sufficient to claim that X is uniformly convex if $\delta_X(1/4) > 0$.

We recall that a pair of Banach spaces (E, X) with E a closed subspace of X is said to have the λ-into-$C(K)$ extension property (λ-EP for short) if for every $C(K)$ space, and every bounded linear map $T : E \to C(K)$, there is an extension $\tilde{T} : X \to C(K)$ of T such that $\|\tilde{T}\| \leq \lambda\|T\|$.

We will say that a separable space E has the λ-universal extension property (λ-UEP) if (E, X) has the λ-EP where E imbeds as a (closed) subspace of a separable space X.

In [1], Speegle showed us that every Banach space satisfying property $\mathcal{A}$ with constant $\alpha \in [0, 1)$ fails the $(1+\varepsilon)$-UEP for any $0 < \varepsilon < \frac{1-\alpha}{1+\alpha}$[1, Theorem 3]. Moreover, two claims were given in [1] without proofs, though the claims are not so clear. Next, we will give the proofs.

Claim 1 Every Banach space E with property $\mathcal{A}$ with constant α fails the $(1+\varepsilon)$-into-c_0 extension property, where $0 < \varepsilon < \frac{1-\alpha}{1+\alpha}$.

Proof For any weak* null sequence $\{e_n^*\}$ in $S(E^*)$, define a map $\Phi : E \to c_0$ by $\Phi(x) = (e_n^*(x))_n$. Obviously, Φ is well defined and linear. Moreover, $\|\Phi\| \le 1$. Suppose, on the contrary, that (E, X) has the $(1+\varepsilon)$-into-c_0 extension property for some X containing E isometrically and some $0 < \varepsilon < \frac{1-\alpha}{1+\alpha}$. That is, there is an extension Ψ, defined on X to c_0, of Φ with $\|\Psi\| \le (1+\varepsilon)\|\Phi\|$. Let $\Psi(x)(n)$ be the nth coordinate of $\Psi(x)$. Define $f_n^* : X \to \mathbf{R}$ by $f_n^*(x) = \Psi(x)(n)$. Obviously, f_n^* is well defined and linear. Moreover, we have

$$\begin{aligned} \|\Psi\| &= \sup_{x \in B(X)} \|\Psi(x)\| = \sup_{x \in B(X)} \sup_n |\Psi(x)(n)| \\ &= \sup_n \sup_{x \in B(X)} |f_n^*(x)| \le (1+\varepsilon)\|\Phi\| \le (1+\varepsilon). \end{aligned}$$

That is, $\|f_n^*\| \le (1+\varepsilon)$ for any $n \in \mathbf{N}$.

We say that f_n^* is an extension of e_n^*. Indeed, $f_n^*(y) = \Psi(y)(n) = \Phi(y)(n) = e_n^*(y)$ for any $y \in E$. By the definition, $\{f_n^*\}$ is obviously convergent to θ in the weak* topology. Hence, we get a weak*-weak* continuous function from E^* into X^* which maps e_n^* to f_n^* and satisfies the conditions described in [1, Proposition 1]. So, (E, X) has the $(1+\varepsilon)$-into-$C(K)$ extension property, which contradict with [1, Theorem 3]. The proof is completed. □

Claim 2 c_0 has property $\mathcal{A}$ with constant α for all $\alpha > 0$ if we consider slices of the extreme points of $B(l_1)$ rather than slices of $B(l_1)$. Hence, c_0 fails the $(2-\varepsilon)$-UEP.

Proof The result is not clear. If we modify the definition of property $\mathcal{A}$ as above, the proof of [1, Theorem 3] will not stand any longer. Because we need the slices to be open in the weak* topology. But the slices of extreme points of $B(l_1)$ are obviously not weak*-open. So, we need to show it in another way.

Let $\{e_n\}$ be the standard unit basis for c_0, and let $S'_n = S(e_n, \alpha) \bigcap \mathrm{ext}B(l_1)$, where

$$\mathrm{ext}B(l_1) = \{(\xi_n) \in S(l_1)\colon \text{there is } n_0 \in \mathbf{N}, \text{ such that } |\xi_{n_0}| = 1\}$$

is the set of all extreme points of $B(l_1)$. Obviously, $S'_n \bigcap -S'_m = \emptyset$ for any integers n and m and any $\alpha \in [0, 1)$. Let

$$K' = B(l_1) \backslash \bigcup_{n=1}^{\infty} S'_n \ \text{ and } K = B(l_1) \backslash \bigcup_{n=1}^{\infty} S(e_n, \alpha).$$

Then $C(K) \subset C_b(K')$ for $K \subset K'$. From the proof of [1, Theorem 3], c_0 can be isometrically imbedded into $C(K)$. Hence, c_0 can be isometrically imbedded into $C_b(K')$. We claim that

$(c_0, C_b(K'))$ fails the $(1+\varepsilon)$-EP. In fact, for e_n^*, the nth coefficient function of the basis $\{e_n\}$ in $B(l_1)$, let Φ_n be any extension of e_n^* defined on $C_b(K')$. Then $\Phi_n|_{C(K)} \stackrel{\triangle}{=} \mu_n$ is an extension of e_n^* defined on $C(K)$. If $\|\Phi_n\| \le (1+\varepsilon)$, then $\|\mu_n\| \le (1+\varepsilon)$. By the proof of [1, Theorem 3], $\mu_n(f)$ is bounded away from 0, where f is the constant one function in $C(K)$. Hence, $\Phi_n(f)$ is also bounded away from 0, if f is considered as an element of $C_b(K')$. That is, the sequence $\{\Phi_n\}$ cannot converge to θ in the weak* topology. By [1, Proposition 2], $(c_0, C_b(K'))$ fails the $(1+\varepsilon)$-EP. The Claim is proved. □

From the proof of Claim 2 above, we can get the following corollary immediately.

Corollary 1 *If (E,X) fails the λ-EP and Z is a superspace of X, then (E, Z) fails the λ-EP either.*

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