Journal of Mathematical Research & Exposition
Jan., 2011, Vol. 31, No. 1, pp. 12–22
DOI:10.3770/j.issn:1000-341X.2011.01.002
Http://jmre.dlut.edu.cn

Efficient Option Pricing Methods Based on Fourier Series Expansions

Deng DING*, **Sio Chong U**
Department of Mathematics, University of Macau, Macao, P. R. China

Abstract A novel option pricing method based on Fourier-cosine series expansion was proposed by Fang and Oosterlee. Developing their idea, three new option pricing methods based on Fourier, Fourier-cosine and Fourier-sine series expansions are presented in this paper, which are more efficient when the option prices are calculated with many strike prices. A series of numerical experiments under different exp-Lévy models are also given to compare these new methods with the Fang and Oosterlee's method and other methods.

Keywords option pricing; Lévy process; Fourier transform; Fourier expansions.

Document code A
MR(2010) Subject Classification 42A10; 62P05; 65T40
Chinese Library Classification O29; O174.21

1. Introduction

We consider the pricing problem for an European call of maturity T and strike price K, written on an underlying asset whose price is modelled by an exp-Lévy process S_t, i.e., the payoff: $(S_T - K)^+$. Under the risk-neutral probability $\mathbf{Q}$, we have well known that the price of this European call is given by

$$C_0 = e^{-rT}\mathbf{E}_{\mathbf{Q}}\big[(S_T - K)^+\big], \tag{1}$$

where $\mathbf{E}_{\mathbf{Q}}[\cdot]$ denotes the mathematical expectation with respect to $\mathbf{Q}$, and $r > 0$ is the interest rate of the risky-free investment [2]. Since the famous Black-Scholes formula of the price (1) was obtained, many efficient numerical methods have been proposed. There are numerical solutions to partial integro-differential equations (PIDEs), Monte Carlo simulation techniques, as well as numerical integration methods. Each of them has its advantages and disadvantages for different financial models and specific applications. In this paper we concentrate on the last group.

Let $q_T(s)$ be the probability density of S_T under the risk-neutral probability $\mathbf{Q}$. Then, the price (1) becomes

$$C_0 = e^{-rT}\int_{-\infty}^{\infty}(s-K)^+ q_T(s)\mathrm{d}s = e^{-rT}\int_{K}^{\infty}(s-K)q_T(s)\mathrm{d}s. \tag{2}$$

Received November 11, 2009; Accepted January 19, 2010
Supported by the Research Grant of University of Macau (Grants Nos. UL020/08-Y3/MAT/JXQ01/FST; RG058/09-10S/DD/FST).
* Corresponding author
E-mail address: dding@umac.mo (D. DING); tony_u_1984@yahoo.com.hk (S. C. U)

Unfortunately, for many relevant pricing processes, their probability densities are usually unknown. On the other hand, the Fourier transforms of these densities, i.e., the characteristic functions, are often available. For instance, from the Lévy-Khinchine theorem the characteristic functions of Lévy processes are known. Hence, the Fourier transform methods for option pricing have been naturally considered by many authors (see [3] and references therein). In recent years, some new numerical integration methods are proposed. The QUAD method was introduced by Andricopoulos et al in [4]. The CONV method was presented by Lord et al in [5]. A fast Hilbert transform approach was considered by Feng and Linetsky in [6]. Meanwhile, a novel numerical method based on Fourier-cosine series expansion was proposed by Fang and Oosterlee in [1], which is called the COS method, and is shown to have the exponential convergence rate and the linear computational complexity. Furthermore, this COS method was also used to price early-exercise and discrete barrier options by Fang and Oosterlee in [7].

In this paper, from the idea of COS method we present three numerical methods, which are based on Fourier-cosine series, Fourier-sine series and Fourier series expansions, respectively. We call these three methods as FCOS, FSIN and FSER, respectively. We must mention that FCOS is different from COS. In fact, our methods are more efficient when the option prices are calculated with many strike prices.

This paper is structured as follows. After this introduction we review the main idea of COS method, and the corresponding numerical algorithm in Section 2. Then, we present FCOS, FSIN and FSER, respectively, and give some results on the coefficients of the responding expansions, in Section 3. Finally, via numerical experiments we compare FSER, FCOS and FSIN, as well as COS, for exponential Lévy Models: the Black-Scholes (BS) model, the Merton's jump-diffusion (MJD) model and the Variance Gamma (VG) model, in Section 4. The numerical results show that the FSER method is the fastest one under a suitable truncated integral interval.

