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Hilbert's Projective Metric and the Norm on a Banach Space

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Abstract In this paper, we establish some relations between the Hilbert's projective metric and the norm on a Banach space and show that the metric and the norm induce equivalent convergences at certain set. As applications, we utilize the main results to discuss the eigenvalue problems for a class of positive homogeneous operators of degree α and the positive solutions for a class of nonlinear algebraic system.

Keywords Hilbert's projective metric; normal and solid cone; norm.

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1. Introduction

The Hilbert's projective metric is particularly useful in proving the existence of a unique fixed point for a positive nonlinear operator defined in Banach space. Elementary accounts of the general theory may be found in Krasnosel'skii, Vainikko, Zabreiko, Rutitskii, and Stetsenko [1] and in Bushell [2]. The properties of the metric and its use in some integral equations can be found in [3–6]. Based upon the Hilbert's projective metric, the authors [4] established several ergodic theorems for nonlinear operators in ordered Banach spaces and the authors [3, 6] proved existence and uniqueness of a solution to several classes of nonlinear integral equations by means of positive homogeneous operators of degree α. In particular, Bushell [7] applied the Hilbert's projective metric to prove that, if T is a real nonsingular $n \times n$ matrix, then there exists a unique real positive definite matrix A such that $T'AT = A^2$ and Koufany [8] formulated the metric on symmetric cones for using the Jordan algebra theory and extended Bushell's theorem to a class of convex cones. In this paper, we establish some relations between Hilbert's projective metric and the norm on Banach spaces. As simple applications, we discuss the eigenvalue problems for a class of positive homogeneous operators of degree α and the positive solutions for a class of nonlinear algebraic system. Therefore, we give the existence, uniqueness of fixed points to

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positive homogeneous operators of degree α and the existence, uniqueness of positive solutions to nonlinear algebraic system.

2. Main results

The following notations are taken from Nussbaum [9], Guo and Lakshmikantham [10]. Let E be a real Banach space and θ be the zero element of E. A closed convex set P in E is called a cone if the following conditions are satisfied:

(i) if $x \in P$, then $\lambda x \in P$ for $\lambda \geq 0$; (ii) if $x \in P$ and $-x \in P$, then $x = \theta$.

A cone P induces a partial ordering $\leq$ in E by

$$x \leq y \text{ if and only if } \quad y - x \in P.$$

A cone P is called normal if there exists a constant N such that

$$\theta \leq x \leq y \text{ implies that } \quad \|x\| \leq N\|y\|,$$

where $\|\cdot\|$ is the norm on E. A cone P is called solid if it contains interior points, i.e., $\mathring{P} \neq \emptyset$.

Lemma 2.1 ([9, 11]) *Let P be a cone in E. Then the following assertions are equivalent.*

(i) P is normal.

(ii) There exists an equivalent norm $\|\cdot\|_1$ on E such that $\theta \leq x \leq y$ implies $\|x\|_1 \leq \|y\|_1$, i.e., $\|\cdot\|_1$ is monotonic.

(iii) $x_n \leq z_n \leq y_n$ $(n = 1, 2, 3 \ldots)$ and $\|x_n - x\| \to 0$, $\|y_n - x\| \to 0$ imply $\|z_n - x\| \to 0$.

Let P be a solid cone in real Banach space E. For given $x, y \in \mathring{P}$, there exist sufficiently small positive number μ and sufficiently large positive number λ such that $x - \mu y \in P$ and $y - \frac{1}{\lambda}x \in P$, i.e., $\mu y \leq x \leq \lambda y$. Hence, we can define

$$m(x, y) = \sup\{\mu > 0 | \mu y \leq x\}, \quad M(x, y) = \inf\{\lambda > 0 | x \leq \lambda y\}.$$

As a result, we have

$$0 < m(x, y) \leq M(x, y) \text{ and } m(x, y)y \leq x \leq M(x, y)y.$$

The Hilbert's projective metric is then defined by

$$d(x, y) = \ln \frac{M(x, y)}{m(x, y)}.$$

Lemma 2.2 ([2, 10]) *$d(x, y)$ is a quasi-metric in $\mathring{P}$, i.e., $d(x, y)$ satisfies the following three conditions:*

(i) $d(x, x) = 0$, $\forall\, x \in \mathring{P}$;

(ii) $d(x, y) = d(y, x)$, $\forall x, y \in \mathring{P}$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in \mathring{P}$.

Moreover we have

(iv) $d(\lambda x, \mu y) = d(x, y)$, $\forall x, y \in \mathring{P}$, $\lambda > 0$, $\mu > 0$;

(v) $d(x, y) = 0$ if and only if $x = \lambda y$, where $\lambda > 0$.

