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Moment Bounds for Strong Mixing Sequences and Their Application *

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Abstract: Some moment inequalities for the strong mixing random variable sequence are established, and applied to discuss the asymptotic normality of the general weight function estimate for the fixed design regression model.

Key words: strong mixing; moment inequality; fixed design; regression; weight estimate; asymptotic normality.

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1. Introduction

Suppose $\{X_i : i \in \mathbf{Z}\}$ is a real-valued random variable sequence on a probability space $(\Omega, \mathcal{B}, P)$. Let $\mathcal{F}_n^m$ denote the σ-fieled generated by $(X_i : m \le i \le n)$. Let

$$\alpha(n) = \sup\{|P(AB) - P(A)P(B)| : A \in \mathcal{F}_{-\infty}^m, B \in \mathcal{F}_{m+n}^\infty\}$$

The sequence $\{X_i\}$ is said to be α-mixing or strong mixing if $\alpha(n) \to 0$ as $n \to \infty$.

The moment inequalities of partial sum $\sum_{i=1}^n X_i$ are important tools in researching asymptotic property of mixing sequences. So many scholars have been trying to develop them. For example, in the ϕ - mixing or ρ - mixing cases, Billingsley[1] and Peligrad[8–10] gave some inqualities for some special order moments. Afterward, Shao Qiman[15,16] and Yang Shanchao [18] obtained some better inequalities for general order moment. However, the resemblance moment inequalities for strong mixing are rarely found. Yokoyama [19] got one only for a strictly stationary strong mixing sequence. That is

$$E|\sum_{i=1}^n X_i|^r \le Cn^{r/2} \quad \text{for} \quad r > 2$$

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as $\alpha(n)$ satisfies some conditions. Therefore, the main purpose of this paper is to develop some more general moment inequalities for strong mixing sequences, and apply them to discuss the asymptotic normality of the nonparametric weight function estimate for the fixed design regression model.

2. Moment Inequality

Throughout this section, it is supposed that $\{X_i : i \geq 1\}$ is a strong mixing random variable sequence with $EX_i = 0$ for all $i \geq 1$, C denotes constant which only depends on some given numbers, $[x]$ denotes the integral part of x, and $||X||_r := (E|X|^r)^{1/r}$.

Theorem 2.1 *Let $\{a_i : i \geq 1\}$ be a real sequence.*

(i) If $E|X_i|^{2+\delta} < \infty$ for some $\delta > 0$, then for all $n \geq 1$

$$E(\sum_{i=1}^{n} a_i X_i)^2 \leq (1 + 20 \sum_{m=1}^{n} \alpha^{\delta/(2+\delta)}(m)) \sum_{i=1}^{n} a_i^2 ||X_i||_{2+\delta}^2. \tag{2.1}$$

(ii) If $|X_i| \leq b_i$ a.s. Then for all $n \geq 1$

$$E(\sum_{i=1}^{n} a_i X_i)^2 \leq (1 + 8 \sum_{m=1}^{n} \alpha(m)) \sum_{i=1}^{n} a_i^2 b_i^2. \tag{2.2}$$

Theorem 2.2 *Let $|X_i| \leq b_i, a.s.$*

(i) If $1 < r \leq 2$, then for given $\theta \in (0,1)$ and all $n \geq 1$

$$E|\sum_{i=1}^{n} X_i|^r \leq C\{n^{\theta(r-1)} \sum_{i=1}^{n} E|X_i|^r + \alpha([n^\theta])(\sum_{i=1}^{n} b_i)^r\}. \tag{2.3}$$

(ii) If $r > 2$ and $\Sigma_{i=1}^{\infty} \alpha(i) < \infty$, then for given $\theta \in (0,1)$ and all $n \geq 1$

$$E|\sum_{i=1}^{n} X_i|^r \leq C\{n^{\theta(r-1)} \sum_{i=1}^{n} E|X_i|^r + \alpha([n^\theta])(\sum_{i=1}^{n} b_i)^r + (\sum_{i=1}^{n} b_i^2)^{r/2}\}. \tag{2.4}$$

Theorem 2.3 *Let $|X_i| \leq b_i$ a.s. $r > 2$ and $\Sigma_{i=1}^{\infty} \alpha(i) < \infty$. If there exist $\lambda \geq 1$ and $0 < \theta < 1$ such that $r/2 - \lambda\theta \geq 0$ and $\alpha(n) = O(n^{-\lambda})$, then for given $\varepsilon > 0$ and all $n \geq 1$*

$$E|\sum_{i=1}^{n} X_i|^r \leq C\{n^\varepsilon \sum_{i=1}^{n} E|X_i|^r + n^{r/2-\lambda\theta}(\sum_{i=1}^{n} b_i^2)^{r/2}\}. \tag{2.5}$$