2. The COS method

We recall that the price of underlying asset is modelled by an exp-Lévy process S_t. Let $\tilde{X}_t = \log(S_t/K)$, where K is the strike price. Consider the call price (1) in the form:

$$C_0(K) = e^{-rT}\mathbf{E}_{\mathbf{Q}}\big[(S_T - K)^+\big] = e^{-rT}K\int_0^\infty (e^x - 1)\tilde{p}_T(x)\mathrm{d}x, \tag{3}$$

where $\tilde{p}_T(x)$ is the probability density of $\tilde{X}_T$ under the risk-neutral probability $\mathbf{Q}$. However, in the most cases, $\tilde{p}_T(x)$ is not available. On the other hand, the characteristic function of $\tilde{X}_T$:

$$\tilde{\phi}_T(z) = \mathbf{E}_{\mathbf{Q}}\big[e^{\mathrm{i}z\tilde{X}_T}\big] = \int_{-\infty}^\infty e^{\mathrm{i}zx}\tilde{p}_T(x)\mathrm{d}x, \quad z \in \mathbf{R}, \tag{4}$$

is usually available, where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. The key point of Fang and Oosterlee's idea is to choose two numbers a and b such that the truncated integral approximates the infinite integral in (4) very well, i.e.,

$$\tilde{\phi}_{1,T}(z) = \int_a^b e^{\mathrm{i}zx}\tilde{p}_T(x)\mathrm{d}x \approx \int_{-\infty}^\infty e^{\mathrm{i}zx}\tilde{p}_T(x)\mathrm{d}x = \tilde{\phi}_T(z), \quad z \in \mathbf{R}. \tag{5}$$

Meanwhile, $\tilde{p}_T(x)$ in $[a,b]$ has the Fourier cosine expansion:

$$\tilde{p}_T(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\Big(n\pi\frac{x-a}{b-a}\Big), \quad x \in [a,\ b], \tag{6}$$

where

$$A_n = \frac{2}{b-a}\int_a^b \tilde{p}_T(x)\cos\Big(n\pi\frac{x-a}{b-a}\Big)\mathrm{d}x, \quad n = 0, 1, 2, \ldots. \tag{7}$$

Comparing the first equation in (5) with the cosine series coefficients of $\tilde{p}_T(x)$ on $[a,b]$ in (7), we have

$$A_n = \frac{2}{b-a}\mathrm{Re}\Big\{\tilde{\phi}_{1,T}\Big(\frac{n\pi}{b-a}\Big)\exp\Big(-\mathrm{i}\frac{na\pi}{b-a}\Big)\Big\}, \quad n = 0, 1, 2, \ldots, \tag{8}$$

where $\mathrm{Re}\{\cdot\}$ denotes taking real part of argument. Then, we get an approximation of $\tilde{p}_T(x)$ by

$$\tilde{p}_T(x) \approx \frac{1}{2}F_0 + \sum_{n=1}^{N-1} F_n \cos\Big(n\pi\frac{x-a}{b-a}\Big), \quad x \in [a,\ b], \tag{9}$$

where

$$F_n = \frac{2}{b-a}\mathrm{Re}\Big\{\tilde{\phi}_T\Big(\frac{n\pi}{b-a}\Big)\exp\Big(-\mathrm{i}\frac{na\pi}{b-a}\Big)\Big\} \approx A_n, \quad n = 0, 1, \ldots, N-1. \tag{10}$$

Now, substituting (9) into (3), we obtain an approximation of the option price (3):

$$C_0(K) \approx Ke^{-rT}\Big\{\frac{F_0}{2}\big(\Phi_0^{(c)}(0,b) - \Psi_0^{(c)}(0,b)\big) + \sum_{n=1}^{N-1} F_n\big(\Phi_n^{(c)}(0,b) - \Psi_n^{(c)}(0,b)\big)\Big\}, \tag{11}$$

where $\Phi_n^{(c)}(0,b)$ and $\Psi_n^{(c)}(0,b)$ are two integrals given by:

$$\Phi_n^{(c)}(c,d) = \int_c^d e^x \cos\Big(n\pi\frac{x-a}{b-a}\Big)\mathrm{d}x \quad \text{and} \quad \Psi_n^{(c)}(c,d) = \int_c^d \cos\Big(n\pi\frac{x-a}{b-a}\Big)\mathrm{d}x,$$

for any $[c,d] \subset [a,b]$, which are analytically given in the following proposition. This method is called COS for the European call under the general underlying processes.

