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A Compact Determinantal Representation for Inverse Difference *

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Abstract: A compact determinantal representation for inverse difference is given. From the representation it is easy to know that the inverse difference of a function at some $n+1$ points $x_0, x_1, \cdots, x_n$ depends on the orderings of these points and it is locally independent of the permutation of first $n-1$ points. Moreover we define reciprocal difference from another point of view, get the relation between inverse difference and reciprocal difference and obtain the property that the reciprocal difference is globally independent of the permutation of the points.

Key words: inverse difference; reciprocal difference; determinant.

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For a given set of real points $X = \{x_0, x_1, \cdots\} \subset [a,b] \subset \mathbb{R}$ and a function $f(x)$ defined in $[a,b]$, let

$$\begin{aligned} R_n(x) &= a_0 + \frac{x-x_0|}{|a_1} + \frac{x-x_1|}{|a_2} + \cdots + \frac{x-x_{n-1}|}{|a_n} \\ &= P_n(x)/Q_n(x), \end{aligned} \tag{1}$$

then $R_n(x)$ is a rational function of type $([\frac{n+1}{2}]/[\frac{n}{2}])$ (which means that $P_n(x)$ and $Q_n(x)$ are polynomials of degree not exceeding $[\frac{n+1}{2}]$ and $[\frac{n}{2}]$ respectively, where $[x]$ denotes the greatest integer not exceeding x) and we have ([2])

$$P_n(x) = a_n P_{n-1}(x) + (x - x_{n-1}) P_{n-2}(x), \tag{2}$$
$$Q_n(x) = a_n Q_{n-1}(x) + (x - x_{n-1}) Q_{n-2}(x). \tag{3}$$

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Therefore

$$\begin{aligned} a_n &= P_n(x_{n-1})/P_{n-1}(x_{n-1}) &(4)\\ &= Q_n(x_{n-1})/Q_{n-1}(x_{n-1}). &(5)\end{aligned}$$

It is not difficult to prove by induction that

$$\begin{aligned} L(P_n(x)) &= 1 \text{ for odd } n,\\ L(Q_n(x)) &= 1 \text{ for even } n,\end{aligned}$$

where $L(P_n(x))$ denotes the leading coefficient of the polynomial $P_n(x)$. Let

$$a_i = \phi[x_0, x_1, \cdots, x_i],\ i = 0, 1, \cdots, n, \tag{6}$$

where $\phi[x_0, x_1, \cdots, x_i]$ is the inverse difference defined as follows

$$\begin{aligned} \phi[x_p] &= f(x_p),\ i = 0, 1, \cdots, n,\\ \phi[x_p, x_q] &= \frac{x_q - x_p}{f(x_q) - f(x_p)},\\ \phi[x_0, \cdots, x_i] &= \frac{x_i - x_{i-1}}{\phi[x_0, \cdots, x_{i-2}, x_i] - \phi[x_0, \cdots, x_{i-2}, x_{i-1}]}.\end{aligned} \tag{7}$$

In this case, one gets

$$R_n(x_i) = f(x_i) = f_i,\ i = 0, 1, \cdots, n. \tag{8}$$

Let

$$N_{2k}(x) = -\frac{\begin{vmatrix} 1 & x_0 & \cdots & x_0^k & f_0 & x_0 f_0 & \cdots & x_0^k f_0\\ 1 & x_1 & \cdots & x_1^k & f_1 & x_1 f_1 & \cdots & x_1^k f_1\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & x_{2k} & \cdots & x_{2k}^k & f_{2k} & x_{2k} f_{2k} & \cdots & x_{2k}^k f_{2k}\\ 1 & x & \cdots & x^k & 0 & 0 & \cdots & 0 \end{vmatrix}}{\begin{vmatrix} 1 & x_0 & \cdots & x_0^k & f_0 & x_0 f_0 & \cdots & x_0^{k-1} f_0\\ 1 & x_1 & \cdots & x_1^k & f_1 & x_1 f_1 & \cdots & x_1^{k-1} f_1\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & x_{2k} & \cdots & x_{2k}^k & f_{2k} & x_{2k} f_{2k} & \cdots & x_{2k}^{k-1} f_{2k} \end{vmatrix}}, \tag{9}$$

$$D_{2k}(x) = \frac{\begin{vmatrix} 1 & x_0 & \cdots & x_0^k & f_0 & x_0 f_0 & \cdots & x_0^k f_0\\ 1 & x_1 & \cdots & x_1^k & f_1 & x_1 f_1 & \cdots & x_1^k f_1\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & x_{2k} & \cdots & x_{2k}^k & f_{2k} & x_{2k} f_{2k} & \cdots & x_{2k}^k f_{2k}\\ 0 & 0 & \cdots & 0 & 1 & x & \cdots & x^k \end{vmatrix}}{\begin{vmatrix} 1 & x_0 & \cdots & x_0^k & f_0 & x_0 f_0 & \cdots & x_0^{k-1} f_0\\ 1 & x_1 & \cdots & x_1^k & f_1 & x_1 f_1 & \cdots & x_1^{k-1} f_1\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & x_{2k} & \cdots & x_{2k}^k & f_{2k} & x_{2k} f_{2k} & \cdots & x_{2k}^{k-1} f_{2k} \end{vmatrix}}, \tag{10}$$