Theorem 2.4 *Let $r > 2$. Suppose there exist $\delta > 0$ and $\lambda > r(r+\delta)/2\delta$ such that $\alpha(n) = O(n^{-\lambda})$ and $E|X_i|^{r+\delta} < \infty$. Then for given $\varepsilon > 0$ and all $n \geq 1$*

$$E|\sum_{i=1}^{n} X_i|^r \leq C\{n^\varepsilon \sum_{i=1}^{n} E|X_i|^r + (\sum_{i=1}^{n} ||X_i||_{r+\delta}^2)^{r/2}\}. \tag{2.6}$$

Further, if $\{X_i : i \geq 1\}$ is a strictly stationary sequence, then for all $n \geq 1$

$$E|\sum_{i=1}^{n} X_i|^r \leq Cn^{r/2} ||X_1||_{r+\delta}^r. \tag{2.7}$$

To prove our theorems, we first give the following lemmas.

Lemma 1[13] *Suppose that ξ and η are $\mathcal{F}_1^k$ - measurable and $\mathcal{F}_{k+n}^{\infty}$ - measurable random variables, respectively.*

(i) If $|\xi| \le C_1$ a.s. and $||\eta| \le C_2$ a.s., *then*

$$|E(\xi\eta) - (E\xi)(E\eta)| \le 4C_1C_2\alpha(n).$$

(ii) If $E|\xi|^p < \infty$, $E|\eta|^p < \infty$ *for some* $p,\ q,\ t > 1$ *with* $1/p + 1/q + 1/t = 1$, *then*

$$|E(\xi\eta) - (E\xi)(E\eta)| \le 10\alpha^{1/t}(n)||\xi||_p \cdot ||\eta||_q.$$

Lemma 2 *For any* $x,\ y \in R^1$, *have*

$$|x+y|^r \le |y|^r + rx|y|^{r-1}\mathrm{sgn}(y) + 2|x|^r, \quad \text{for } 1 < r \le 2, \tag{2.8}$$

$$|x+y|^r \le |y|^r + d_1|x|^r + d_2x|y|^{r-1}\mathrm{sgn}(y) + d_3x^2|y|^{r-2}, \quad \text{for } r > 2, \tag{2.9}$$

where $d_1 = 2^r,\ d_2 = r,\ d_3 = 2^r \cdot r^2$.

Proof It is easy to show that $|1+t|^r \le 1 + rt + 2|t|^r$ for $r \in (1,2], t \in R^1$ and $|1+t|^r \le 1 + d_1|t|^r + d_2t + d_3t^2$ for $r > 2, t \in R^1$. This implies lemma 2.

Proof of Theorem 2.1 By lemma 1(ii)

$$\begin{aligned}E(\sum_{i=1}^n a_iX_i^2) &\le \sum_{i=1}^n a_i^2EX_i^2 + 20\sum_{i=1}^{n-1}\sum_{j=i+1}^n |a_ia_j|\alpha^{\delta/(2+\delta)}(j-i)||X_i||_{2+\delta}||X_j||_{2+\delta}\\ &\le (1 + 20\sum_{m=1}^n \alpha^{\delta/(2+\delta)}(m))\sum_{i=1}^n a_i^2||X_i||_{2+\delta}^2.\end{aligned}$$

Hence (2.1) holds. Simarly, we get (2.2) by lemma 1(i). Completing the proof.

Fix n and redefine $X_i,\ b_i$ as $X_i = X_i, b_i = b_i$ for $1 \le i \le n$ and $X_i = 0, b_i = 0$ for $i > n$. Let

$$k = [(n/2)^\theta], \quad m = [(n/2)^{1-\theta}], \tag{2.10}$$

where $0 < \theta < 1$. Clearly,

$$n < 2(m+1)k, \quad [n^\theta] < k < 2n^\theta, \quad m < n^{1-\theta}. \tag{2.11}$$