Proposition 1 (Result 3.1 in [1]) *Let $[c,d] \subset [a,b]$. Then, for $n = 0, 1, 2, \ldots$*

$$\begin{aligned}\Phi_n^{(c)}(c,d) =& \frac{1}{1+\left(\frac{n\pi}{b-a}\right)^2}\Big[\cos\Big(n\pi\frac{d-a}{b-a}\Big)e^d - \cos\Big(n\pi\frac{c-a}{b-a}\Big)e^c + \\ & \frac{n\pi}{b-a}\sin\Big(n\pi\frac{d-a}{b-a}\Big)e^d - \frac{n\pi}{b-a}\sin\Big(n\pi\frac{c-a}{b-a}\Big)e^c\Big],\end{aligned}$$

$$\Psi_n^{(c)}(c,d) = \begin{cases} \Big[\sin\Big(n\pi\frac{d-a}{b-a}\Big) - \sin\Big(n\pi\frac{c-a}{b-a}\Big)\Big]\frac{b-a}{n\pi} & n \neq 0, \\ (d-c) & n = 0 \end{cases}.$$

Fang and Oosterlee also showed in their paper [1] that, in most cases, the convergence rate of the COS method (11) is exponential and the computational complexity is linear. They also discussed the truncation range for COS method, and gave a general formula to determine the interval of integration $[a,b]$ in that paper.

However, we can see the density $\tilde{p}_T(x)$ in (3) depends on the strike price K, and so that the interval of integration $[a,b]$ also depends on K. This leads to the inconveniency and extra complexity when we calculate the option prices with many different strike prices. In the next section, we will modify the Fang and Oosterlee's idea so that we can improve COS in this aspect.

3. FCOS, FSIN and FSER methods

Let X_t be log price of the underlying asset, i.e., $X_t = \log S_t$, and let $k = \log K$. Then, the option price (3) becomes

$$C_0(k) = e^{-rT}\mathbf{E}_{\mathbf{Q}}\big[\big(e^{X_T} - e^k\big)^+\big] = e^{-rT}\int_k^\infty (e^x - e^k)p_T(x)\mathrm{d}x, \tag{12}$$

where $p_T(x)$ is the density of X_T under the risk-neutral probability $\mathbf{Q}$. The characteristic function of X_T is given by

$$\phi_T(z) = \mathbf{E}_{\mathbf{Q}}\big[e^{\mathrm{i}zX_T}\big] = \int_{-\infty}^\infty e^{\mathrm{i}zx}p_T(x)\mathrm{d}x, \quad z \in \mathbf{R}. \tag{13}$$

Similarly to the Fang and Oosterlee's idea, we can choose the interval of integration $[a, b]$ such that

$$\phi_{1,T}(z) = \int_a^b e^{\mathrm{i}zx}p_T(x)\mathrm{d}x \approx \phi_T(z), \tag{14}$$

and by using the Fourier-cosine expansion of $p_T(x)$ in $[a, b]$, we have

$$p_T(x) \approx \frac{1}{2}F_0^{(c)} + \sum_{n=1}^{N-1} F_n^{(c)} \cos\Big(n\pi\frac{x-a}{b-a}\Big), \quad x \in [a, b], \tag{15}$$

where

$$F_n^{(c)} = \frac{2}{b-a}\mathrm{Re}\Big\{\phi_T\Big(\frac{n\pi}{b-a}\Big)\exp\Big(-\mathrm{i}\frac{na\pi}{b-a}\Big)\Big\}, \quad n = 0, 1, \ldots. \tag{16}$$

Substituting (15) into (12), we get an approximation of the option price (12):

$$\begin{aligned} C_0(k) \approx & e^{-rT}\frac{1}{2}F_0^{(c)}\big(\Phi_0^{(c)}(k, b) - e^k\Psi_0^{(c)}(k, b)\big) + \\ & e^{-rT}\sum_{n=1}^{N-1} F_n^{(c)}\big(\Phi_n^{(c)}(k, b) - e^k\Psi_n^{(c)}(k, b)\big), \end{aligned} \tag{17}$$

where $\Phi_n^{(c)}(k, b)$ and $\Psi_n^{(c)}(k, b)$ are given by Proposition 1. We call this method as FCOS.

Since the density $p_T(x)$ is independent of the log strike price k, we do not need to change the interval of integration $[a, b]$ in the approximation (14) whenever k changes. In fact, we only need to change the exact integrals $\Phi_n^{(c)}(k, b)$ and $\Psi_n^{(c)}(k, b)$, and hence, the calculation must be more convenient than the COS formula (11).

If we change to use the Fourier-sine expansions to approximate $p_T(x)$ in $[a, b]$, then (15) becomes

$$p_T(x) \approx \sum_{n=1}^{N-1} F_n^{(s)} \sin\Big(n\pi\frac{x-a}{b-a}\Big), \quad x \in [a, b], \tag{18}$$

where

$$F_n^{(s)} = \frac{2}{b-a}\mathrm{Im}\Big\{\phi_T\Big(\frac{n\pi}{b-a}\Big)\exp\Big(-\mathrm{i}\frac{na\pi}{b-a}\Big)\Big\}, \quad n = 0, 1, \ldots, N-1. \tag{19}$$

Here $\mathrm{Im}\{\cdot\}$ denotes taking imaginary part of argument. Substituting (18) into (12), we obtain another approximation of the option price (12):

$$C_0(k) \approx e^{-rT} \sum_{n=1}^{N-1} F_n^{(s)} \big(\Phi_n^{(s)}(k,b) - e^k \Psi_n^{(s)}(k,b)\big), \tag{20}$$

where $\Phi_n^{(s)}(k,b)$ and $\Psi_n^{(s)}(k,b)$ are two integrals:

$$\Phi_n^{(s)}(k,b) = \int_k^b e^x \sin\Big(n\pi \frac{x-a}{b-a}\Big) \mathrm{d}x \quad \text{and} \quad \Psi_n^{(s)}(k,b) = \int_k^b \sin\Big(n\pi \frac{x-a}{b-a}\Big) \mathrm{d}x,$$

which are analytically given in the following proposition. We call this method as FSIN.