From Lemma 2.2 we know that $(\mathring{P}\bigcap S_r, d)$ is a metric space, where $S_r = \{x \in E | \|x\| = r\}$, $\forall r > 0$. Moreover, we have the following.

Theorem 2.1 *Suppose that the norm on E is monotonic, that is, $\theta \le x \le y$ implies $\|x\| \le \|y\|$. Then $(\mathring{P}\bigcap S_r, d)$ is a complete metric space.*

Proof The completeness of $(\mathring{P}\bigcap S_r, d)$ in case $r = 1$ has been proved by Guo and Lakshmikantham [10]. To prove the general case, suppose that $\{x_n\}$ is a Cauchy sequence in $(\mathring{P}\bigcap S_r, d)$. From Lemma 2.2(iv), we know that $\{\frac{x_n}{r}\}$ is a Cauchy sequence in $(\mathring{P}\bigcap S_1, d)$. Therefore, there exists $z \in \mathring{P}\bigcap S_1$ such that $d(\frac{x_n}{r}, z) \to 0$ $(n \to \infty)$. It follows from Lemma 2.2 (iv) that $d(x_n, rz) \to 0$ as $n \to \infty$, so $\{x_n\}$ converges to rz in $(\mathring{P}\bigcap S_r, d)$. $(\mathring{P}\bigcap S_r, d)$ is a complete metric space. □

Theorem 2.2 *Suppose that P is normal and solid. Then $(\mathring{P}\bigcap S_r, d)$ is a complete metric space.*

Proof Since P is normal, from Lemma 2.1, we know that there exists a norm $\|\cdot\|_1$ on E which satisfies the following two conditions:

(A_1) $\|\cdot\|_1$ is equivalent to $\|\cdot\|$, i.e., there exist $\delta > \beta > 0$ such that $\beta\|x\| \le \|x\|_1 \le \delta\|x\|$ for any $x \in E$;

(A_2) Norm $\|\cdot\|_1$ is monotonic.

By Theorem 2.1, $(\mathring{P}\bigcap S_r^{(1)}, d)$ is a complete metric space, where $S_r^{(1)} = \{x \in E | \|x\|_1 = r\}$. Now we prove that $(\mathring{P}\bigcap S_r, d)$ is a complete metric space too. Let $\{x_n\} \in \mathring{P}\bigcap S_r$ and $d(x_n, x_m) \to 0 (n, m \to \infty)$. Since $\|x_n\| = r$, we have from (A_1) that $0 < \beta r \le \|x_n\|_1 \le \delta r$ $(n = 1, 2, \ldots)$. Setting $z_n = \frac{r x_n}{\|x_n\|_1}$, we see $z_n \in \mathring{P}\bigcap S_r^{(1)}$ and

$$d(z_n, z_m) = d(\frac{r x_n}{\|x_n\|_1}, \frac{r x_m}{\|x_m\|_1}) = d(x_n, x_m) \to 0, \quad n, m \to \infty.$$

Thus by the completeness of $(\mathring{P}\bigcap S_r^{(1)}, d)$, there exists $z^* \in \mathring{P}\bigcap S_r^{(1)}$ such that $d(z_n, z^*) \to 0$ $(n \to \infty)$. Since $\|z^*\|_1 = r$ and $\beta\|z^*\| \le \|z^*\|_1 \le \delta\|z^*\|$, we have

$$\frac{r}{\delta} \le \|z^*\| \le \frac{r}{\beta}.$$

Let $x^* = \frac{r z^*}{\|z^*\|}$. Then $x^* \in \mathring{P}\bigcap S_r$ and

$$d(x_n, x^*) = d(\frac{\|x_n\|_1}{r} z_n, \frac{r z^*}{\|z^*\|}) = d(z_n, z^*) \to 0, \quad n \to \infty.$$

Hence, $(\mathring{P}\bigcap S_r, d)$ is complete and our theorem is proved. □

Now let $e \in E$ and $e > \theta$. Set

$$E_e = \{x \in E | \text{ there exists } \lambda > 0 \text{ such that } -\lambda e \le x \le \lambda e\}$$

and

$$\|x\|_e = \inf\{\lambda > 0 | -\lambda e \le x \le \lambda e\}, \quad \forall\, x \in E_e.$$

It is easy to see that E_e becomes a normed linear space under the norm $\|\cdot\|_e$, and $\|x\|_e$ is called the e-norm of the element $x \in E_e$.