Denote $M_i = \{2(i-1)k+1,\ 2(i-1)k+2,\ \ldots,\ (2i-1)k,\}$, $\widetilde{M_i} = \{(2i-1)k+1,\ (2i-1)k+2,\ \ldots,\ 2ik\}$ for $i = 1,\ 2,\ \ldots,\ m+1$. Let

$$Y_i = \sum_{j\in M_i} X_j,\ Z_i = \sum_{j\in\widetilde{M_i}} X_j,\ A_i = \sum_{j\in M_i} b_j,\ \widetilde{A_i} = \sum_{j\in\widetilde{M_i}} b_j,\ B_i = \sum_{j\in M_i} b_j^2,$$
$$\widetilde{B_i} = \sum_{j\in\widetilde{M_i}} b_j^2,\ E_i = \sum_{j\in M} E|X_j|^r,\ \widetilde{E_i} = \sum_{j\in\widetilde{M_i}} E|X_j|^r,$$

for $i = 1,\ 2,\ \ldots,\ m+1$. Thus

$$\sum_{i=1}^{n} X_i = \sum_{i=1}^{m+1} Y_i + \sum_{i=1}^{m+1} Z_i, \tag{2.12}$$

$$\sum_{i=1}^{n} b_i = \sum_{i=1}^{m+1} A_i + \sum_{i=1}^{m+1} \widetilde{A_i}, \tag{2.13}$$

$$\sum_{i=1}^{n} b_i^2 = \sum_{i=1}^{m+1} B_i + \sum_{i=1}^{m+1} \widetilde{B_i}, \tag{2.14}$$

$$\sum_{i=1}^{n} E|X_i|^r = \sum_{i=1}^{m+1} E_i + \sum_{i=1}^{m+1} \widetilde{E_i}, \tag{2.15}$$

Proof of Theorem 2.2 As $1 < r \le 2$, form (2.8), lemma 1 (i) and (2.13)

$$\begin{aligned} E|\sum_{i=1}^{m+1} Y_i|^r = |Y_1 + \sum_{i=2}^{m+1} Y_1|^r &\le E\{|\sum_{i=2}^{m+1} Y_i|^r + rY_1|\sum_{i=2}^{m+1} Y_i|^{r-1}\mathrm{sgn}(\sum_{i=2}^{m+1} Y_i) + 2E|Y_1|^r \\ &\le E|\sum_{i=2}^{m+1} Y_i|^r + 4r\alpha(k)A_1(\sum_{i=1}^{n} b_i)^{r-1} + 2E|Y_1|^2 \\ &\le \cdots \\ &\le 2\sum_{i=1}^{m+1} E|Y_i|^r + 4r\alpha(k)(\sum_{i=1}^{m} A_i)(\sum_{i=1}^{n} b_i)^{r-1} \\ &\le 2\sum_{i=1}^{m+1} E|Y_i|^r + 4r\alpha(k)(\sum_{i=1}^{n} b_i)^r. \end{aligned} \tag{2.16}$$

In the same way

$$E|\sum_{i=1}^{m+1} Z_i|^r \le 2\sum_{i=1}^{m+1} E|Z_i|^r + 4r\alpha(k)(\sum_{i=1}^{n} b_i)^r. \tag{2.17}$$

Combining (2.12), (2.16) and (2.17)

$$E|\sum_{i=1}^{n} X_i|^r \le 2\sum_{i=1}^{m+1}(E|Y_i|^r + E|Z_i|^r) + 8r\alpha(k)(\sum_{i=1}^{n} b_i)^r.$$

Using Minkowski inequality to $E|Y_i|^r$ and $E|Z_i|^r$ and noting (2.11), we get (2.3).

As $r > 2$, from (2.9) and lemma 1(i)

$$\begin{aligned} E|\sum_{i=1}^{m+1} Y_i|^r &= E|Y_1 + \sum_{i=2}^{m+1} Y_i|^r \\ &\le E\{|\sum_{i=2}^{m+1} Y_i|^r + d_1|Y_1|^r + d_2Y_1|\sum_{i=2}^{m+1} Y_i|^{r-1}\mathrm{sgn}(\sum_{i=2}^{m+1} Y_i) + d_3Y_1^2|\sum_{i=2}^{m+1} Y_i|^{r-2}\} \\ &\le E|\sum_{i=2}^{m+1} Y_i|^r + d_1E|Y_1|^r + 4d_2\alpha(k)A_1(\sum_{i=1}^{n} b_i)^{r-1} + d_3EY_1^2E|\sum_{i=2}^{m+1} Y_i|^{r-2} + \\ &\quad 4d_3\alpha(k)A_1^2(\sum_{i=1}^{n} b_i)^{r-2} \\ &\le \cdots \end{aligned}$$

$$\leq d_1 \sum_{i=1}^{m+1} E|Y_i|^r + 8d_3\alpha(k)(\sum_{i=1}^{n} b_i)^r + d_3\triangle, \tag{2.18}$$

where $\triangle := \sum_{j=1}^{m} EY_j^2 E|\sum_{i=j+1}^{m+1} Y_i|^{r-2}$. Now we first prove the following

$$E|\sum_{i=1}^{m+1} Y_i|^r \leq C\{\sum_{i=1}^{m+1} E|Y_i|^r + \alpha(k)(\sum_{i=1}^{n} b_i)^r + (\sum_{i=1}^{n} b_i^2)^{r/2}\}. \tag{2.19}$$