Proposition 2 *Let $[c,d] \subset [a,b]$. Then, for $n = 1, 2, \ldots$*

$$\begin{aligned}\Phi_n^{(s)}(c,d) =& \frac{1}{1+\left(\frac{n\pi}{b-a}\right)^2} \Big[\sin\Big(n\pi \frac{d-a}{b-a}\Big) e^d - \sin\Big(n\pi \frac{c-a}{b-a}\Big) e^c - \\ & \frac{n\pi}{b-a} \cos\Big(n\pi \frac{d-a}{b-a}\Big) e^d + \frac{n\pi}{b-a} \cos\Big(n\pi \frac{c-a}{b-a}\Big) e^c \Big], \\ \Psi_n^{(s)}(c,d) =& - \Big[\cos\Big(n\pi \frac{d-a}{b-a}\Big) - \cos\Big(n\pi \frac{c-a}{b-a}\Big) \Big] \frac{b-a}{n\pi}.\end{aligned}$$

By a simple calculation one can verify this proposition. Finally, applying the Fourier-series expansion to approximate $p_T(x)$ in $[a,b]$, we have

$$p_T(x) \approx \frac{1}{2} F_0^{(f,c)} + \sum_{n=1}^{N-1} \Big[F_n^{(f,c)} \cos\Big(n\pi \frac{2x-a-b}{b-a}\Big) + F_n^{(f,s)} \sin\Big(n\pi \frac{2x-a-b}{b-a}\Big) \Big], \tag{21}$$

for all $x \in [a,b]$, where

$$\begin{aligned} F_n^{(f,c)} &= \frac{2}{b-a} \mathrm{Re}\Big\{ \phi_T\Big(\frac{2n\pi}{b-a}\Big) \exp\Big(- in\pi \frac{a+b}{b-a}\Big) \Big\}, \quad n = 0, 1, 2, \ldots, \\ F_n^{(f,s)} &= \frac{2}{b-a} \mathrm{Im}\Big\{ \phi_T\Big(\frac{2n\pi}{b-a}\Big) \exp\Big(- in\pi \frac{a+b}{b-a}\Big) \Big\}, \quad n = 1, 2, \ldots. \end{aligned}$$

Substituting (21) into (12), we derive a method, which is called FSER, to approximate the option price (12):

$$\begin{aligned} C_0(k) \approx & e^{-rT} \frac{1}{2} F_0^{(f,c)} \big(\Phi_0^{(f,c)}(k,b) - e^k \Psi_0^{(f,c)}(k,b)\big) + \\ & e^{-rT} \sum_{n=1}^{N-1} \Big[F_n^{(f,c)} \big(\Phi_n^{(f,c)}(k,b) - e^k \Psi_n^{(f,c)}(k,b)\big) + \\ & F_n^{(f,s)} \big(\Phi_n^{(f,s)}(k,b) - e^k \Psi_n^{(f,s)}(k,b)\big) \Big], \end{aligned} \tag{22}$$

where four integrals:

$$\begin{aligned} \Phi_n^{(f,c)}(k,b) &= \int_k^b e^x \cos\Big(n\pi \frac{2x-a-b}{b-a}\Big) \mathrm{d}x, \quad & \Psi_n^{(f,c)}(k,b) &= \int_k^b \cos\Big(n\pi \frac{2x-a-b}{b-a}\Big) \mathrm{d}x, \\ \Phi_n^{(f,s)}(k,b) &= \int_k^b e^x \sin\Big(n\pi \frac{2x-a-b}{b-a}\Big) \mathrm{d}x, \quad & \Psi_n^{(f,s)}(k,b) &= \int_k^b \sin\Big(n\pi \frac{2x-a-b}{b-a}\Big) \mathrm{d}x, \end{aligned}$$

are analytically given in the following proposition.