Lemma 2.3 ([12]) *Let cone P be normal. Then*

(i) E_e is a Banach space with e-norm, and there exists a constant $\omega > 0$ such that $\|x\| \leq \omega\|x\|_e$ for any $x \in E_e$;

(ii) $P_e = E_e \bigcap P$ is a normal solid cone of E_e;

(iii) if P is solid and $e \in \mathring{P}$, then $E_e = E$ and the e-norm $\|\cdot\|_e$ is equivalent to the original norm $\|\cdot\|$.

Theorem 2.3 *Let P be normal and solid and $\{x_n\} \in \mathring{P} \bigcap S_r$, $x \in \mathring{P} \bigcap S_r$. Then $d(x_n, x) \to 0$ $(n \to \infty)$ if and only if $\|x_n - x\| \to 0$ $(n \to \infty)$.*

Proof Suppose that $d(x_n, x) \to 0$ $(n \to \infty)$. Then

$$\frac{M(x_n, x)}{m(x_n, x)} \to 1, \quad n \to \infty. \tag{2.1}$$

We know

$$m(x_n, x)x \leq x_n \leq M(x_n, x)x.$$

That is

$$x \leq \frac{x_n}{m(x_n, x)} \leq \frac{M(x_n, x)}{m(x_n, x)}x. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\theta \leq \frac{x_n}{m(x_n, x)} - x \leq \frac{M(x_n, x)}{m(x_n, x)}x - x.$$

Since P is normal, we have

$$\|\frac{x_n}{m(x_n, x)} - x\| \leq N\|\frac{M(x_n, x)}{m(x_n, x)}x - x\| = N|\frac{M(x_n, x)}{m(x_n, x)} - 1| \cdot \|x\| \to 0, \quad n \to \infty, \tag{2.3}$$

where N is the normal constant of cone P. Thus

$$\frac{\|x_n\|}{m(x_n, x)} \to \|x\|, \quad n \to \infty. \tag{2.4}$$

Note that $\|x_n\| = \|x\| = r$, from (2.4), we have

$$m(x_n, x) \to 1, \quad n \to \infty. \tag{2.5}$$

Therefore, from (2.3) and (2.5), we can get

$$\begin{aligned} \|x_n - x\| &\leq \|x_n - \frac{x_n}{m(x_n, x)}\| + \|\frac{x_n}{m(x_n, x)} - x\| \\ &= |r - \frac{r}{m(x_n, x)}| + \|\frac{x_n}{m(x_n, x)} - x\| \to 0, \quad n \to \infty. \end{aligned}$$

Hence, we have proved that $d(x_n, x) \to 0$ $(n \to \infty)$ implies $\|x_n - x\| \to 0$ $(n \to \infty)$.

In the following we prove the converse conclusion. Suppose $\|x_n - x\| \to 0$ $(n \to \infty)$. Take $e \in \mathring{P}$, by Lemma 2.3, we know $E_e = E$ and the e-norm is equivalent to the original norm $\|\cdot\|$ and thus

$$\varepsilon_n = \|x_n - x\|_e \to 0, \quad n \to \infty, \tag{2.6}$$

and

$$-\varepsilon_n e \le x_n - x \le \varepsilon_n e. \tag{2.7}$$

Note that $x \in \mathring{P}$, we can choose a small positive number γ such that $x \ge \gamma e$. It follows from (2.7) that

$$(1-\frac{\varepsilon_n}{\gamma})x \le x - \varepsilon_n e \le x_n \le x + \varepsilon_n e \le (1+\frac{\varepsilon_n}{\gamma})x.$$

This shows that

$$1-\frac{\varepsilon_n}{\gamma} \le m(x_n,x) \le M(x_n,x) \le 1+\frac{\varepsilon_n}{\gamma}.$$

Therefore

$$d(x_n,x) = \ln\frac{M(x_n,x)}{m(x_n,x)} \le \ln\frac{1+\frac{\varepsilon_n}{\gamma}}{1-\frac{\varepsilon_n}{\gamma}} \to 0, \quad n\to\infty.$$

Hence, we have proved that $\|x_n - x\| \to 0$ $(n\to\infty)$ implies $d(x_n,x)\to 0$ $(n\to\infty)$. □

Corollary 2.4 ([10]) *Let P be normal and solid and $\{x_n\}\subset \mathring{P}$, $x\in\mathring{P}$. Then $\|x_n - x\|\to 0$ if and only if $d(x_n,x)\to 0$ with $\|x_n\|\to\|x\|$.*

Proof Suppose that $\|x_n - x\|\to 0$ $(n\to\infty)$. Then, $\|x_n\|\to\|x\|$ and

$$\|\frac{x_n}{\|x_n\|}-\frac{x}{\|x\|}\|\to 0, \quad n\to\infty.$$

Since $\frac{x_n}{\|x_n\|}, \frac{x}{\|x\|}\in\mathring{P}\bigcap S_1$, by Theorem 2.3 in the case $r=1$, we get $d(\frac{x_n}{\|x_n\|},\frac{x}{\|x\|})\to 0$. Note that $d(x_n,x) = d(\frac{x_n}{\|x_n\|},\frac{x}{\|x\|})$, thus $d(x_n,x)\to 0$ as $n\to\infty$.

