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Simultaneous Approximation by Modified Bernstein-Durrmeyer Operators *

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Abstract: The aim of the present paper is to prove direct and converse results for simultaneous approximation by modified Bernstein-Durrmeyer operators. A point-wise equivalence characterization of simultaneous approximation is obtained.

Key words: Modified Bernstein-Durrmeyer operators; simultaneous approximation; direct theorem; converse theorem.

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1. Introduction

Let $P_{n,k}(x) = C_n^k x^k (1-x)^{n-k}, x \in [0,1], n \in N$, we denote the Bernstein operators by $B_n f = B_n(f,x) = \sum_{k=0}^n P_{n,k}(x) f(\frac{n}{k})$. It is well-known that the operators are positive linear contractions in $C[0,1]$ and reproduce linear functions. The Bernstein-Durrmeyer operators defined by

$$D_n f = D_n(f,x) = \sum_{k=0}^{n} P_{n,k}(x)(n+1)\int_0^1 P_{n,k}(t)f(t)\mathrm{d}t$$

were introduced by Durrmeyer[2] and studied by Derriennic[3], Heilmann[4] and Ditzian et al.[5] It was shown that $D_n f$ are positive linear contractions in $L_p[0,1]$, self-adjoint and commute (i.e., $D_k D_n f = D_n D_k f$). But these operators do not reproduce linear functions. Thus, Chen[1] modified the Bernstein-Durrmeyer operators as $M_n f = M_n(f,x) = \sum_{k=0}^n P_{n,k}(x)\Phi_{n,k}(f)$, where

$$\Phi_{n,k}(f) = \begin{cases} f(0), & k=0; \\ (n-1)\int_0^1 P_{n-2,k-1}(t)f(t)\mathrm{d}t, & k = 1,2,\cdots,n-1; \\ f(1), & k=n. \end{cases}$$

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Obviously, the operators M_nf are linear positive contractions in $C[0,1]$ and reproduce linear functions. Here we will study the direct and converse results on simultaneous approximation by the modified operators M_nf. In the point-wise sense, an equivalence characterization of simultaneous approximation by M_nf will be given.

2. Main results and lemma

Let $\Delta_t f(x) = f(x+t) - f(x-t)(x \pm t \in [0,1])$ be the usual symmetric difference of f. We denote the modulus of smoothness of f by $\omega(f,h) = \sup_{0<t\le h} \|\Delta_t f\|$. The main results of this paper are as follows.

Theorem 1 *For $f^{(r)} \in C[0,1], r \in N$, we have*

$$|M_n^{(r)}(f,x) - f^{(r)}(x)| \le C\{\omega(f^{(r)},(x(1-x)/n + 1/n^2)^{1/2}) + |f^{(r)}(x)|/n\},$$

here and in the following C denotes a constant independent of n and f, but its value may be different at different occurrence.

Theorem 2 *Let $f^{(r)} \in C[0,1], r \in N$ and $r < \alpha < r+1$. Then*

$$|M_n^{(r)}(f,x) - f^{(r)}(x)| \le C(x(1-x)/n + 1/n^2)^{(\alpha-r)/2}$$

if and only if $\omega(f^{(r)},h) = O(h^{\alpha-r})$.

To prove above theorems, we first give a lemma.

Lemma *For $f^{(r)} \in C[0,1], r \in N$, we have*

$$M_n^{(r)}(f,x) = \frac{n!(n-1)!}{(n-r)!(n+r-2)!}\sum_{k=0}^{n-r} P_{n-r,k}(x)\int_0^1 f^{(r)}(t)P_{n+r-2,k+r-1}(t)\mathrm{d}t.$$

Proof For convenience we suppose that $\Phi_{n,k}(f) = 0, P_{n,k}(x) \equiv 0$ for $k < 0$ or $k > 0$. Since

$$P'_{n,k}(x) = n(P_{n-1,k-1}(x) - P_{n-1,k}(x)),$$
$$P''_{n,k}(x) = n(n-1)(P_{n-2,k}(x) - 2P_{n-2,k-1}(x) + P_{n-2,k-2}(x)).$$

