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A Formula Containing Four Independent Bases and Its Applications *

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Abstract: A transformation formula containing four independent bases is found by a special inversion formula and it is applied to derive a few summation formulas for basic hypergeometric series only by elementary method. The hypergeometric series, the limits of those formulas are also obtained.

Key words: inverse relations; basic hypergeometric series; bibasic summation formulas; transformation formulas.

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All the notation and terminology is adopted from [3, p1–6]. The (generalized) hypergeometric series is defined by

$$ {}_{r+1}F_r\left[\begin{matrix} a_1,\cdots,a_r & ;z \\ b_1,\cdots,b_s & \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_r)_n}{n!(b_1)_n\cdots(b_s)_n}z^n, $$

where the shifted factorial $(a)_n$ is given by $(a)_n := a(a+1)(a+2)\cdots(a+n-1), n \geq 1, (a)_0 := 1$. We shall also use the compact Gasper-Raman notation

$$ (a_1,a_2,\ldots,a_r)_n = (a_1)_n(a_2)_n\cdots(a_r)_n. $$

Given a (fixed) complex number q with $|q|<1$, the basic hypergeometric series is defined by

$$ {}_r\varphi_s\left[\begin{matrix} a_1,\cdots,a_r & ;q,z \\ b_1,\cdots,b_s & \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(a_1;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}((-1)^n q^{\binom{n}{2}})^{s-r+1}z^n, $$

where the q-shifted factorial $(a;q)_n$ is given by

$$ (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), n \geq 1, (a;q)_0 := 1. $$

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The infinite q-shifted factorial $(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^{n-1})$ can be used to extend (finite) q-shifted factorial by defining $(a;q)_\beta := (a;q)_\infty/(aq^\beta;q)_\infty, \beta$ arbitrary. A basic hypergeometric series ${}_{r+1}\varphi_r$ is called very well-poised if $a_i b_i = qa_0$ for $i=1,2,\ldots,r$, and among the parameters a_i occur both $q\sqrt{a_0}$ and $-q\sqrt{a_0}$. We use the standard abbreviation for very well-poised basic hypergeometric series,

$$ {}_{r+1}W_r(a_0;a_3,a_4,\cdots,a_r;q,z) := {}_{r+1}\varphi_r\left[\begin{matrix} a_0, q\sqrt{a_0}, -q\sqrt{a_0}, a_3, a_4, \cdots, a_r \\ \sqrt{a_0}, -\sqrt{a_0}, qa_0/a_3, qa_0/a_4, \cdots, qa_0/a_r \end{matrix} ;q,z\right]. $$

We use short notation in the basic hypergeometric context which is analogous to the hypergeometric ones.

All identities in our paper are subject to suitable conditions on the parameters such that the involved hypergeometric or basic hypergeometric series converge. We shall not state these conditions for each identity. The reader should consult [3, p4-5]. We also use some elementary identities in [3, Appendix I] to prove the identities in our paper.

Recently, the authors[4] got a pair of inverse series relation including two independent bases:

$$ f(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{(1-Aq^{2k}/B)(Apq^k, Bpq^{-k};p)_{n-1}}{(Aq^k/B;q)_{n+1}} g(k), $$

$$ g(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{(1-Ap^kq^k)(1-Bp^kq^{-k})(Aq^n/B;q)_k q^{\binom{n-k}{2}}}{(Apq^n, Bpq^{-n};p)_k} f(k), \tag{1} $$

where p and q are independent bases and A, B are arbitrary parameters. It was shown that this formula could be used in the formula in [2, (2.8)] to derived the inverse form

$$ \frac{(a,b;p)_n}{(dq, adq/b;q)_n} = \sum_{k=0}^{n} \frac{(-1)^k(1-d)(1-ad/b)(1-ad^2q^{2k}/b)(ad^2/b;q)_k(adq^k, b/dq^k;p)_n q^{\binom{k+1}{2}}}{(1-dq^k)(1-adq^k/b)(q;q)_k(q;q)_{n-k}(ad^2/b;q)_{n+k+1}}, $$

$$ n = 0,1,2,\ldots. \tag{2} $$

Moreover, this formula could be used to derive some summation formulas and transformation formulas.