$$N_{2k+1}(x) = (-1)^{k+1} \frac{\begin{vmatrix} 1 & x_0 & \cdots & x_0^{k+1} & f_0 & x_0 f_0 & \cdots & x_0^k f_0 \\ 1 & x_1 & \cdots & x_1^{k+1} & f_1 & x_1 f_1 & \cdots & x_1^k f_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2k+1} & \cdots & x_{2k+1}^{k+1} & f_{2k+1} & x_{2k+1} f_{2k+1} & \cdots & x_{2k+1}^k f_{2k+1} \\ 1 & x & \cdots & x^{k+1} & 0 & 0 & \cdots & 0 \end{vmatrix}}{\begin{vmatrix} 1 & x_0 & \cdots & x_0^{k} & f_0 & x_0 f_0 & \cdots & x_0^k f_0 \\ 1 & x_1 & \cdots & x_1^{k} & f_1 & x_1 f_1 & \cdots & x_1^k f_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2k+1} & \cdots & x_{2k+1}^{k} & f_{2k+1} & x_{2k+1} f_{2k+1} & \cdots & x_{2k+1}^k f_{2k+1} \end{vmatrix}}, \tag{11}$$

$$D_{2k+1}(x) = (-1)^{k} \frac{\begin{vmatrix} 1 & x_0 & \cdots & x_0^{k+1} & f_0 & x_0 f_0 & \cdots & x_0^k f_0 \\ 1 & x_1 & \cdots & x_1^{k+1} & f_1 & x_1 f_1 & \cdots & x_1^k f_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2k+1} & \cdots & x_{2k+1}^{k+1} & f_{2k+1} & x_{2k+1} f_{2k+1} & \cdots & x_{2k+1}^k f_{2k+1} \\ 0 & 0 & \cdots & 0 & 1 & x & \cdots & x^{k} \end{vmatrix}}{\begin{vmatrix} 1 & x_0 & \cdots & x_0^{k} & f_0 & x_0 f_0 & \cdots & x_0^k f_0 \\ 1 & x_1 & \cdots & x_1^{k} & f_1 & x_1 f_1 & \cdots & x_1^k f_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2k+1} & \cdots & x_{2k+1}^{k} & f_{2k+1} & x_{2k+1} f_{2k+1} & \cdots & x_{2k+1}^k f_{2k+1} \end{vmatrix}}, \tag{12}$$

From the fact that $N_n(x)/D_n(x)$ determined by (9)–(12) is a rational function of type $([n+1/2]/[n/2])$, $N_n(x_i)/D_n(x_i) = f_i$, $i = 0, 1, \cdots, n$ (see [1], [4]) and $L(N_{2k+1}(x)) = 1, L(Q_{2k}(x)) = 1$ it follows

$$P_n(x) \equiv N_n(x), \;\; Q_n(x) \equiv D_n(x). \;\; n = 0, 1, 2, \cdots. \tag{13}$$

Let

$$D^{(k,l)} = \begin{vmatrix} 1 & x_0 & \cdots & x_0^{k} & f_0 & x_0 f_0 & \cdots & x_0^l f_0 \\ 1 & x_1 & \cdots & x_1^{k} & f_1 & x_1 f_1 & \cdots & x_1^l f_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{k+l+1} & \cdots & x_{k+l+1}^{k} & f_{k+l+1} & x_{k+l+1} f_{k+l+1} & \cdots & x_{k+l+1}^l f_{k+l+1} \end{vmatrix} \tag{14}$$

and denote by $D_{p;q}^{(k,l)}$ the determinant formed by deleting the p^{th} row and the q^{th} column from $D^{(k,l)}$. Then we have

$$\phi[x_0, x_1, \cdots, x_n] \;\; = \;\; N_n(x_{n-1})/N_{n-1}(x_{n-1})$$

$$= (-1)^{[\frac{n}{2}]}(x_n - x_{n-1})\frac{D^{([\frac{n-1}{2}],[\frac{n-2}{2}])}D_{n;n+1}^{([\frac{n-1}{2}],[\frac{n}{2}])}}{D^{([\frac{n-2}{2}],[\frac{n-3}{2}])}D^{([\frac{n}{2}],[\frac{n-1}{2}])}}. \tag{15}$$

