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The Correction of Extension of Steffensen's Inequality*

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Abstract

Richard Bellman pointed out an extension of Steffensen's inequality in [1] and [2]. But its result and proof are wrong. In this paper, these mistakes are corrected.

The well-known Steffensen's inequality is the following.

Let (a) $f(t)$ be nonnegative and monotone decreasing in $[a,b]$; (b) $g(t)$ satisfy the constraint $0\le g(t)\le 1$, $t\in[a,b]$. Then

$$\int_{b-c}^{b} f(t)\,dt\le\int_a^b f(t)\,g(t)\,dt\le\int_a^{a+c} f(t)\,dt,$$

where $c=\int_a^b g(t)\,dt$.

Its proof refers to [1] or [2]. Olkin's inequality can be easily proved by applying Steffensen's inequality.

R. Ballman raises an extension in [2] as follows:

Theorem (Bellman). Let (a) $f(t)$ be nonnegative and monotone decreasing in $[a,b]$, $f\in L^p[a,b]$; (b) $g(t)$ satisfy relation $g(t)\ge 0$ in $[a,b]$ and $\int_a^b g^q\,dt\le 1$, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$. Then

$$\left(\int_a^b f(t)g(t)\,dt\right)^p\le\int_a^{a+c} f^p(t)\,dt,$$

where $c=\left(\int_a^b g(t)\,dt\right)^{p/q}$.

Remark In [1] and [2], $c=a+\left(\int_a^b g(t)\,dt\right)^p$. But, from context, it is clearly a slip of the pen.

The following example shows that the theorem cannot be established.

Example 1 Let $a=0$, $b=1$, $p=q=2$,

$$f(t)=\begin{cases}(2n(1-nt))^{1/2} & 0\le t<\frac{1}{n}\\ 0 & \frac{1}{n}\le t\le 1,\end{cases}$$

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where is a real number greater than one. Again, let $g(t)=f(t)$. So

$$\int_a^b (g(t))^q \mathrm{d}t=\int_0^{\frac{1}{n}} 2n(1-nt)\mathrm{d}t=1.$$

Then $f(t)$ and $g(t)$ satisfy Theorem's conditions. thus

$$c=\left(\int_a^b g(t)\mathrm{d}t\right)^p=\left(\int_0^{\frac{1}{n}}(2n(1-nt))^{\frac{1}{2}}\mathrm{d}t\right)^2=\frac{8}{9n},$$

$$\left(\int_a^b f(t)g(t)\mathrm{d}t\right)^p=\left(\int_0^{\frac{1}{n}}(2n(1-nt))\mathrm{d}t\right)^2-1,$$

$$\int_a^{a+c} f^p(t)\mathrm{d}t=\int_0^{\frac{8}{9n}}(2n(1-nt))\mathrm{d}t=1-\left(\frac{1}{9}\right)^2,$$

therefore

$$\left(\int_a^b f(t)g(t)\mathrm{d}t\right)^p>\int_a^{a+c} f^p(t)\mathrm{d}t,$$

which is contrary to Bellman's Theorem.

We give a correction of the theorem as follows.

In Bellman's theorem

$$c=\begin{cases}\left(\dfrac{f(a+0)}{f(b-0)}\right)^{p-1}\left(\int_a^b g(t)\mathrm{d}t\right)^p, & f(b-0)>0\\ b-a & f(b-0)=0.\end{cases}$$

Proof Applying Hölder inequality

$$\left(\int_a^t f(x)g(x)\mathrm{d}x\right)^p\leq\left(\int_a^t f^p(x)\mathrm{d}x\right)\left(\int_a^t g^q(x)\mathrm{d}x\right)^{p/q}\leq\int_a^t f^p(x)\mathrm{d}x. \quad (*)$$

Defining function $u=u(t)$ in $[a,b]$ determined by the following equation

$$\left(\int_a^t f(x)g(x)\mathrm{d}x\right)^p=\int_a^{u(t)} f^p(x)\mathrm{d}x,$$

clearly, $u(t)$ satisfies relation $u(t)\leq t$ and $u(a)=a$. For this reason, differential equation

$$f^p(u)\frac{\mathrm{d}u}{\mathrm{d}t}=pf(t)g(t)\left(\int_a^t f(x)g(x)\mathrm{d}x\right)^{p-1}$$

is established almost everywhere.

If $f(b-0)>0$, we have

$$\frac{\mathrm{d}u}{\mathrm{d}t}\leq p\left(\frac{f(a+0)}{f(b-0)}\right)^{p-1}g(t)\left(\int_a^t g(x)\mathrm{d}x\right)^{p-1},$$

as a result

$$u(t)\leq a+\left(\frac{f(a+0)}{f(b-0)}\right)^{p-1}\left(\int_a^t g(x)\mathrm{d}x\right)^p.$$

If $f(b-0)=0$, from expression $(*)$, we know $u(t)\leq b$ is always held, while example 1 shows that equality can be held. Thus our proof is complete.

Example 2 Let $f(t)=\mathrm{d}>0$ (constant),
$g(t)=e>0$ (constant),

where $e \leq \left(\frac{1}{b-a}\right)^{1/q}$, so

$$c = \left(\frac{f(a+0)}{f(b-0)}\right)^{p-1}\left(\int_a^b g(t)\,\mathrm{d}t\right)^p = (c(b-a))^p,$$

$$\left(\int_a^b f(t)g(t)\,\mathrm{d}t\right)^p = (\mathrm{d}e(b-a))^p,$$

$$\int_a^{a+c} f^p(t)\,\mathrm{d}t = (\mathrm{d}e(b-a))^p,$$

therefore

$$\left(\int_a^b f(t)g(t)\,\mathrm{d}t\right)^p = \int_a^{a+c} f^p(t)\,\mathrm{d}t.$$

Example 1 and example 2 indicate that the expression of c is optimal.

References

[1] Beckenbach, E. F. and Bellman, R., Inequalities, Fourth Printing, Springer-Verlag, Berlin Heidelberg New york Tokyo, 48—49, 1983.

[2] Bellman, R., Proc. Amer. Math. Soc. 10, 807—809, (1959).

关于Steffensen不等式推广的注记

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摘　要

Richard Bellman在[1]与[2]中给出了Steffensen不等式的一个推广．但结果与证明有误．本文更正了这些错误．