In this paper we use (1) to derive the transformation formula

$$ \begin{aligned} &\frac{(1-AP^nQ^n)(1-BP^nQ^{-n})}{(1-ap^nq^n)(1-bp^nq^{-n})(1-A)(1-B)}\times \\ &\quad \frac{(A,B;P)_n(C,A/BC;Q)_nQ^n}{(Q,AQ/B;Q)_n(AP/C,BCP;P)_n} \frac{(ap/c,bcp;p)_n(q;q)_n}{(c,a/bc;q)_n} \\ &= \sum_{k=0}^{n} \frac{(q^{-n},q)_k(apq^k,bpq^{-k};p)_{n-1}(cq,aq/bc;q)_k(1-aq^{2k}/b)q^{n(k+1)-k}}{(q;q)_k(aq^k/b;q)_{n+1}(c,a/bc;q)_k(1-apq^k)(1-bpq^{-k})}\times \\ &\quad \sum_{j=0}^{k} \frac{(1-ap^{j+1}q^j)(1-bp^{j+1}q^{-j})(q^{-k},aq^k/b;q)_j(ap/c,bcp;p)_jq^j}{(ap^2q^k,bp^2q^{-k};p)_j(cq,aq/bc;q)_j}\times \\ &\quad \frac{(CQ,AQ/BC;Q)_j(AP,BP;P)_j}{(AP/C,BCP;P)_j(Q,AQ/B;Q)_j}, \end{aligned} \tag{3} $$

where p, q, P and Q are independent bases and a, b, c, A, B, C are arbitrary parameters. Formula (3) is the inverse form of [1, (2.10)].

Observe that if $C = 1$, then formula (3) can be summed by [2, (2.8)], giving formula

$$(1-a)(1-b)\sum_{k=0}^{n}\frac{(q;q)_n(apq^k, bpq^{-k};p)_{n-1}(1-aq^{2k}/b)(-1)^k q^{n+\binom{k}{2}}}{(q;q)_k(q;q)_{n-k}(aq^k/b;q)_{n+1}} = \delta_{n,0}, \tag{4}$$

where $n = 0, 1, \ldots$, and $\delta_{n,k}$ is the Kronecker delta function. By switching p and q in (4) and replacing a, b by $a/q, b/q$, we find that

$$(1-a/q)(1-b/q)\sum_{k=0}^{n}\frac{(p;p)_n(ap^k, bp^{-k};q)_{n-1}(1-ap^{2k}/b)(-1)^k p^{n+\binom{k}{2}}}{(p;p)_k(p;p)_{n-k}(ap^k/b;p)_{n+1}} = \delta_{n,0}, \tag{5}$$

which is equivalent to Gasper's formula [1, (3.5)].

2. Derivation of (3)

Take $F(n) = (1-ap^n q^n)(1-bq^{-n}p^n)q^n f(n)$ and $G(n) = q^{-\binom{n}{2}}g(n)$ in (1) and then replace Ap and Bp by a and b to get

$$\begin{aligned} F(n) =&(1-ap^{n-1}q^n)(1-bp^{n-1}q^{-n})\times \\ &\sum_{k=0}^{n}\frac{(q^{-n};q)_k(1-aq^{2k}/b)(aq^k, bq^{-k};p)_{n-1}q^{nk+1}}{(q;q)_k(aq^k/b;q)_{n+1}}G(k), \\ G(n) =& \sum_{k=0}^{n}\frac{(q^{-n};q)_k(aq^n/b;q)_k}{(q;q)_k(aq^n, bq^{-n};p)_k}F(k). \end{aligned} \tag{6}$$

By replacing a, b, c by $1/ap^n q^n, 1/bp^n q^{-n}, 1/cq^n$ in [1, (2.10)], we find that

$$\begin{aligned} &\sum_{k=0}^{n}\frac{p^{2k}(1-ap^{n-k}q^{n-k})(1-bp^{n-k}q^{-n+k})}{(1-ap^n q^n)(1-bp^n q^{-n})}\times \\ &\frac{(1/ap^n q^n, 1/bp^n q^{-n};p)_k(1/cq^n, bc/aq^n;q)_k}{(q, b/aq^{2n-1};q)_k(c/ap^{n-1}, 1/cbp^{n-1};p)_k}\times \\ &\frac{(AP^{n-k+1}/C, BCP^{n-k+1};P)_k(Q^{n-k+1}, BQ^{n-k+1}/A;Q)_k q^k}{(CQ^{n-k+1}, AQ^{n-k+1}/BC;Q)_k(AP^{n-k+1}, BP^{n-k+1};P)_k} \\ =&\frac{(1/ap^{n-1}q^n, 1/bp^{n-1}q^{-n};p)_n(1/cq^{n-1}, bc/aq^{n-1};q)_n}{(q, b/aq^{2n-1};q)_n(c/ap^{n-1}, 1/cbp^{n-1};p)_n}\times \\ &\frac{(Q, AQ/B;Q)_n(AP/C, BCP;P)_n}{(AP, BP;P)_n(CQ, AQ/BC;Q)_n}\times \\ &\sum_{k=0}^{n}\frac{(1-AP^kQ^k)(1-BP^kQ^{-k})}{(1-A)(1-B)}\frac{(A, B;P)_k(C, A/BC;Q)_k}{(Q, AQ/B;Q)_k(AP/C, BCP;P)_k}\times \\ &\frac{(a/c, bc;p)_k(q^{-n}, aq^n/b;q)_k Q^k}{(c, a/bc;q)_k(aq^n, bq^{-n};p)_k}. \end{aligned}$$