On the other hand, by (2), (3) and (6) we have

$$\begin{aligned} \phi[x_0, x_1, \cdots, x_n] &= \frac{P_n(x)Q_{n-2}(x) - P_{n-2}(x)Q_n(x)}{P_{n-1}(x)Q_{n-2}(x) - P_{n-2}(x)Q_{n-1}(x)} \\ &= (-1)^n \frac{P_n(x)Q_{n-2}(x) - P_{n-2}(x)Q_n(x)}{(x-x_0)(x-x_1)\cdots(x-x_{n-2})}. \end{aligned} \tag{16}$$

Since both $P_n(x)Q_{n-2}(x)$ and $P_{n-2}(x)Q_n(x)$ are polynomials of degree $n-1$, we get

$$\phi[x_0, x_1, \cdots, x_n] = (-1)^n L(P_n(x)Q_{n-2}(x) - P_{n-2}(x)Q_n(x)). \tag{17}$$

With the relation (13) in mind, we obtain

$$\begin{aligned} \phi[x_0, x_1, \cdots, x_{2k}] &= L(N_{2k}(x)) - L(N_{2k-2}(x)), \\ \phi[x_0, x_1, \cdots, x_{2k+1}] &= L(D_{2k+1}(x)) - L(D_{2k-1}(x)). \end{aligned} \tag{18}$$

Let us define

$$\begin{aligned} \rho[x_0, x_1, \cdots, x_{2k}] &= L(N_{2k}(x)), \\ \rho[x_0, x_1, \cdots, x_{2k+1}] &= L(D_{2k+1}(x)), \end{aligned} \tag{19}$$

then by (9) and (12) we have

$$\begin{aligned} \rho[x_0, x_1, \cdots, x_{2k}] &= (-1)^k \frac{D^{(k-1,k)}}{D^{(k,k-1)}}, \\ \rho[x_0, x_1, \cdots, x_{2k+1}] &= (-1)^k \frac{D^{(k+1,k)}}{D^{(k,k)}}, \end{aligned} \tag{20}$$

and thus we derive another representation for inverse difference $\phi[x_0, x_1, \cdots, x_n]$

$$\begin{aligned} \phi[x_0, x_1, \cdots, x_{2k}] &= \rho[x_0, x_1, \cdots, x_{2k}] - \rho[x_0, x_1, \cdots, x_{2k-2}] \\ &= (-1)^k \left(\frac{D^{(k-1,k)}}{D^{(k,k-1)}} + \frac{D^{(k-2,k-1)}}{D^{(k-1,k-2)}}\right), \end{aligned} \tag{21}$$

$$\begin{aligned} \phi[x_0, x_1, \cdots, x_{2k+1}] &= \rho[x_0, x_1, \cdots, x_{2k+1}] - \rho[x_0, x_1, \cdots, x_{2k-1}] \\ &= (-1)^k \left(\frac{D^{(k+1,k)}}{D^{(k,k)}} + \frac{D^{(k,k-1)}}{D^{(k-1,k-1)}}\right). \end{aligned} \tag{22}$$

From relations (21) and (22), one observes that $\rho[x_0, x_1, \cdots, x_n]$ defined by (19) is the very reciprocal difference of $f(x)$ at points $x_0, x_1, \cdots, x_n$ (see [3]). From the determinantal expressions (15) and (20) it is clear that the inverse difference $\phi[x_0, x_1, \cdots, x_n]$ depends on the orderings of the points $x_0, x_1, \cdots, x_n$, but is independent of the permutations of $x_0, x_1, \cdots, x_{n-2}$ while the reciprocal difference $\rho[x_0, x_1, \cdots, x_n]$ is independent of the global permutations of all the points $x_0, x_1, \cdots, x_n$. As a by-product, from (15), (21) and (22) we acquire the following identical relations

$$\begin{aligned} &D^{(k-1,k)}D^{(k-1,k-2)} + D^{(k-2,k-1)}D^{(k,k-1)} \\ &\quad = (x_{2k} - x_{2k-1})D^{(k-1,k-1)}D^{(k-1,k)}_{2k;2k+1}, \\ &D^{(k+1,k)}D^{(k-1,k-1)} + D^{(k,k-1)}D^{(k,k)} \\ &\quad = (x_{2k+1} - x_{2k})D^{(k,k-1)}D^{(k,k)}_{2k+1;2k+2}. \end{aligned} \tag{23}$$

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逆差商的一个紧凑行列式表示

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摘 要: 本文给出了逆差商的一个紧凑行列式表示. 从该表示式易知，一个函数在某 $n+1$ 个点的 n 阶逆差商与这些点的排序有关，但与前 n-1 个点的局部换序无关. 此外，还从另一角度定义倒差商，得出了倒差商与逆差商之间的关系以及倒差商的整体换序无关性.