Proposition 3 *Let $[c,d] \subset [a,b]$. Then, for $n = 0, 1, 2, \ldots$*

$$\Phi_n^{(f,c)}(c,d) = \frac{1}{1+\left(\frac{2n\pi}{b-a}\right)^2}\Big[\cos\Big(n\pi\frac{2d-a-b}{b-a}\Big)e^d - \cos\Big(n\pi\frac{2c-a-b}{b-a}\Big)e^c + \frac{2n\pi}{b-a}\sin\Big(n\pi\frac{2d-a-b}{b-a}\Big)e^d - \frac{2n\pi}{b-a}\sin\Big(n\pi\frac{2c-a-b}{b-a}\Big)e^c\Big],$$

$$\Psi_n^{(f,c)}(c,d) = \begin{cases} \Big[\sin\Big(n\pi\frac{2d-a-b}{b-a}\Big) - \sin\Big(n\pi\frac{2c-a-b}{b-a}\Big)\Big]\frac{b-a}{2n\pi} & n \neq 0 \\ d-c & n = 0 \end{cases}$$

$$\Phi_n^{(f,s)}(c,d) = \frac{1}{1+\left(\frac{2n\pi}{b-a}\right)^2}\Big[\sin\Big(n\pi\frac{2d-a-b}{b-a}\Big)e^d - \sin\Big(n\pi\frac{2c-a-b}{b-a}\Big)e^c - \frac{2n\pi}{b-a}\cos\Big(n\pi\frac{2d-a-b}{b-a}\Big)e^d + \frac{2n\pi}{b-a}\cos\Big(n\pi\frac{2c-a-b}{b-a}\Big)e^c\Big],$$

$$\Psi_n^{(f,s)}(c,d) = \begin{cases} -\Big[\cos\Big(n\pi\frac{2d-a-b}{b-a}\Big) - \cos\Big(n\pi\frac{2c-a-b}{b-a}\Big)\Big]\frac{b-a}{2n\pi} & n \neq 0 \\ 0 & n = 0 \end{cases}.$$

The proposition is easy to be proved by a simple calculation.

4. Numerical experiments

In this section via numerical experiments we compare the accuracy and CPU times among FCOS, FSIN and FSER, which are respectively given by (17), (20) and (22) in Section 3. Here we consider the error:

$$\epsilon := \max_k \big|C_0(k) - \tilde{C}_0(k)\big|, \tag{23}$$

where $k = \log K$, $C_0(k)$ is the exact price, and $\tilde{C}_0(k)$ is the corresponding approximate price. The computer used for all these experiments has a Pentium(R) 4 CPU 3.6GHz and the code is written in MATHLAB 7.

First, we compare the errors between the those methods and COS (11) for the BS model. Let B_t be a standard Brownian motion. Under the risk-neutral probability $\mathbf{Q}$, the price of the underlying asset in the BS model is of the form:

$$S_t = S_0 \exp\Big\{\big(r - \frac{1}{2}\sigma^2\big)t + \sigma B_t\Big\}, \quad 0 \le t \le T, \tag{24}$$

where r is the interest rate of the risk-free investment. The characteristic function of the log price $X_t = \log S_t$ is now given by

$$\phi_t(z) = \exp\Big\{\mathrm{i}z\Big[x_0 + \big(r - \frac{1}{2}\sigma^2\big)t\Big] - \frac{1}{2}\sigma^2 z^2 t\Big\}, \quad z \in \mathbf{R}, \tag{25}$$

where $x_0 = \log S_0$. The exact price of the European call option in the BS model is given by the famous Black-Scholes formula.

Table 1 gives the errors and CPU times of different methods by taking $K = 80$, 100, 120 (i.e., $k = \log 80$, $\log 100$, $\log 120$) for the BS model with $S_0 = 100$, $r = 0.05$, $T = 0.1$ and $\sigma = 0.2$.

Also, as in [1] we choose

$$[a,b]=\Big[c_1-10\sqrt{c_2+\sqrt{c_4}},\quad c_1+10\sqrt{c_2+\sqrt{c_4}}\Big],$$

in FCOS (17), FSIN (20) and FSER (22) with the parameters:

$$c_1=\log S_0+\big(r-\frac{1}{2}\sigma^2\big)T,\quad c_2=\sigma^2T,\quad \text{and}\quad c_4=0.$$

On the other hand, we must mention here that we use the parameter c_1:

$$c_1=\log\frac{S_0}{K}+\big(r-\frac{1}{2}\sigma^2\big)T$$

in COS (11), and we obtain almost the same errors as in the Table 2 in [1]. We have the different CPU times with them because the used computers are different.

