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On the Adjacent Strong Edge Coloring of Halin Graphs *

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Abstract: A proper k-edge coloring f of graph $G(V,E)$ is said to be a k-adjacent strong edge coloring of graph $G(V,E)$ iff every $uv \in E(G)$ satisfy $f[u] \neq f[v]$, where $f[u] = \{f(uw)|uw \in E(G)\}$ then f is called **k-adjacent strong edge coloring** of G, is abbreviated k-ASEC; and $\chi'_{as}(G) = \min\{k|k\text{-ASEC of } G\}$ is called the **adjacent strong edge chromatic number.** In this paper, we study the $\chi'_{as}(G)$ of Halin graphs with $\Delta(G) \geq 5$.

Key words: adjacent strong edge coloring; adjacent strong edge chromatics number; Halin graph.

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1. Introduction

The coloring of graphs is widely applied in practice. In some network problem, some problem can be converted to the strong edge coloring and adjacent strong edge coloring.

Definition 1[1,2] *Suppose $G(V,E)$ is a 3-connected plane graph, if remove all edges on the boundary of a face f_0 (the degree of vertices of f_0 is 3), $G(V,E)$ become a tree T, then $G(V,E)$ is called a* **Halin graph**, *and f_0 is called the* **outer face of** G **(others the interior face)**, *vertices in $V(f_0)$ is called* **outer vertex (others the interior vertex)**.

In practice, there are more and more coloring problems having particular propertis[3]. In [6,7], one kind of edge strong coloring have been studied and obtained a larger bounds. In this paper, we discuss the adjacent strong edge-coloring of a graph and obtained the adjacent strong edge chromatic number of Halin graph.

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Definition 2[4] *A proper k-edge coloring f of graph $G(V,E)$ is said to be a k-adjacent strong edge coloring of graph $G(V,E)$ (is abreviated a k-ASEC of graph $G(V,E)$) iff every $uv \in E(G)$ satisfy $f[u] \neq f[v]$, where $f[u] = \{f(uw)|uw \in E(G)\}$ then f is called* **k-adjacent strong edge coloring** *of G, is abbreviated k-ASEC; and*

$$\chi'_{as}(G) = \min\{k|k - \text{ASEC of } G\}$$

is called the **adjacent strong edge chromatic number**.

Based on some results of some special graphs, a conjecture was presented as follow.

Conjecture[4] For a connected graph $G(V,E)$ with $|V(G)| \geq 3$, and $G \neq C_5$ have

$$\Delta(G) \leq \chi'_{as}(G) \leq \Delta(G) + 2.$$

We have proved this conjecture hold for cycle, fan graph, wheel graph, tree, complete graph, complete r-partite graph ($r \geq 2$), outer plane graph, 1-tree, high degree plane graph and Series-parallel graph.

The existence of $\chi'_{as}(G)$ for simple connected graphs which order is at least 3 is obviously, and $\chi'_{as}(G) \geq \Delta(G)$. Where $\Delta(G)$ is the maximum degree of graph G.

In this paper, we study the adjacent strong edge coloring of Halin graphs with $\Delta(G) = 5$.

The other terminologies we refer to References [3,5].

2. Main results

Lemma 1[2] *Let G be a Halin graph, then*

(1) The degree of all outer vertices is 3, and any two interior faces have and just have one common edge, so do any interior and outer face;

(2) If $G \ncong W_p$ (wheel of order p), there are at least two interior vertices of G, and there always exists an interior vertex w which is only adjacent to one interior vertex;

(3) If $G \ncong W_p$ (wheel of order p), w is the vertex of (2), let $N(w) = \{u, v_1, v_2, \ldots, k\}$, where u is the unique interior vertex adjacent to w, and $v_1, v_2, \ldots, v_k$ are outer vertices adjacent to w, and x is the outer vertex adjacent to v_1, and x_0, x_1 are other two adjacent vertices of x; y is the outer vertex adjacent to v_k, and y_0, y_1 are other two adjacent vertices of y, We consider the graph

$$G_0^1 = G - \{v_1, v_2, \ldots, v_k\} + \{xw, yw\},$$

$$G_0^2 = G - \{v_i, v_{i+1}, \ldots, v_j\} + \{v_{i-1}w, v_{j+1}w\},$$

then G_0^1 and G_0^1 is also a Halin graph, and $\Delta(G_0^1) = \Delta(G_0^1) = \Delta(G)$.