(1) As $2 < r \leq 4$, by Jessen inequality and theorem 1 (ii)

$$\triangle \leq \sum_{j=1}^{m} EY_j^2\{E(\sum_{i=j+1}^{m+1} Y_i)^2\}^{(r-2)/2} \leq C\sum_{j=1}^{m} B_j(\sum_{i=1}^{n} b_i^2)^{(r-2)/2} \leq C(\sum_{i=1}^{n} b_i^2)^{r/2}. \tag{2.20}$$

Hence (2.18) implies (2.19).

(2) Suppose (2.19) hold for $r-2$ with $r > 4$. Then

$$\triangle \leq C\sum_{j=1}^{m} EY_j^2\{\sum_{i=1}^{m+1} E|Y_i|^{r-2} + \alpha(k)(\sum_{i=1}^{n} b_i)^{r-2} + (\sum_{i=1}^{n} b_i^2)^{(r-2)/2}\}$$

$$\leq C\sum_{j=1}^{m} EY_j^2 \sum_{i=1}^{m+1} E|Y_i|^{r-2} + \alpha(k)(\sum_{i=1}^{n} b_i)^r + (\sum_{i=1}^{n} b_i^2)^{r/2} \tag{2.21}$$

follows from theorem 1 (ii). And

$$\sum_{j=1}^{m} EY_j^2 \sum_{i=1}^{m+1} E|Y_i|^{r-2} \leq \sum_{j=1}^{m} EY_j^2\{(\sum_{i=1}^{m+1} EY_j^2)^{2/(r-2)}(\sum_{i=1}^{m+1} E|Y_i|^r)^{(r-4)/(r-2)}\}$$

$$\leq (\sum_{i=1}^{m+1} EY_i^2)^{r/2} + \sum_{i=1}^{m+1} E|Y_i|^r \leq C(\sum_{i=1}^{n} b_i^2)^{r/2} + \sum_{i=1}^{m+1} E|Y_i|^r \tag{2.22}$$

follows by Hölder inequality and theorem 1(ii). From (2.18), (2.21) and (2.22), we have (2.19). That implies that (2.19) hold for all $r > 2$. Similarly, $E|\Sigma_{i=1}^{m+1} Z_i|^r$ has the same bound on the right-hand side of (2.19). Thus

$$E|\sum_{i=1}^{n} X_i|^r \leq C\{\sum_{i=1}^{m+1}(E|Y_i|^r + E|Z_i|^r) + \alpha(k)(\sum_{i=1}^{n} b_i)^r + (\sum_{i=1}^{n} b_i^2)^{r/2}\}. \tag{2.23}$$

Therefore, (2.4) follows by Minkowski inequality.

Proof of Theorem 2.3 Since $\alpha(k) \leq Ck^{-\lambda} \leq Cn^{-\lambda\theta}$, so

$$\alpha(k)(\sum_{i=1}^{n} b_i)^r \leq Cn^{r/2-\lambda\theta}(\sum_{i=1}^{n} b_i^2)^{r/2}.$$

Hence, from (2.23)

$$E|\sum_{i=1}^{n} X_i|^r \leq C\{\sum_{i=1}^{m+1}(E|Y_i|^r + E|Z_i|^r) + n^{r/2-\lambda\theta}(\sum_{i=1}^{n} b_i^2)^{r/2}\}, \tag{2.24}$$

for $r > 2$. By Minkowski inequality, (2.11) and (2.15)

$$E|\sum_{i=1}^{n} X_i|^r \le C\{n^{\theta(r-1)}\sum_{i=1}^{n} E|X_i|^r + n^{r/2-\lambda\theta}(\sum_{i=1}^{n} b_i^2)^{r/2}\}.$$