Let

$$w(n):=\sum_{k=0}^{n}\frac{p^{2k}(1-ap^{n-k}q^{n-k})(1-bp^{n-k}q^{-n+k})}{(1-ap^nq^n)(1-bp^nq^{-n})}\times$$
$$\frac{(1/ap^nq^n,1/bp^nq^{-n};p)_k(1/cq^n,bc/aq^n;q)_k}{(q,b/aq^{2n-1};q)_k(c/ap^{n-1},1/cbp^{n-1};p)_k}\times$$
$$\frac{(AP^{n-k+1}/C,BCP^{n-k+1};P)_k(Q^{n-k+1},BQ^{n-k+1}/A;Q)_kq^k}{(CQ^{n-k+1},AQ^{n-k+1}/BC;Q)_k(AP^{n-k+1},BP^{n-k+1};P)_k},$$
$$h(n):=\frac{(1/ap^{n-1}q^n,1/bp^{n-1}q^{-n};p)_n(1/cq^{n-1},bc/aq^{n-1};q)_n}{(q,b/aq^{2n-1};q)_n(c/ap^{n-1},1/cbp^{n-1};p)_n}\times$$
$$\frac{(Q,AQ/B;Q)_n(AP/C,BCP;P)_n}{(AP,BP;P)_n(CQ,AQ/BC;Q)_n},$$
$$F(n):=\frac{(1-AP^nQ^n)(1-BP^nQ^{-n})}{(1-A)(1-B)}\frac{(a/c,bc;p)_n(q;q)_n}{(c,a/bc;q)_n}\times$$
$$\frac{(A,B;P)_n(C,A/BC;Q)_nQ^n}{(Q,AQ/B;Q)_n(AP/C,BCP;P)_n}.$$

Then, we have

$$w(n)/h(n)=\sum_{k=0}^{n}\frac{(q^{-n},aq^n/b;q)_k}{(q;q)_k(aq^n,bq^{-n};p)_k}F(k). \tag{8}$$

If applying inverse relation (6) to (8), we can get

$$F(n)=(1-ap^{n-1}q^n)(1-bp^{n-1}q^{-n})\times$$
$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k(1-aq^{2k}/b)(aq^k,bq^{-k};p)_{n-1}q^{n(k+1)}}{(q;q)_k(aq^k/b;q)_{n+1}}w(k)/h(k). \tag{9}$$

By straight forward calculations we can obtain

$$F(n)/(1-ap^{n-1}q^n)(1-bp^{n-1}q^{-n})$$
$$=\sum_{k=0}^{n}\frac{(q^{-n};q)_k(aq^k,bq^{-k};p)_{n-1}(cq,aq/bc;q)_k(1-aq^{2k}/b)q^{n(k+1)-k}}{(q;q)_k(aq^k/b;q)_{n+1}(c,q/bc;q)_k(1-aq^k)(1-bq^{-k})}\times$$
$$\sum_{j=0}^{k}\frac{q^j(1-ap^jq^j)(1-bp^jq^{-j})(q^{-k},aq^k/b;q)_j(a/c,bc;p)_j}{(apq^k,bpq^{-k};p)_j(cq,aq/bc;q)_j}\times$$
$$\frac{(AP,BP;P)_j(CQ,AQ/BC;Q)_j}{(Q,AQ/B;Q)_j(AP/C,BCP;P)_j}.$$

The formula (3) follows from above formula by replacing a,b by ap,bp, respectively.