We have from the induction that $P_{n,k}^{(r)}(x) = \frac{n!}{(n-r)!}\sum_{j=0}^{r} C_r^j(-1)^j P_{n-r,k-r+j}(x)$, which implies

$$\begin{aligned}
M_n^{(r)}(f,x) &= \frac{n!}{(n-r)!}\sum_{j=0}^{r} C_r^j(-1)^j \sum_{k=0}^{n} \Phi_{n,k}(f)P_{n-r,k-r+j}(x) \\
&= \frac{n!}{(n-r)!}\sum_{j=0}^{r} C_r^j(-1)^j \sum_{k=0}^{n-r+j} \Phi_{n,k+r-j}(f)P_{n-r,k}(x) \\
&= \frac{n!}{(n-r)!}\sum_{k=0}^{n-r} P_{n-r,k}(x)\sum_{j=0}^{r}(-1)^{r-j}C_r^j\Phi_{n,k+j}(f) \\
&= \frac{n!}{(n-r)!}\sum_{k=0}^{n-r}(-1)^r P_{n-r,k}(x)\delta_{nk}^{(r)}(f),
\end{aligned}$$

where $\delta_{nk}^{(r)}(f)=\sum_{j=0}^{r}C_r^j(-1)^j\Phi_{n,k+j}(f)$. Now, to finish the proof of Lemma 1, we only need to prove for $k=0,1,2,\cdots,n$ that

$$\delta_{nk}^{(r)}(f)=(-1)^r\frac{(n-1)!}{(n+r-2)!}\int_0^1 f^{(r)}(t)P_{n+r-2,k+r-1}(t)\mathrm{d}t.$$

In fact, for $k=0$, we have after integrating by part

$$\begin{aligned}\delta_{nk}^{(r)}(f)&=f(0)+\sum_{j=1}^{r}C_r^j(-1)^j\Phi_{n,j}(f)\\&=\sum_{j=1}^{r}C_r^j(-1)^j(n-1)\int_0^1(f(t)-f(0))P_{n-2,j-1}(t)\mathrm{d}t\\&=\frac{(n-1)!}{(n+r-2)!}\int_0^1(f(t)-f(0))P_{n+r-2,r-1}^{(r)}(t)\mathrm{d}t\\&=(-1)^r+\frac{(n-1)!}{(n+r-2)!}\int_0^1 f^{(r)}(t)P_{n+r-2,r-1}(t)\mathrm{d}t.\end{aligned}$$

Similarly, the case for $k=n$ can be proved. For $1\le k\le n-1$, we know

$$\begin{aligned}\delta_{nk}^{(r)}&=(n-1)\int_0^1(\sum_{j=0}^{r}C_r^j(-1)^jP_{n-2,k+j-1}(t))f(t)\mathrm{d}t\\&=\frac{(n-1)!}{(n+r-2)!}\int_0^1 P_{n+r-2,k+r-1}(t)\mathrm{d}t\\&=(-1)^r\frac{(n-1)!}{(n+r-2)!}\int_0^1 f^{(r)}(t)P_{n+r-2,k+r-1}(t)\mathrm{d}t.\end{aligned}$$

The proof of Lemma 1 is complete.

3. Proof of theorems

Proof of Theorem 1 By direct computation, we can derive

$$\begin{aligned}Q(n,x)&=\frac{n!(n-1)!}{(n-r)!(n+r-2)!}\sum_{k=0}^{n-r}P_{n-r,k}(x)n\int_0^1(t-x)^2P_{n+r-2,k+r-1}(t)\mathrm{d}t\\&\le C(x(1-x)/n+1/n^2).\end{aligned}$$