Applying the inequality above to $E|Y_i|^r$ and $E|Z_i|^r$ in (2.24), and noting (2.11) and (2.15),

$$E|\sum_{i=1}^{n} X_i|^r \le C\{\sum_{i=1}^{m+1}(k^{\theta(r-1)}(E_i+\widetilde{E}_i)+k^{r/2-\lambda\theta}(B_i^{r/2}+\widetilde{B}_i^{r/2}))+n^{r/2-\lambda\theta}(\sum_{i=1}^{n} b_i^2)^{r/2}\}$$

$$\le C\{n^{\theta^2(r-1)}\sum_{i=1}^{n} E|X_i|^r + n^{r/2-\lambda\theta}(\sum_{i=1}^{n} b_i^2)^{r/2}\}.$$

Again, applying the inequality above to $E|Y_i|^r$ and $E|Z_i|^r$ in (2.24), and repesting t times in this way, we have

$$E|\sum_{i=1}^{n} X_i|^r \le C\{n^{\theta^t(r-1)}\sum_{i=1}^{n} E|X_i|^r + n^{r/2-\lambda\theta}(\sum_{i=1}^{n} b_i^2)^{r/2}\}, \tag{2.25}$$

for integer $t \ge 1$. Since $0 < \theta < 1$, $\theta^t(r-1) < \varepsilon$ for some $t > 1$. Thus (2.5) holde.

Proof of Theorem 2.4 Denote $\beta_0 = \delta/r(r+\delta)$, $V_i = \sum_{j\in M_i} ||X_j||_{r+\delta}^2$ for $i = 1, 2, \cdots, m+1$ and $f(n) = 1 + 10d_2(k^{1/2}\alpha^{\beta_0}(k))^{r/(r-2)} + 10d_3(k\alpha^{2\beta_0}(k))^{r/(r-2)}$. By Minkowski inequality and Hölder inequality

$$\alpha^{\beta_0}(k)||Y_1||_{r+\delta}||\sum_{i=2}^{m+1} Y_i||_r^{r-1} \le \alpha^{\beta_0}(k)k^{1/2}V_1^{1/2}||\sum_{i=2}^{m+1} Y_i||_r^{r-1}$$

$$\le V_1^{1/2} + (k^{1/2}\alpha^{\beta_0}(k))^{r/(r-1)}E|\sum_{i=2}^{m+1} Y_i|^r, \tag{2.26}$$

$$\alpha^{2\beta_0}(k)||Y_1||_{r+\delta}^2||\sum_{i=2}^{m+1} Y_i||_r^{r-2} \le \alpha^{2\beta_0}(k)kV_1||\sum_{i=2}^{m+1} Y_i||_r^{r-2}$$

$$\le V_1^{1/2} + (k\alpha^{2\beta_0}(k))^{r/(r-2)}E|\sum_{i=2}^{m+1} Y_i|^r. \tag{2.27}$$

Since $\lambda > r(r+\delta)/2\delta = 1/2\beta_0$, so $\lambda\beta_0 > 1/2$. In (2.10), choose $\theta = (1+(\lambda\beta_0-1/2)r/(r-1))^{-1}$. Then from (2.11)

$$m(k^{1/2}\alpha^{\beta_0}(k))^{r/(r-1)} \le Cn^{1-\theta(1+(\lambda\beta_0-1/2)r/(r-1))} \le C, \tag{2.28}$$

$$m(k\alpha^{2\beta_0}(k))^{r/(r-2)} \le Cn^{1-\theta(1+(\lambda\beta_0-1/2)2r/(r-2))} \le C. \tag{2.29}$$

(2.28) and (2.29) implies

$$(f(n))^m \le C. \tag{2.30}$$

Apply lemma 1(ii) to the first inequality of (2.18) and use (2.26), (2.27) and (2.30),

$$E|\sum_{i=1}^{m+1} Y_i|^r \le E|\sum_{i=2}^{m+1} Y_i|^r + d_1E|Y_1|^r + 10d_2\alpha^{\beta_0}(k)||Y_1||_{r+\delta}||\sum_{i=2}^{m+1} Y_i||_r^{r-1} +$$

$$d_3EY_1^2E|\sum_{i=2}^{m+1} Y_i|^{r-2} + 10d_3\alpha^{2\beta_0}(k)||Y_1||_{r+\delta}^2||\sum_{i=2}^{m+1} Y_i||_r^{r-2}$$

