The series $\sum_{n=1}^{\infty}\frac{1}{n^{k+1}}e^{\frac{-z^{2k}}{n^{2k}}}$ (odd k) and Riemann Zeta function

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J. Tennenbaum[1] discussed the function $\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-z/n}$ in 1977. Zhang Nanyue[2] discussed the function $\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-z^2/n^2}$ in 1983. Now we discuss the functions $\sum_{n=1}^{\infty}\frac{1}{n^{k+1}}\cdot e^{-z^{2k}/n^{2k}}$ (k positive odd) in this paper which finds representations of two integrales about Riemann Zeta function

$$\Gamma(s)\zeta(s)=\frac{\cos\frac{\pi(k-s)}{4k}}{\cos\frac{\pi s}{2}}\int_0^{\infty}\Big[\sum_{m=0}^{k-1}(-1)^m\frac{v_m\operatorname{sh}(x\lambda_m)-u_m\sin(x\tau_m)}{\operatorname{ch}(x\lambda_m)-\cos(x\tau_m)}-\frac{1}{\sqrt{2}}\Big]x^{s-1}dx \quad (\sigma>0).$$

$$\Gamma(s)\zeta(s)=\frac{\cos\frac{\pi(k-s)}{4k}}{\cos\frac{\pi s}{2}}\cdot\int_0^{\infty}\Big[\sum_{m=0}^{k-1}(-1)^m\frac{v_m\operatorname{sh}(x\lambda_m)-u_m\sin(x\tau_m)}{\operatorname{ch}(x\lambda_m)-\cos(x\tau_m)}\Big]x^{s-1}dx,$$
$$(-k<\sigma_k<0,\ s\neq-1,-3,\cdots,-k+2).$$

$$\Gamma(s)\zeta(s)=\frac{\sin\frac{\pi(k-s)}{4k}}{\cos\frac{\pi s}{2}}\cdot\int_0^{\infty}\Big[\sum_{m=0}^{k-1}(-1)^m\frac{v_m\operatorname{sh}(x\lambda_m)+u_m\sin(x\tau_m)}{\operatorname{ch}(x\lambda_m)-\cos(x\tau_m)}-\frac{1}{\sqrt{2}}\Big]x^{s-1}dx,\quad (\sigma>1).$$

where k is any positive odd number, $\operatorname{Re}(s)=\sigma$ or σ_k, $\lambda_m=\sin\frac{(2m+1)\pi}{4k}$, $\tau_m=\cos\frac{(2m+1)\pi}{4k}$, $u_m=\sin\frac{(2m+1)\pi}{4}$, $v_m=\cos\frac{(2m+1)\pi}{4}$; and obtain three diffrent proofs about the equation of Riemann Zeta function for any positive odd number k. This is Zhang Nanyue's result when k = 1.

References

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