3. Some consequences of formula (3)

When $P=Q=p=q$, $A=a,c=d$ and $n=0,1,\ldots$, after some simplification, (3) reduce to the summation formula

$$\frac{(B,C,a/BC,aq/d,bdq,aq/b;q)_n}{(aq/B,aq/C,BCq,d,a/bd,b;q)_n}$$
$$=\sum_{k=0}^{n}\frac{(a/b,aq/bd,dq,q^{-n},aq^n,1/bq;q)_k(1-aq^{2k}/b)q^k}{(q,d,a/bd,aq^{n+1},1/bq^{n-1},aq^2;q)_k(1-a/b)}\times$$
$${}_{10}W_9(aq;Bq,Cq,bdq,aq/d,aq/BC,aq^k/b,q^{-k};q,q). \tag{10}$$

If letting $n\to\infty$ in (10) , then we get a summation formula

$$\frac{(B,C,a/BC,aq/b,aq/d,bdq;q)_\infty}{(aq/B,aq/C,BCq,b,d,a/bd;q)_\infty}$$
$$=\sum_{k=0}^{\infty}\frac{(a/b,1/bq,dq,aq/bd;q)_k(1-aq^{2k}/b)b^k}{(q,aq^2,a/bd,d;q)_k(1-a/b)}\times$$
$${}_{10}W_9(aq;Bq,Cq,bdq,aq/d,aq/BC,aq^k/b,q^{-k};q,q). \tag{11}$$

Formula (3) can also be used to derive a summation formula containing two independent bases and shifted factorials by replacing A,B,C by P^A,P^B,P^C, setting $P=Q^\mu$, and letting $Q\to\infty$ to get

$$\frac{(A+n+n/\mu)(B+n-n/\mu)}{(1-ap^nq^n)(1-bp^nq^{-n})AB}\frac{(ap/c,bcp;p)_n(q;q)_n}{(c,a/bc;q)_n}\times$$
$$\frac{(A)_n(B)_n(C\mu)_n(A\mu-B\mu-C\mu)_n}{n!(A\mu+1-B\mu)_n(A+1-C)_n(B+C+1)_n}$$
$$=\sum_{k=0}^{n}\frac{(q^{-n};q)_k(apq^k,bpq^{-k};p)_{n-1}(1-cq^k)(1-aq^k/bc)(1-aq^{2k}/b)q^{n(k+1)-k}}{(q;q)_k(aq^k/b;q)_{n+1}(1-c)(1-a/bc)(1-apq^k)(1-bpq^{-k})}\times$$
$$\sum_{j=0}^{k}\frac{(1-ap^{j+1}q^j)(1-bp^{j+1}q^{-j})(aq^k/b;q)_j(ap/c,bcp;p)_j(q^{-k};q)_jq^j}{(ap^2q^k,bp^2q^{-k};p)_j(cq,aq/bc;q)_j}\times$$
$$\frac{(A+1)_j(B+1)_j(C\mu+1)_j(A\mu+1-B\mu-C\mu)_j}{j!(A\mu+1-B\mu)_j(A+1-C)_j(B+C+1)_j}, \tag{12}$$

which is the reverse form of the transformation formula in [1, (2.12)].

Similarly, replacing a,b,c in (12) by p^a,p^b,p^c, setting $p=q^\lambda$, and letting $q\to\infty$, we obtain

$$\frac{(A+n+n/\mu)(B+n-n/\mu)}{AB(a+n+n/\lambda)(b+n-n/\lambda)}\frac{(a+1-c)_n(b+c+1)_n}{(c\lambda)_n(A\lambda-B\lambda-C\lambda)_n}\times$$
$$\frac{(A)_n(B)_n(C\mu)(A\mu-B\mu-C\mu)_n}{(A\mu+1-B\mu)_n(A+1-C)_n(B+C+1)_n}$$
$$=\sum_{k=0}^{n}\frac{(-n)_k(1+k/c\lambda)(1+k/\lambda(a-b-c))(a\lambda-b\lambda+2k)(1+a+k/\lambda)_{n-1}(1+b-k/\lambda)_n}{k!(a\lambda+\lambda+k)(b\lambda+\lambda-k)(a\lambda-b\lambda+k)_{n+1}}$$
$$\sum_{j=0}^{k}\frac{(a\lambda+j\lambda+j+\lambda)(b\lambda+j\lambda-j+\lambda)(a\lambda-b\lambda+k)_j(a+1-c)_j(b+1+c)_j(-k)_j}{(c\lambda+1)_j(a\lambda-b\lambda-c\lambda+1)_j(2+a+k/\lambda)_j(2+b-k/\lambda)_j}\times$$

$$\frac{(A+1)_j(B+1)_j(C\mu+1)_j(A\mu+1-B\mu-C\mu)_j}{j!(A\mu+1-B\mu)_j(A+1-C)_j(B+C+1)_j}, \tag{13}$$

which is the reverse form of the rather strange looking ${}_{10}F_9$ transformation formula in [1, (2.13)].

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一个含四个独立基的变换公式及其应用

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摘 要: 本文利用反演的方法得到了一个四个独立基的变换公式并由此得到了几个新的基本超几何级数求和公式和超几何级数求和公式.

关键词: 反演关系；基本超几何级数；双基求和公式；变换公式.