Lemma 2 *For wheel graph W_p, have*

$$\chi'_{as}(G) = \Delta(G) = p - 1.$$

If following process, we denote $V_\Delta = \{v|v \in V(G), d(v) = \Delta(G)\}$, and $G[V_\Delta]$ the induced subgraph by V_Δ.

Theorem 1 *For Halin graph G of $\Delta(G) = 5$, have $5 \le \chi'_{as}(G) \le 6$. And $\chi'_{as}(G) = 6$ iff*

$$E(G[V_\Delta]) \neq \emptyset.$$

Proof We use induction on $p = |V(G)|$ to prove there exists a 5-ASEC of G.

If $E(G[V_\Delta]) = \emptyset$, we prove $\chi'_{as}(G) = 5$.

Obviously, $\chi'_{as}(G) \ge 5$. $C = \{c_1, c_2, c_3, c_4, c_5\}$ denotes the set of five colors. If $p = \Delta(G) + 1$, then $G = W_p$, by lemma 2, the conclusion is true. Suppose the conclusion is true for $|V(G)| < p$.We now to prove the conclusion is true for $|V(G)| = p$. We quote W denotethe set of all vertices w (exactly adjacent to one interior vertex) satisfy the condition of (2) and (3) of Lemma 1.

In following process, $g[v] = \{g(vu)|vu \in E(G')\}$, and the meaning of v_1, v_2, $\ldots$, v_k, u, x, y, x_0, x_1, y_0, y_1 same as Lemma 1. We consider the $w \in W$ which have minimal degree.

Case 1 If $d(w) = 3$, let $N(w) = \{u, v_1, v_2\}$. We consider the graph

$$G' = G - \{v_1, v_2\} + \{xw, yw\},$$

then by Lemma 1, G' is also a Halin graph and $\Delta(G') = \Delta(G)$. By the induction hypothesis, G' has a 5-ASEC g. We now based on g to construct a 5-ASEC f of G. If $d(u) = 4$ or 5, the proving is easily than this case of $d(u) = 3$, hence assume $d(u) = 3$.

Subcase 1.1 If $g(wy) \in g[x], g(xw) \in g[y]$, without lose of generality, assume that

$$g(xx_1) = g(wy), g(yy_1) = g(wx),$$

then

$$g(wu) \notin \{g(xx_0), g(yy_0)\},\ \{g(xw), g(yw)\} \not\subset g[u].$$

Without lose of generality, we assume $g(xw) \notin g[u]$. Let

$$f(xv_1) = f(wv_2) = g(xw),$$

then no matter what color the edge v_1w colored, $f[w] \neq f[u]$ is always true. Let $f(wv_1) \in C \setminus \{g(xx_0), g(xx_1), f(xv_1), g(wu)\}$, then no matter what color the edge v_1v_2 colored, $f[x] \neq f[v_1]$ is always true, so can let $f(v_1v_2) = g(wu)$. The colors of others elements of $E(G)$ is same as G', obviously f is a 5-ASEC of G, so the conclusion is true.

Subcase 1.2 If $g(wy) \in g[x]$ and $g(wx) \notin g[y]$ (about the cases $g(wy) \notin g[x]$ and $g(wx) \in g[y]$, or $g(wy) \notin g[x]$ and $g(wx) \notin g[y]$, the proof is samilar to $g(wy) \in g[x]$ and $g(wx) \notin g[y]$ and omitted), then $g(wu) \notin g[x]$. Let

$$f(xv_1) = f(v_2w) = g(xw), f(v_1v_2) = g(wu).$$

Because of $f(v_1v_2) = g(wu) \notin g[x]$ and $f(v_2w) = g(xw) \notin g[y]$, hence no matter what color the edge v_1w is colored, $f[x] \neq f[v_1]$ and $f[v_2] \neq f[y]$ is always hold. Hence can let

$f(v_1w) \in C \setminus \{f(xv_1), f(v_1v_2), f(v_2w)\}$. Obviously f is a 5-ASEC of $G(V, E)$.