	COS		FSER		FCOS		FSIN	
N	ϵ	time	ϵ	time	ϵ	time	ϵ	time
16	0.0074	0.444	2.4285e-7	0.451	0.0074	0.335	0.1086	0.353
32	2.4285e-7	0.834	1.3768e-13	0.884	2.4285e-7	0.664	3.3981e-6	0.615
64	9.5923e-14	1.701	1.3768e-13	1.734	4.4645e-14	1.264	1.1369e-13	1.217
128	9.5923e-14	2.995	1.3768e-13	3.213	4.4645e-14	2.247	1.1369e-13	2.136
256	9.5923e-14	5.992	1.3768e-13	6.187	4.4645e-14	4.492	1.1369e-13	4.371

Table 1 Maximum errors and CPU times (ms) with strike price $K=80,100,120$ in the BS model

Table 1 shows that all methods have almost the same errors for the BS model, and FCOS and FSIN seem to be faster than COS and FSER. Next, we consider the MJD model, which is the first successful financial model with jumps introduced by Merton [8]. Let N_t be a Poisson process of intensity λ, and $\{\xi_i\}$ be a sequence of independent and identically normally distributed random variables with the mean μ_J and the standard variation σ_J, such that N_t, $\{\xi_i\}$ and the standard Brownian motion B_t are mutually independent. Under a risk-neutral probability $\mathbf{Q}_M$, the price of the underlying asset in the MJD model is of the form:

$$S_t=S_0\exp\Big\{\mu_Mt+\sigma B_t+\sum_{i=1}^{N_t}\xi_i\Big\},\quad 0\le t\le T, \tag{26}$$

where

$$\mu_M=r-\frac{1}{2}\sigma^2-\lambda\big(e^{\mu_J+\frac{1}{2}\sigma_J^2}-1\big).$$

This MJD model explains the jump part as the financial market response to outside news: good news and bad news. The news arrives according to the Poisson process N_t and the stock price changes in response according to the jump size ξ_i. The characteristic function of the log price $X_t=\log S_t$ is now given by

$$\phi_t(z)=\exp\Big\{\lambda\big(e^{\mathrm{i}\mu_Jz-\frac{1}{2}\sigma_J^2z^2}-1\big)t+\mathrm{i}z\big(x_0+\mu_Mt\big)-\frac{1}{2}\sigma^2z^2t\Big\},\quad z\in\mathbf{R}, \tag{27}$$

where $x_0=\log S_0$. For each non-negative integer $n=0,1,2,\ldots$, set:

$$S_0^{(n)}=S_0e^{n\mu_J+\frac{1}{2}n\sigma_J^2},\quad r_n=r-\lambda\big(e^{\mu_J+\frac{1}{2}\sigma_J^2}-1\big)\quad\text{and}\quad \sigma_n^2=\sigma^2+n\sigma_J^2/T.$$

Merton in [8] derived an exact pricing formula for the European call option:

$$C_0(S_0) = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}\big(\lambda(T-t)\big)^n}{n!} C_0^{\mathrm{BS}}\big(S_0^{(n)}; r_n, \sigma_n\big). \tag{28}$$

Here, for each n, $C_0^{\mathrm{BS}}\big(S_0^{(n)}; r_n, \sigma_n\big)$ is the price of European call $C = (S_T^{(n)} - K)^+$ in the BS model in which the price of underlying asset is given by:

$$S_t^{(n)} = S_0^{(n)} \exp\Big\{\big(r_n - \frac{1}{2}\sigma_n^2\big)t + \sigma_n B_t\Big\}, \quad 0 \le t \le T,$$

and the interest rate is r_n. Here we use the formula (28) up to the 7^{th} term to calculate the exact prices $C_0(k)$. Tables 2, 3 and 4 give the errors and CPU times of COS, FSER, FCOS and FSIN for the MJD model with different intervals of integration $[a, b]$. Here we consider the date set of log strike price $k = \log K$:

$$k_j = a + j\Delta_k, \quad j = 0, 1, \ldots, 2^{10} - 1, \quad \Delta_k = (b-a)/2^{10},$$

and the parameters:

$$S_0 = 50, \quad T = 20/252, \quad r = 0.05, \quad \sigma = 0.2, \quad \lambda = 1, \quad \mu_J = -0.1 \quad \text{and} \quad \sigma_J = 0.1.$$

We also set:

$$c_1 = \lambda\mu_J T + \log S_0 + \Big(r - \frac{1}{2}\sigma^2 - \lambda\big(e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1\big)\Big)T,$$

for FCOS (17), FSIN (20) and FSER (22), and

$$c_1 = \lambda\mu_J T + \log\frac{S_0}{K} + \Big(r - \frac{1}{2}\sigma^2 - \lambda\big(e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1\big)\Big)T,$$

for COS (11). In these tables, if an error ϵ is greater than 50, then it is denoted by ”–”.

	COS		FSER		FCOS		FSIN	
N	ϵ	time	ϵ	time	ϵ	time	ϵ	time
16	0.1162	0.0651	0.0090	0.0552	0.1162	0.0284	0.5864	0.0311
32	0.0014	0.1326	8.4077e-7	0.1141	0.0014	0.0560	0.0078	0.0650
64	5.8023e-9	0.2673	7.8027e-7	0.2294	5.8023e-9	0.1111	2.4712e-8	0.1305
128	6.8860e-9	0.5350	7.8027e-7	0.4622	6.8860e-9	0.2220	6.8287e-9	0.2610
256	6.8860e-9	1.0844	7.8027e-7	0.9364	6.8860e-9	0.4447	6.8287e-9	0.5222

Table 2 Maximum errors and CPU times (s) in MJD model with $[a, b] = [c_1 - 1,\ c_1 + 1]$.