$$\leq f(n)E|\sum_{i=2}^{m+1} Y_i|^r + d_1 E|Y_1|^r + 20d_3 V_1^{r/2} + d_3 EY_1^2 E|\sum_{i=2}^{m+1} Y_i|^{r-2}$$
$$\leq \cdots$$
$$\leq (f(n))^m \{d_1 \sum_{i=1}^{m+1} E|Y_i|^r + 20d_3 \sum_{i=1}^{m+1} V_i^{r/2} + d_3 \triangle\}$$
$$\leq C\{\sum_{i=1}^{m+1} E|Y_i|^r + (\sum_{i=1}^{n} ||X_i||_{r+\delta}^2)^{r/2} + \triangle\}, \tag{2.31}$$

where $\triangle := \Sigma_{j=1}^{m} EY_j^2 E|\Sigma_{i=j+1}^{m+1} Y_i|^{r-2}$. From theorem 1 (ii) and using the resembalance ways of (2.20), (2.21), and (2.22), we have

$$\triangle \leq C\{\sum_{i=1}^{m+1} E|Y_i|^r + (\sum_{i=1}^{n} ||X_i||_{r+\delta}^2)^{r/2}\}. \tag{2.32}$$

Combining (2.31) and (2.32),

$$E|\sum_{i=1}^{m+1} Y_i|^r \leq C\{\sum_{i=1}^{m+1} E|Y_i|^r + (\sum_{i=1}^{n} ||X_i||_{r+\delta}^2)^{r/2}\}.$$

Thus

$$E|\sum_{i=1}^{n} X_i|^r \leq C\{\sum_{i=1}^{m+1} (E|Y_i|^r + E|Z_i|^r) + (\sum_{i=1}^{n} ||X_i||_{r+\delta}^2)^{r/2}\}. \tag{2.33}$$

From (2.33) and the same way of proving (2.25),

$$E|\sum_{i=1}^{n} X_i|^r \leq C\{n^{\theta^t(r-1)} \sum_{i=1}^{n} E|Y_i|^r + (\sum_{i=1}^{n} ||X_i||_{r+\delta}^2)^{r/2}\}, \tag{2.34}$$

for integer $t \geq 1$. Since $0 < \theta < 1$, hence (2.34) implies (2.6).

3. Application

To show the application of the inequalities in section 2, here we discuss the asymptotic normality of the general linear estimator for the fixed design regression. Therefore, consider observations

$$Y_{ni} = g(x_{ni}) + \varepsilon_{ni},\ 1 \leq i \leq n, \tag{3.1}$$

where the design points $x_{n1}, \ldots, x_{nn} \in A$, which is a compact set of R^d, g is a bounded real valued function on A, and $\varepsilon_{n1}, \ldots, \varepsilon_{nn}$ are regression errors with zero mean and finite variance σ^2. A common estimate of g is

$$g_n(x) = \sum_{i=1}^{n} w_{ni}(x) Y_{ni}, \tag{3.2}$$

where weight function $w_{ni}(x)$, $i = 1, 2, \ldots, n$, depend on the fixed design points $x_{n1}, \ldots, x_{nn}$ and on the number of observations n.

In the independent case, the estimate (3.2) has been considered by much literature, such as, Priestly and Chao[11], Clark[2], Georgiev[4-7] and the references therein. In various dependence cases, $g_n(x)$ has been also researched very much. Fox example, Fan[3], Roussas[12], Roussas, Tran and Ioannides[14], Tran, Roussas, Yakowitz and Van[17] and the references therein.

Under the strong mixing condition, asymptotic normality of (3.2) has been established by [14]. Here our purpose is to use the moment inequalities in section 2 to give some more weaker conditions for asymptotic normality of the estimate (3.2). Adapting the basic assumptions of [14], we assume the followings.

Assumption (A1). (i) g:$A \to R$ is a bounded function defined on the compact subset A of R^d; (ii) $\{\xi_t : t = 0, \pm 1, \ldots\}$ is a strictly stationary and α - mixing time series with $E\xi_1 = 0$, $\mathrm{var}(\xi_1) = \sigma^2 \in (0, \infty)$; (iii) For each n, the joint distribution of $\{\varepsilon_{ni} : 1 \le i \le n\}$ is the same as that of $\{\xi_1, \ldots, \xi_n\}$.

Denote

$$w_n(x) := \max\{|w_{ni}(x)| : 1 \le i \le n\}, \quad \sigma_n^2(x) := \mathrm{Var}(g_n(x)). \tag{3.3}$$

Assumption (A2). (i) $\Sigma_{i=1}^n |w_{ni}(x)| \le C$ for all $n \ge 1$; (ii) $w_n(x) = O(\Sigma_{i=1}^n w_{ni}^2(x))$; (iii) $\Sigma_{i=1}^n w_{ni}^2(x) = O(\sigma_n^2(x))$.