Case 2 If $d(w) = 4$, let $N(w) = \{u, v_1, v_2, v_3\}$. We consider the graph

$$G' = G - \{v_1, v_2, v_3\} + \{xw, yw\},$$

then by Lemma 1, G' is also a Halin graph and $\Delta(G') = \Delta(G)$. By the induction hypothesis, G' has a 5-ASEC g. We now based on g to construct a 5-ASEC f of G. If $d(u) = 3$ or 5, the proving is easily than this case of $d(u) = 4$, hence assume $d(u) = 4$, let

$$f(xv_1) = f(wv_3) = g(xw), \quad f(yv_3) = f(v_1w) = g(wy), f(v_1v_2) = g(wu).$$

- If $\{f(v_1w), f(v_3w)\} \subset g[u]$, let $f(v_2w) \in C \setminus g[u]$.
- If $\{f(v_1w), f(v_3w)\} \not\subset g[u]$, let $f(v_2w) \in C \setminus \{f(v_1w), f(v_3w), g(wu)\}$.

Obviously $f[y] = g[y]$ and $\{g(xw), g(wu)\} \not\subset g[y]$.

- If $f(v_1v_2) = g(wu) \in g[y] = f[y]$, then $f(v_3w) = g(xw) \notin g[y]$. Let $f(v_2v_3) \in C \setminus \{f(v_1v_2), f(v_3w), f(v_2w), f(v_3y)\}$.

  Obviously f is a 5-ASEC of G, so the conclusion is true.
- If $f(v_3w) = g(xw) \in g[y] = f[y]$, then $f(v_1v_2) = g(wu) \notin g[y] = f[y]$.
  - If $f(v_2w) \in g[y] = f[y]$, because $f(v_2w) \neq f(v_1w) = f(v_3w)$, hence we can let

    $$f(v_2v_3) \in C \setminus \{f(v_1v_2), f(v_3w), f(v_2w), f(v_3y)\}.$$

    Obviously f is a 5-ASEC of G, so the conclusion is true.
  - If $f(v_2w) \notin g[y] = f[y]$, because

    $$f(wv_2) \neq f(v_1w) = f(v_3y), f(v_3y) \neq f(v_1v_2) = g(wu), f(v_3w) \neq f(v_2w),$$

    hence we can exchange the colors of v_2w and v_3w. After this process, we have $f(v_3w) \notin g[y] = f[y]$, and can let

    $$f(v_2v_3) \in C \setminus \{f(v_1v_2), f(v_3w), f(v_2w), f(v_3y)\}.$$

    Obviously f is a 5-ASEC of G, so the conclusion is true.

Case 3 If $d(w) = 5$, because $E(G[V_\Delta]) = \emptyset$, hence we have $d(u) = 3$ or 4, we consider graph $G' = G - \{v_2, v_3\} + \{v_1v_4\}$, then by Lemma 1, G' is also a Halin graph and $\Delta(G') = \Delta(G)$. By the induction hypothesis, G' has a 5-ASEC g. We now based on g to construct a 5-ASEC f of G.

Subcase 3.1 If $g(v_1v_4) = g(wu)$, then let

$$\begin{aligned} f(v_1v_2) =& f(v_3v_4) = g(v_1v_4), f(v_2v_3) = g(v_1w), \\ f(v_2w) \in& C \setminus \{f(xv_1), g(v_1w), g(v_4w), g(wv_4)\}, \\ f(v_3w) \in& C \setminus \{g(v_1w), f(v_2w), g(v_4w), f(wu)\}. \end{aligned}$$

Obviously,

$$f(v_2v_3) = g(v_1w) \notin \{g(v_4w), g(v_4y)\}, f(v_2w) \notin \{g(xv_1), f(v_1v_2), g(v_1w)\}$$

and

$$f(v_2w) \notin \{f(v_2v_3), f(v_3w), f(v_3y)\},$$

hence $f[v_3] \neq f[v_4], f[v_1] \neq f[v_2]$ and $f[v_2] \neq f[v_3]$. The colors of others elements of $E(G)$ is same as G', obviously f is a 5-ASEC of G, so the conclusion is true.