	COS		FSER		FCOS		FSIN	
N	ϵ	time	ϵ	time	ϵ	time	ϵ	time
64	1.9497	0.2509	0.3945	0.2248	1.9497	0.1086	14.8728	0.1280
128	0.0822	0.5147	1.8707e-4	0.4532	0.0822	0.2178	0.8118	0.2563
256	1.0589e-5	1.0541	9.1021e-11	0.9057	1.0589e-5	0.4327	4.6232e-5	0.5119
512	9.4595e-11	2.1374	9.1021e-11	1.8139	9.4929e-11	0.8668	9.2868e-11	1.0224
1024	9.4595e-11	4.3102	9.1021e-11	3.6230	9.4929e-11	1.7361	9.2868e-11	2.0490

Table 3 Maximum errors and CPU times (s) in MJD model with $[a, b] = [c_1 - 5,\ c_1 + 5]$.

	COS		FSER		FCOS		FSIN	
N	ϵ	time	ϵ	time	ϵ	time	ϵ	time
128	–	0.5059	28.6501	0.4594	–	0.2205	–	0.2584
256	5.9793	1.0446	0.0134	0.9150	5.9793	0.4396	–	0.5176
512	7.6740e-4	2.1242	6.7720e-10	1.8375	7.6740e-4	0.8742	0.0040	1.0333
1024	3.8627e-10	4.2969	6.7720e-10	3.6802	3.1665e-10	1.7514	6.9950e-10	2.0656
2048	3.8627e-10	8.6876	6.7720e-10	7.3485	3.1665e-10	3.5250	6.9950e-10	4.1517

Table 4 Maximum errors and CPU times (s) in MJD model with $[a,b] = [c_1 - 10,\ c_1 + 10]$.

Tables 2, 3 and 4 show that all of these methods have almost the same accuracy for MJD model, and COS and FSER take about double CPU times of FCOS or FSIN. In fact, we must note that the interval of integration $[a,b]$ for COS depends on K so that it takes more time than FCOS and FSIN, and in FSER we need to calculate the double terms of what are in FCOS or FSIN. Meanwhile, we can see that as the interval of integration becomes bigger, we need to take a larger N to obtain a stable error, and a suitable interval of integration can improve the accuracy of these methods.

Finally, we consider another exponential Lévy model: the VG model. The Variance Gamma process was first introduced to finance community as a model for asset returns (log price increments) and option pricing by Madan and Seneta [9]. Recently, some researches show that it is an efficient model for the real stock markets, and many authors have researched the option pricing problems by using this model [1, 3, 5, 10].

In the VG model, the price of the underlying asset is given by $S_t = S_0 e^{L_t}$, where L_t is a Variance Gamma process, i.e., a pure jump Lévy process with infinite arrival rate of jumps [10]. Under a risk-neutral probability $\mathbf{Q}$, the price S_t can be expressed as

$$S_t = S_0 \exp\Big\{\Big(r + \frac{1}{\kappa}\log\big(1 - \theta\kappa - \frac{1}{2}\sigma^2\kappa\big)\Big)t + Z_t(\theta,\sigma,\kappa)\Big\}, \quad 0 \le t \le T. \tag{29}$$

Here $Z_t(\theta,\sigma,\kappa) = \theta G_t + \sigma B_{G_t}$ is a Variance Gamma process, which is created by random time changing a Brownian motion with drift process: $\theta t + \sigma B_t$ by a tempered gamma process G_t with unit mean rate and variance rate κ, where κ, θ and σ are parameters. The characteristic function of the log price $X_t = \log S_t$ can be obtained by the use of the subordination theorem [2] in the form:

$$\phi_t(z) = \exp\Big\{\mathrm{i}z\Big[x_0 + \Big(r + \frac{1}{\kappa}\log\big(1 - \theta\kappa - \frac{1}{2}\sigma^2\kappa\big)\Big)t\Big]\Big\}\Big(1 + \frac{1}{2}\sigma^2\kappa z^2 - \mathrm{i}\theta\kappa z\Big)^{-t/\kappa}, \tag{30}$$

where $x_0 = \log S_0$.