Assumption (A3). There exist positive integers $p := p(n)$ and $q := q(n)$ such that $p + q \le n$ for sufficiently large n and as $n \to \infty$,

$$qp^{-1} \to o, \quad np^{-1}\alpha(q) \to 0, \quad nqp^{-1}\sum_{i=1}^{n} w_{ni}^2(x) \to 0, \tag{3.4}$$

$$p\sum_{i=1}^{n} w_{ni}^2(x) \to 0. \tag{3.5}$$

Here we will prove the following results.

Theorem 3.1 *Let Assumptions (A1) $\sim$ (A3) be satisfied. If $P(|\xi_1| \le C) = 1$ and*

$$\alpha(n) = O(n^{-\lambda}) \quad \textit{for some} \quad \lambda > 1, \tag{3.6}$$

then

$$\frac{g_n(x) - Eg_n(x)}{\sigma_n(x)} \xrightarrow{d} N(0,\ 1). \tag{3.7}$$

Theorem 3.2 *Let Assumptions (A1) $\sim$ (A3) be satisfied. If for some $s > 0$, $E|\xi_1|^{2+s} < \infty$, and*

$$\alpha(n) = O(n^{-\lambda}) \quad \textit{for some} \quad \lambda > (2+s)/s, \tag{3.8}$$

then (3.7) holds.

Remark 3.1 Theorem 2.1 of [14] is a corollary of theorem 3.1 here. Indeed, conditions (3.5) and (3.6) are substituted by the followings in theorem 2.1 of [14]

$$p^2\sum_{i=1}^{n} w_{ni}^2(x) \to 0 \quad (\text{as} \quad n \to \infty), \tag{3.9}$$

$$\sum_{i=1}^{\infty} \alpha^{s/(2+s)}(i) < \infty \quad \text{for some } s > 0. \tag{3.10}$$

Remark 3.2 Compare theorem 3.2 here with theorem 3.1 in [14]. Theorem 3.1 of [14] uses the conditions (3.9) and (3.10). Clearly, (3.9) is stronger than (3.5). Furthermore, (3.8) is almost as weak as (3.10). Our proof of theorem 3.1 and theorem 3.2 is much more simple then that of [14].

To prove the theorems, now we first give some denotions. For convenient writing, omit everywhere the argument x and set $S_n = \sigma_n^{-1}(g_n - Eg_n)$, $Z_{ni} = \sigma_n^{-1} w_{ni}\varepsilon_{ni}$ for $i = 1, 2, \ldots, n$, so that $S_n = \sum_{i=1}^n Z_{ni}$. Let $k = [n/(p+q)]$. Then S_n may be split as $S_n = S_n' + S_n'' + S_n'''$, where

$$S_n' = \sum_{m=1}^k y_{nm}, \quad S_n'' = \sum_{m=1}^k y_{nm}', \quad S_n''' = y_{nk+1}',$$

$$y_{nm} = \sum_{i=k_m}^{k_m+p-1} Z_{ni}, \quad y_{nm}' = \sum_{j=l_m}^{l_m+q-1} Z_{nj}, \quad y_{nk+1}' = \sum_{i=(p+q)+1}^{n} Z_{ni},$$

$k_m = (m-1)(p+q)+1$, $l_m = (m-1)(p+q)+p+1$, $m = 1, \ldots, k$. Thus, to prove (3.7), it suffices to show that $E(S_n'')^2 \to 0$, $E(S_n''')^2 \to 0$, and

$$S_n' \xrightarrow{d} N(0, 1). \tag{3.11}$$

Proof of Theorem 3.1 By Theorem 2.1(ii), Assumption (A2) (ii) and (iii), and (3.4), we have

$$E(S_n'')^2 \le C \sum_{m=1}^{k} \sum_{i=k_m}^{k_m+q-1} \sigma_n^{-2} w_{ni}^2 \le Ckq\sigma_n^{-2} w_{ni}^2 \le C\tfrac{n}{p+q} q w_n$$

$$\le C(1+qp^{-1})^{-1} nqp^{-1} \sum_{i=1}^n w_{ni}^2 \to 0, \tag{3.12}$$

$$E(S_n''')^2 = E(y_{nk+1}')^2 \le C \sum_{i=k(p+q)+1}^{n} \sigma_n^{-2} w_{ni}^2 \le C(n-k(p+q))\sigma_n^{-2} w_n^2$$

$$\le C(\frac{n}{p+q} - 1)(p+q) w_n \le C(1+qp^{-1})p \sum_{i=1}^n w_{ni}^2 \to 0. \tag{3.13}$$

