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Existence of Positive Solutions to Second Order Neumann Boundary Value Problems *

JIANG Da-qing[1], *LIU Hui-zhao*[2]

(1. Dept. of Math., Northeast Normal University, Changchun 130024, China ;
2. Dept. of Appl. Math., Heibei University of Technology, Tianjin 300130, China)

Abstract: The existence of positive solutions to second-order Neumann BVPs $-u''+Mu=f(t,u), u'(0)=u'(1)=0$ and $u''+Mu=f(t,u), u'(0)=u'(1)$ is proved by a simple application of a Fixed Point Theorem in cones due to Krasnoselskii[1,6].

Key words: Neumann BVP; positive solutions; the fixed point theorem.

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1. Introduction

Recently, Neumann boundary value problems have been studied by many different methods[2-4], and Krasnoselskii fixed point theorem has been used to establish the existence of positive solutions to non-periodic BVPs by many authors [5 7-9]. However, as far as we know, Neumann boundary value problems have not been studied by applying the Krasnoselskii fixed point theorem.

In this paper we apply the Krasnoselskii fixed point theorem [1,6] to establish the existence of positive solutions to the following equations

$$-u''+Mu=f(t,u), \quad 0<t<1, \tag{1.1}$$

$$u''+Mu=f(t,u), \quad 0<t<1 \tag{1.2}$$

with the Neumann boundary value problems

$$u'(0)=0, \qquad u'(1)=0. \tag{1.3}$$

The main results of the present paper are as follows.

Theorem 1 *Assume that $f(t,u):[0,1]\times[0,\infty)$ is nonegative and continuous. Then the*

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Biography: JIANG Da-qing (1965-), male, associate professor.

Neumann boundary value problem (1.1) (1.3) (resp. (1.2) (1.3)) has a positive solution, provided $M > 0$ (resp. $M \in (0, \frac{\pi^2}{4})$) and one of the following conditions holds:

(i) $\lim_{u\downarrow 0} \max_{t\in[0,1]} \frac{f(t,u)}{u} = 0$ *and* $\lim_{u\uparrow\infty} \min_{t\in[0,1]} \frac{f(t,u)}{u} = +\infty$,

or

(ii) $\lim_{u\downarrow 0} \min_{t\in[0,1]} \frac{f(t,u)}{u} = +\infty$ *and* $\lim_{u\uparrow\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u} = 0$.

The proof of Theorem 1 is based on an application of the following

Krasnoselskii fixed point theorem[1, 6] Let E be a Banach space, and let K be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let

$$\Phi : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|\Phi u\| \le \|u\| \quad \forall u \in K \cap \partial\Omega_1$ and $\|\Phi u\| \ge \|u\| \quad \forall u \in K \cap \partial\Omega_2$, or

(ii) $\|\Phi u\| \ge \|u\| \quad \forall u \in K \cap \partial\Omega_1$ and $\|\Phi u\| \le \|u\| \quad \forall u \in K \cap \partial\Omega_2$.

Then Φ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

2. The proof of Theorem 1

In this section we suppose that the condition in Theorem 1 hold.

First we prove Theorem 1 for the boundary value problem (1.1)(1.3).

It is easy to see that problem (1.1) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) f(s, u(s)) \mathrm{d}s, \tag{2.1}$$

where $m = \sqrt{M}$ and

$$G(t,s) := \begin{cases} \dfrac{\operatorname{ch} m(1-t) \operatorname{ch} ms}{m \operatorname{sh} m}, & 0 \le s \le t \le 1, \\ \dfrac{\operatorname{ch} m(1-s) \operatorname{ch} mt}{m \operatorname{sh} m}, & 0 \le t \le s \le 1. \end{cases}$$

where $\operatorname{ch} x = \frac{e^x + e^{-x}}{2}$; $\operatorname{sh} x = \frac{e^x - e^{-x}}{2}$, and consequently, it is equivalent to the fixed point equation $u = \Phi u$ in $E = C[0,1]$, with $\Phi : E \to E$ given by

$$\Phi u = \int_0^1 G(t,s) f(s, u(s)) \mathrm{d}s. \tag{2.2}$$

It is obvious that Φ is completely continuous.