Madan et al also obtained a closed form expression for the European call price in terms of the modified Bessel function of the second kind and the degenerate hypergeometric function (Theorem 2 in [10]). However, it is not easy to obtain the exact price by this closed form formula. Here we only compare FCOS and FSER with a semi-FFT numerical method, which was proposed by the authors in [11], for the VG model. Figures 1, 2 and 3 plot the price function with respect to the strike price K by FCOS, FSER and semi-FFT ($\Delta_k = 0.05$, $N = 2^{14}$ and $\alpha = 1.5$) with different $N = 128$, 256, 512, respectively. The corresponding parameters are given by

$$S_0 = 50, \quad \sigma = 0.2, \quad r = 0.05, \quad T = 20/252, \quad \theta = -0.1 \quad \text{and} \quad \kappa = 0.1.$$

We also take $k_j = a + j\Delta_k$ with $\Delta_k = (b-a)/2^{12}$, and set the interval of integration $[a, b] = [c_1 - 5,\ c_1 + 5]$ for FCOS and FSER, where

$$c_1 = \log(S_0) + \Big(r + \frac{1}{\kappa}\log(1 - \theta\kappa - \frac{1}{2}\kappa\sigma^2)\Big)T.$$

From these figures we can see that both of FCOS and FSER are globally convergent to semi-FFT as N increases, and FSER seems to be faster than FCOS.

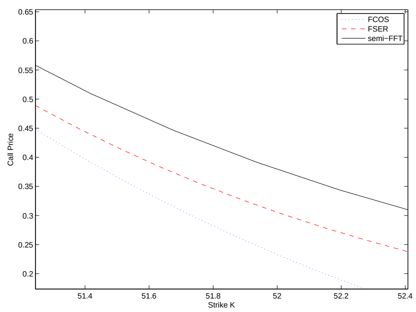

Figure 1 FSER, FCOS with $N = 128$ and semi-FFT for VG model

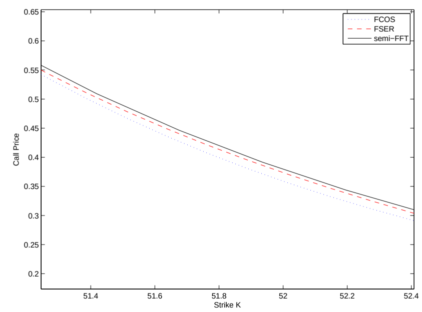

Figure 2 FSER, FCOS with $N = 256$ and semi-FFT for VG model

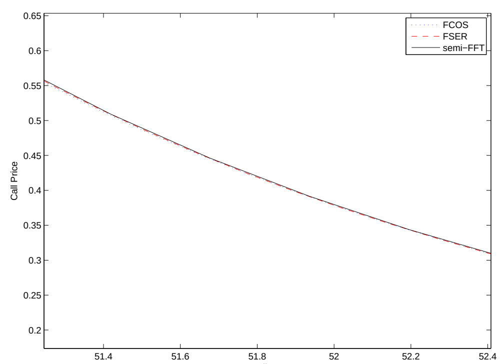

Figure 3 FSER, FCOS with $N = 512$ and semi-FFT for VG model

References

[1] FANG F, OOSTERLEE C W. *A novel pricing method for European options based on Fourier-cosine series expansions* [J]. SIAM J. Sci. Comput., 2008/09, **31**(2): 826–848.

[2] CONT R, TANKOV P. *Financial Modelling with Jump Processes* [M]. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[3] CARR P P, MADAN D B. *Option valuation using the fast Fourier transform* [J]. J. Comput. Finance, 1999, **2**(4): 61–73.

[4] ANDRICOPOULOS A D, WIDDICKS M, DUCK P D, NEWTON P D. *Universal option valuation using quadrature methods* [J]. J. Financial Econom., 2003, **67**: 447–471.

[5] LORD R, FANG F, BERVOETS F. et al. *A fast and accurate FFT-based method for pricing early-exercise options under Lévy processes* [J]. SIAM J. Sci. Comput., 2008, **30**(4): 1678–1705.

[6] FENG Liming, LINETSKY V. *Pricing discretely monitored barrier options and defaultable bonds in Lévy process models: a fast Hilbert transform approach* [J]. Math. Finance, 2008, **18**(3): 337–384

[7] FANG F, OOSTERLEE C W. *Pricing early-exercise and discrete barrier options by Fourier-cosine series expansions* [J]. Numer. Math., 2009, **114**(1): 27–62.

[8] MERTON C R. *Option pricing when underlying stock returns and discountinuous* [J]. J. Financial Econom., 1976, **3**: 125–144.

[9] MADAN D B, SENETA E. *The Variance-Gamma (V.G.) model for share market returns* [J]. Journal of Business, 1990, **63**: 511–524.

[10] MADAN D B, CARR P P, CHANG E C. *The variance gamma process and option pricing* [J]. European Finance Review, 1998, **2**: 79–105.

[11] DING Deng, U Sion Chong. *An accurate and stable FFT-based method for pricing options under exp-Lévy processes* [C]. in the Proceeding of ISCM II - EPMESC XII 2009, American Institute of Physics, Melville, NY, USA.