Subcase 3.2 If $g(v_1v_4) \neq g(wu)$, then let

$$\begin{aligned} f(v_3v_4) =& f(v_1w) = g(v_1v_4), f(v_1v_2) = f(v_3v_4) = g(v_1w), f(v_2v_3) = g(wu), \\ f(v_2w) \in& C \setminus \{f(v_1v_2), f(v_1w), f(v_4w), g(wu)\}. \end{aligned}$$

The colors of others elements of $E(G)$ is same as G'. Since

$$\{f(v_2w), f(v_2v_3)\} \bigcap \{f(v_1w), f(v_1v_2)\} = \emptyset, \{f(v_2v_3), f(v_3w)\} \bigcap \{f(v_3v_4), f(v_4w)\} = \emptyset$$

and

$$f(v_2w) \neq f(v_3v_4),$$

hence $f[v_1] \neq f[v_2], f[v_3] \neq f[v_4]$ and $f[v_2] \neq f[v_3]$. Obviously f is a 5-ASEC of G, so the conclusion is true.

Combining above process, $G(V, E)$ have a 5-ASEC, by the induction hypothesis principle, the conclusion is true.

If $E(G[V_\Delta]) \neq \emptyset$, we prove $\chi'_{as}(G) = 6$.

Obviously, $\chi'_{as}(G) \geq 6$. $C = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ denotes the set of six colors. If $p = 2\Delta(G)$, by enumeration, the conclusion is true. Suppose the conclusion is true for $|V(G)| < p$.We now to prove the conclusion is true for $|V(G)| = p$. We quote W denotes the set of all vertices w (exactly adjacent to one interior vertex) satisfy the condition of (2) and (3) of Lemma 1.

If $d(w) = 3, 4$, the proof is same Case 1–2. If $d(w) = 5$ and $d(u) = 3, 4$, the proof is same as Case 3. If $d(w) = d(u) = 5$, the proof is samilar to the case of $d(w) = 5$ and $d(u) = 3, 4$, the proof is omited. Hence $\chi'_{as}(G) = 6$ iff $E(G[V_\Delta]) \neq \emptyset$, the conclusion is true.

By the prove of Theorem 1, following theorem is obviously true.

Theorem 2 *For Halin-graph G with $\Delta(G) \leq 4$, have $4 \leq \chi'_{as}(G) \leq 5$. And $\chi'_{as}(G) = 5$ if $E(G[V_\Delta]) \neq \emptyset$.*

For 3-regular Halin graph G, $\chi'_{as}(G) = 5$ if $G = K_3$, and $\chi'_{as}(G) = 4$ if $|V(G)| = 6$.

Theorem 3 *For Halin graph G of $\Delta(G) \geq 6$, have*

$$\Delta(G) \leq \chi'_{as}(G) \leq \Delta(G) + 1.$$

And

$$\chi'_{as}(G) = \Delta(G) + 1 \text{ iff } E(G[V_\Delta]) \neq \emptyset.$$

Proof The proof is similar to Theorem 1. Using induction on $|V(G)|$, the terminologys are same Theorem 1.

If $d(w) = 3, 4, 5$, the proof is same as the case 1–3 of Theorem 1 and the proof is more easily. If $d(w) \geq 6$, construct graph G' as Case 3 of Theorem 1, the proof is similar to Case 3 of Theorem 1, and the proof is omitted.

Open problem:

If H is a proper subgraph of G, in which case that $\chi'_{as}(H) \leq \chi'_{as}(G)$ is true?

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Halin- 图的邻强边染色

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摘 要: 图 $G(V,E)$ 的正常 k- 边染色 f 叫做图 $G(V,E)$ 的 k- 邻强边染色当且仅当任意 $uv \in E(G)$ 满足 $f[u] \neq f[v]$, 其中 $f[u] = \{f(uw)|uw \in E(G)\}$, 称 f 是 G 的 k- 临强边染色, 简记为 k-ASEC. 并且 $\chi'_{as}(G) = \min\{k|k\text{-ASEC of } G\}$ 叫做 $G(V,E)$ 的 邻强边色数. 本文研究了 $\Delta(G) \geq 5$ 的 Halin- 图的邻强边色数.

关键词: 领强边染色; 邻强边色数; Halin- 图.