Thus, from Lemma 1 it follows that

$$\begin{aligned}&|M_n^{(r)}(f,x)-f^{(r)}(x)|\\&\le n!(n-1)!(n-r)!(n+r-2)!\sum_{k=0}^{n-r}P_{n-r,k}(x)\int_0^1|f^{(r)}(t)-f^{(r)}(x)|P_{n+r-2,k+r-1}(t)\mathrm{d}t+\\&\quad|f^{(r)}(x)(1-\frac{n!(n-1)!}{(n-r)!(n+r-1)!})|\\&\le\frac{n!(n-1)!}{(n-r)!(n+r-2)!}\sum_{k=0}^{n-r}P_{n,k}(x)\int_0^1\omega(f^{(r)},|t-x|)P_{n+r-2,k+r-1}(t)\mathrm{d}t+\frac{C}{n}|f^{(r)}(x)|\end{aligned}$$

$$\le \omega(f^{(r)},\delta)\{1+\frac{\delta^{-1}n!(n-1)!}{(n-r)!(n+r-2)!}\sum_{k=0}^{n-r}P_{n-r,k}(x)\int_0^1|t-x|P_{n+r-2,k+r-1}(t)\mathrm{d}t\}+$$
$$\frac{C}{n}|f^{(r)}(x)|$$
$$\le \omega(f^{(r)},\delta)\{1+C\delta^{-1}(Q(n,x))^{1/2}\}+\frac{C}{n}|f^{(r)}(x)|$$
$$\le \omega(f^{(r)},\delta)\{1+\delta^{-1}C(x(1-x)/n+1/n^2)^{1/2}\}+\frac{C}{n}|f^{(r)}(x)|.$$

Putting $\delta=(\frac{x(1-x)}{n}+\frac{1}{n^2})^{1/2}$, then Theorem 1 follows.

Proof of Theorem 2 From Theorem 1, it is sufficient to prove the inverse part. Let $h\in(0,\frac{1}{8}), 0<t\le h, n\in N$. Set

$$\delta(n,x,t)=\max\{1/n,\varphi(x+t)/\sqrt{n},\varphi(x)/\sqrt{n}\}, \varphi(x)=(x(1-x))^{1/2}.$$

Then

$$|f^{(r)}(x+t)-f^{(r)}(x)|\le|M_n^{(r)}(f,x+1)-f^{(r)}(x+t)|+|M_n^{(r)}(f,x)-f^{(r)}(x)|+$$
$$|\int_x^{x+t}M_n^{(r+1)}(f,u)\mathrm{d}u|\le C\delta^{\alpha-r}(n,x,t)+I$$

with $I=|\int_x^{x+t}M_n^{(r+1)}(f,u)\mathrm{d}u|$. Let $f_h(x)=\frac{1}{h}\int_{-h/2}^{h/2}f^*(x+u)\mathrm{d}u, h>0$, where

$$f^*(x)=\begin{cases} f^{(r)}(x), & 0\le x\le 1;\\ f^{(r)}(0), & x<0;\\ f^{(r)}(1), & x>1.\end{cases}$$

Then by a simple computation, we see

$$\|f^{(r)}-f_h\|\le\omega(f^{(r)},h/2), \|f_h'\|\le\omega(f^{(r)},h)/h. \tag{2.1}$$

Now, we estimate I. Writing

$$M_{n,r}(g,x)=\frac{n!(n-1)!}{(n-r)!(n+r-2)!}\sum_{k=0}^{n-r}P_{n-r,k}(x)\int_0^1 g(t)P_{n+r-2,k+r-1}(t)\mathrm{d}t, \tag{2.2}$$

we can obtain

$$\|M_{n,r}g\|\le C\|g\|,\ \|M_{n,r}'g\|\le Cn\|g\|,\quad g\in C[0,1], \tag{2.3}$$
$$\|M_{n,r}'g\|\le C\|g'\|,\quad g'\in C[0,1]. \tag{2.4}$$