Definition let E be a Banach space, and K a closed, nonempty subset of E. K is a cone provided (i) $\alpha u + \beta u \in K$, for all $u, V \in K$ and all $\alpha, \beta \ge 0$, and (ii) $u, -u \in K$ imply $u = 0$.

Now we define

$$K = \{u \in E : u(t) \ge 0 \quad \text{and} \quad \min_{t\in[0,1]} u(t) \ge \sigma\|u\|\}, \tag{2.3}$$

where $||u|| = \sup_{t\in[0,1]}\{|u(t)| : u \in E\}$ and $\sigma = \frac{1}{\mathrm{ch}^2 m}$. It is obvious that K is a cone in $E = C[0,1]$.

Lemma 2.1 $\Phi(K) \subset K$.

Proof A direct calculation shows that

$$\frac{1}{m\,\mathrm{sh}\,m} = A \le G(t,s) \le B = \frac{\mathrm{ch}^2 m}{m\,\mathrm{sh}\,m}, \quad \text{i.e.} \quad 1 \ge \frac{G(t,s)}{B} \ge \sigma \quad \text{for all} \quad 0 \le t, s \le 1. \tag{2.4}$$

Hence, for $u \in K$ we have

$$\begin{aligned}\min_{0\le t\le 1}(\Phi u)(t) &= \min_{0\le t\le 1}\int_0^1 G(t,s)f(s,u(s))\mathrm{d}s \\ &\ge \sigma\int_0^1 Bf(s,u(s))\mathrm{d}s \ge \sigma\max_{0\le t\le 1}\int_0^1 G(t,s)f(s,u(s))\mathrm{d}s \\ &\ge \sigma||\Phi u||.\end{aligned}$$

First we suppose (i) hold. Since $\lim_{u\downarrow 0}\max_{t\in[0,1]}\frac{f(t,u)}{u} = 0$, so, for fixed $\varepsilon > 0$ $B\varepsilon < 1$, we may choose $R_1 > 0$ such that

$$0 \le f(t,u) \le \varepsilon u \quad \text{whenever} \quad 0 \le u \le R_1, \quad t \in [0,1]. \tag{2.5}$$

Thus, if $u \in K$ and $||u|| = R_1$, then it follows from (2.4) and (2.5) that

$$||\Phi u|| \le B\int_0^1 f(s,u(s))\mathrm{d}s \le B\varepsilon||u|| < R_1 = ||u|| \tag{2.6}$$

Now, we let $\Omega_1 := \{u \in E : ||u|| < R_1\}$, then (2.6) shows that

$$||\Phi u|| < ||u|| \quad \forall u \in K \cap \partial\Omega_1.$$

On the other hand, also from (i), we have $\lim_{u\uparrow\infty}\min_{t\in[0,1]}\frac{f(t,u)}{u} = +\infty$, so, for fixed $\eta > 0$, $A\sigma\eta > 1$, we can chose $R_2 > R_1/\sigma$, such that

$$f(t,u) \ge \eta u \quad \text{whenever} \quad u \ge \sigma R_2, \quad t \in [0,1] \tag{2.7}$$

If $u \in K$ with $||u|| = R_2$, then it follows from (2.4), (2.7) and (2.3) that

$$||\Phi u|| \ge A\int_0^1 f(s,u(s))\mathrm{d}s \ge A\sigma\eta||u|| > R_2 = ||u||$$

Let $\Omega_2 := \{u \in E : ||u|| < R_2\}$, then we get

$$||\Phi u|| > ||u|| \quad \forall u \in K \cap \partial\Omega_2. \tag{2.8}$$

Therefore, by the first part of the Krasnoselskii fixed point theorem, it follows that Φ has a fixed point $u \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. Furthermore, $R_1 \le ||u|| \le R_2$. Since $G(t,s) > 0$, it follows that $u(t) > 0$ for $0 \le t \le 1$.