Let $s_n^2 = \Sigma_{m=1}^k \mathrm{var}(y_{nm})$. From Lemma 2.2 of [14]

$$E(S_n')^2 \to 1 \text{ and } s_n^2 \to 1. \tag{3.14}$$

Let Φ_x stand for the characteristic function of the $r.\ v.\ X$. Then, by Theorem 7.2 in Roussas and Ioannides[13] and (3.4),

$$|\Phi_{s_n'}(t) - \prod_{m=1}^{k} \Phi_{y_{nm}}(t)| \le C(k-1)\alpha(q) \le Cnp^{-1}\alpha(q) \to 0. \tag{3.15}$$

Hence, $\{y_{nm} : m = 1, \ldots, k\}$ may be assumed to be independent random variablds. From (3.14) and according to Berry-Esseen central limit theorem, for (3.11) it suffices to show that

$$\sum_{m=1}^{k} E|y_{nm}|^r \to 0 \text{ for some } r > 2. \tag{3.16}$$

Since $\lambda > 1$ in (3.6), we may choose $0 < \theta < 1$ such that $\lambda\theta > 1$. Let $r = 2\lambda\theta$. Thus $r > 2$ and $r/2 - \lambda\theta = 0$. Given positive $\varepsilon < (r-2)/2$. By Theorem 2.3 and Assumption (A3), we get

$$\begin{aligned}\sum_{m=1}^{k} E|y_{nm}|^r &\le C\sum_{m=1}^{k}\{p^\varepsilon \sum_{i=k_m}^{k_m+p-1} E|Z_{ni}|^r + (\sum_{i=k_m}^{k_m+p-1} \sigma_n^{-2} w_{ni}^2)^{r/2}\} \\ &\le C\{p^\varepsilon \sum_{i=1}^{n} \sigma_n^{-r}|w_{ni}|^r + p^{(r-2)/2}\sum_{i=l}^{n}|w_{ni}|^{r/2}\} \le Cp^{(r-2)/2}\sum_{i=l}^{n}|w_{ni}|^{r/2} \\ &\le Cp^{(r-2)/2} w_n^{(r-2)/2} \le C(p\sum_{i=1}^{n} w_{ni}^2)^{(r-2)/2} \to 0.\end{aligned}$$

Completing the proof.

Proof of Theorem 3.2 By Theorem 2.1 (i), it is in the same way to get (3.12) and (3.13). And (3.14) and (3.15) also hlod. So we only have to prove (3.16). Since $\lambda > (2+s)/s$ in (3.8), hence we may choose positive t such that $0 < t < s/2$ and $(2+s)/s < (1+t)(2+s)/(s-2t) < \lambda$. Let $r = 2(1+t)$ and $\delta = s - 2t$. Then $r + \delta = 2 + s$ and

$$\frac{r(r+\delta)}{2\delta} = \frac{(1+t)(2+s)}{s-2t} < \lambda.$$

Given positive $\varepsilon < (r-2)/2$. Using Theorem 2.4, Assumption (A2) and (3.5), we have

$$\begin{aligned}\sum_{m=1}^{k} E|y_{nm}|^r &\le C\sum_{m=1}^{k}\{p^\varepsilon \sum_{i=k_m}^{k_m+p-1} E|Z_{ni}|^r + (\sum_{i=k_m}^{k_m+p-1} \sigma_n^{-2} w_{ni}^2 ||\xi_1||_{r+\delta}^2)^{r/2}\} \\ &\le C\{p^\varepsilon \sum_{i=1}^{n}|w_{ni}|^{r/2} + p^{(r-2)/2}\sum_{i=1}^{n}|w_{ni}|^{r/2}\} \\ &\le Cp^{(r-2)/2}\sum_{i=1}^{n}|w_{ni}|^{r/2} \le Cp^{(r-2)/2} w_n^{(r-2)/2}\sum_{i=1}^{n}|w_{ni}| \\ &\le C(p\sum_{i=1}^{n} w_{ni}^2)^{(r-2)/2} \to 0.\end{aligned}$$

Completing the proof.

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强混合序列的矩不等式及其应用

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摘 要: 对强混合随机变量序列建立一些矩不等式，并应用这些不等式研究固定设计回归模型的一般加权函数估计的渐近正态性。