Since

$$I\le\int_x^{x+t}|M_{n,r}'(f^{(r)}-f_h,u)|\mathrm{d}u+\int_x^{x+t}|M_{n,r}'(f_h)|\mathrm{d}u=I_1+I_2,$$

from (2.2) and (2.4) it follows that

$$I_2\le t\omega(f^{(r)},h)/h. \tag{2.5}$$

Moreover, we can prove

$$I_1 \le Ct(\min\{n,(n/\varphi^2(x+t))^{1/2},(n/\varphi^2(x))^{1/2}\})\omega(f^{(r)},h). \tag{2.6}$$

In fact, by (2.2) and (2.3), it is not difficult to deduce

$$I_1 \le Cnt\|f^{(r)}-f_h\| \le Cnt\omega(f^{(r)},h). \tag{2.7}$$

On the other hand, for $0 \le \varphi^2(u) \le 1/n$, we have

$$J = |\varphi(u)M'_{n,r}(f^{(r)}-f_h,u)| \le C\varphi(u)n\|f^{(r)}-f_h\| \le C\sqrt{n}\omega(f^{(r)},h).$$

For $\varphi^2(u) > 1/n$, noting that $P'_{n,k}(x) = \frac{n}{\varphi^2(x)}(\frac{k}{n}-x)P_{n,k}(x)$, we know

$$\begin{aligned} J \le&(n-r)\varphi^{-1}(u)\frac{n!(n-1)!}{(n-r)!(n+r-2)!}\sum_{k=0}^{n-r}|\frac{k}{n-r}-u|\times \\ &P_{n-r,k}(u)\int_0^1 P_{n+r-2,k+r-1}(t)|f^{(r)}(t)-f_h(t)|\mathrm{d}t \\ \le&C(n-r)\|f^{(r)}-f_h\|\varphi^{-1}(u)\sum_{k=0}^{n-r}|\frac{k}{n-r}-u|P_{n-r,k}(u) \\ \le&C(n-r)\|f^{(r)}-f_h\|\varphi^{-1}(u)(B_{n-r}((t-u)^2,u))^{1/2} \le Cn^{1/2}\omega(f^{(r)},h). \end{aligned}$$

Hence, using (see [6]) $\int_x^{x+t}\varphi^{-1}(u)\mathrm{d}u \le 4t\max(\varphi^{-1}(x),\varphi^{-1}(x+t))$, we have

$$I_1 \le Ctn^{1/2}\omega(f^{(r)},h)\max(\varphi^{-1}(x),\varphi^{-1}(x+t)). \tag{2.8}$$

Combining (2.7) and (2.8), we obtain (2.6). So, putting $\delta(n,x,t)=h$, it follows from (2.5) and (2.6) that

$$|\Delta_t f^{(r)}(x)| \le C(\delta^{\alpha-r}(n,x,t)+t\delta^{-1}(n,x,t)\omega(f^{(r)},h)).$$

Note that for $n \ge 2$, there holds

$$\delta(n,x,t) < \delta(n-1,x,y) \le 2\delta(n,x,t).$$

Thus for any $\delta \in (0,1/8)$, we can choose an n such that

$$\delta(n,x,t) \le \delta < \delta(n-1,x,t) \le 2\delta(n,x,t).$$

Therefore,

$$|\Delta_t f^{(r)}(x)| \le C(\delta^{\alpha-r}+\frac{h}{\delta}\omega(f^{(r)},\delta)).$$

Hence,

$$\omega(f^{(r)},h) \le C(\delta^{\alpha-r}+\frac{h}{\delta}\omega(f^{(r)},\delta)),$$

which implies from the Berens-Lorentz lemma (see [7]) that

$$\omega(f^{(r)},h) = O(h^{\alpha-r}).$$

The proof of Theorem 2 is complete.

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修正的 Bernstein-Durrmeyer 算子的同时逼近

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摘 要: 本文的目的是证明修正的 Bernstein-Durrmeyer 算子同时逼近的正逆定理，在点态意义下，我们得到了一个同时逼近的等价特征刻画.

关键词: 修正的 Bernstein-Durrmeyer 算子；同时逼近；正定理；逆定理.