Next we suppose (ii) hold. Since $\lim_{u\downarrow 0}\min_{t\in[0,1]}\frac{f(t,u)}{u}=+\infty$, so, for fixed $\eta>0$, $A\sigma\eta>1$, we may choose $R_1>0$ so that

$$f(t,u)\geq \eta u \quad \text{whenever} \quad 0\leq u\leq R_1. \tag{2.9}$$

Thus, for $u\in K$ with $\|u\|=R_1$, by (2.4) and (2.9), we have

$$\|\Phi u\|\geq A\int_0^1 f(s,u(s))\mathrm{d}s\geq A\sigma\eta\|u\|>R_1=\|u\|.$$

If we put $\Omega_1:=\{u\in E:\|u\|<R_1\}$, then (2.1) shows that

$$\|\Phi u\|>\|u\| \quad \forall u\in K\cap\partial\Omega_1. \tag{2.10}$$

On the other hand, also from (ii), since $\lim_{u\uparrow\infty}\max_{t\in[0,1]}\frac{f(t,u)}{u}=0$, there exists $R_0>0$ such that

$$f(t,u)\leq\varepsilon u \quad \text{whenever} \quad u\geq R_0, \quad t\in[0,1], \tag{2.11}$$

where $\varepsilon>0$ satisfies $B\varepsilon<1$.

If $\max_{t\in[0,1]}f(t,u)$ is unbounded for $u\in[0,+\infty)$, then we choose $R_2>R_0+R_1$ so that

$$f(t,u)\leq\max_{t\in[0,1]}f(t,R_2)\leq\varepsilon R_2, \quad \text{for} \quad u\in(0,R_0], \quad t\in[0,1],$$
$$f(t,u)\leq\varepsilon u\leq\varepsilon R_2, \quad \text{for} \quad u\in[R_0R_2], \quad t\in[0,1].$$

Thus we have

$$f(t,u)\leq\varepsilon R_2 \quad \text{for} \quad u\in[0R_2], \quad t\in[0,1]. \tag{2.12}$$

For $u\in K$ with $\|u\|=R_2$, it follows from (2.11) and (2.12) that

$$\|\Phi u\|\geq B\int_0^1 f(s,u(s))ds\leq B\varepsilon R_2<R_2=\|u\|.$$

If $\max_{t\in[0,1]}f(t,u)$ is bounded on $[0,+\infty)$, say

$$f(t,u)\leq N \quad \text{for all} \quad u\geq 0, \quad t\in[0,1]. \tag{2.13}$$

In this case, we let $R_2>R_1+N/\varepsilon$. For $u\in K$ with $\|u\|=R_2$, from (2.13) we have

$$\|\Phi u\|\geq B\int_0^1 f(s,u(s))ds\leq BN\leq B\varepsilon R_2<R_2=\|u\|.$$

Therefore, in either case we may put $\Omega_2:=\{u\in E:\|u\|<R_2\}$, and we have

$$\|\Phi u\|<\|u\| \quad \forall u\in K\cap\partial\Omega_2.$$

By the second part of the Krasnoselskii fixed point theorem, it follows that (1.1) (1.3) has a positive solution. This completes the proof of Theorem 1 for problem (1.1) (1.3).

If we consider

$$G(t,s):=\begin{cases}\dfrac{\cos m(1-t)\cos ms}{m\sin m}, & 0\leq s\leq t\leq 1,\\[2mm] \dfrac{\cos m(1-s)\cos mt}{m\sin m}, & 0\leq t\leq s\leq 1.\end{cases}$$

and

$$K = \{u \in E : u(t) \geq 0 \quad \text{and} \quad \min_{t\in[0,1]} u(t) \geq \sigma||u||\},$$

where $||u|| = sup_{t\in[0,1]}\{|u(t)| : u \in E\}$ and $\sigma = \cos^2 m.$, we can prove theorem 1 for problem (1.2) (1.3) similarly as for problem (1.1), (1.2). This complete the proof of Theorem 1.

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二阶微分方程 Neumann 边值问题正解存在性

蒋 达 清[1], 刘 辉 昭[2]

(1. 东北师范大学数学系，长春 130024; 2. 河北工业大学应用数学系，天津 300130)

摘 要: 本文利用锥不动点定理证明了 $-u'' + Mu = f(t,u), u'(0) = u'(1) = 0$ 和 $u'' + Mu = f(t,u), u'(0) = u'(1) = 0$ 两个二阶微分方程 Neumann 边值问题正解的存在性.