Conversely, let $d(x_n,x)\to 0$ with $\|x_n\|\to\|x\|$. Then

$$d(\frac{x_n}{\|x_n\|},\frac{x}{\|x\|}) = d(x_n,x)\to 0.$$

By Theorem 2.3 in the case $r=1$, we have $\|\frac{x_n}{\|x_n\|}-\frac{x}{\|x\|}\|\to 0$. Moreover, we obtain

$$\|x_n - x\| = \|x_n\|\cdot\|\frac{x_n}{\|x_n\|}-\frac{x}{\|x_n\|}\| \le \|x_n\|(\|\frac{x_n}{\|x_n\|}-\frac{x}{\|x\|}\| + \|\frac{x}{\|x\|}-\frac{x}{\|x_n\|}\|).$$

It follows from $\|x_n\|\to\|x\|$ that $\|x_n - x\|\to 0$ $(n\to\infty)$. □

Remark 2.1 Theorem 2.3 and Corollary 2.4 show that the convergence in Hilbert's projective metric and the convergence in norm are equivalent on $\mathring{P}\bigcap S_r$ or $\mathring{P}$. Under some circumstances, Hilbert's projective metric has its own excellent privilege. For instance, let $E = C[0,1]$ and $P = \{f\in E | f(x)\ge 0,\ x\in[0,1]\}$. It is easy to see that P is solid, the norm on E is monotonic and $\mathring{P} = \{f\in E | f(x) > 0,\ x\in[0,1]\}$. For $\forall r>0$, set $S_r = \{f\in E|\ \|f\| = r\}$. Then by Theorem 2.1, $(\mathring{P}\bigcap S_r, d)$ is a complete metric space. However, for usual metric

$$d_1(x,y) = \max_{t\in[0,1]}|x(t)-y(t)|,$$

$(\mathring{P}\bigcap S_r, d_1)$ is not complete. In addition, even if $d_1(x_n,x)\not\to 0$, $d(x_n,x)\to 0$ is possible. For example, let $x_n(t) = 2r - \frac{2r}{n}t$, $x(t) = r$ $(r>0)$. We have

$$d_1(x_n,x) = \max_{t\in[0,1]}|x_n(t)-x(t)| = \max_{t\in[0,1]}|r-\frac{2r}{n}t|\not\to 0,$$

but for Hilbert's projective metric

$$d(x_n, x) = d(\frac{x_n}{2}, x) \to 0.$$

3. Applications

In this section, we discuss the eigenvalue problems for a class of positive homogeneous operators of degree α and give the existence, uniqueness of fixed points to positive homogeneous operators of degree α by using Theorem 2.3. A class of nonlinear algebraic system is also considered. We also assume that E is a real Banach space and $P \subset E$ is a solid cone. Let A be an operator from $\mathring{P}$ to $\mathring{P}$. Recall the following definition from [2].

Definition 3.1 *If $A(\lambda x) = \lambda^\alpha Ax$ for all $x \in \mathring{P}$, $\lambda > 0$, we say that A is positive homogeneous of degree α in $\mathring{P}$.*

Remark 3.1 Let $\alpha \in (0, 1)$ and P be normal, and let operator $A : \mathring{P} \to \mathring{P}$ be increasing, general positive homogeneous of degree α. Then operator $A : \mathring{P} \to \mathring{P}$ is continuous [10].

Lemma 3.1 ([13]) *Let (E, d) be a metric space and $f : E \to E$ be contractive (i.e., $x \neq y$ implies $d(f(x), f(y)) < d(x, y)$). Then each cluster point $\xi \in E$ of the sequence $\{f^n(x)\}$ is a unique fixed point of f and $f^n(x) \to \xi$.*

Now we can state and prove the following eigenvalue and fixed-point theorem by using Lemma 3.1 and Theorem 2.3.

Theorem 3.1 *Let $\alpha \in (0, 1)$ and P be normal, and let operator $A : \mathring{P} \to \mathring{P}$ be increasing and positive homogeneous of degree α. Suppose that: (Q) for some $x_0 \in \mathring{P}$, the sequence $\{A^n x_0\}_0^\infty$ (denote $A^0 x_0 = x_0$) has a limit point $\xi \in \mathring{P}$. Then*

(a) $\forall r > 0$, $\exists \xi_r \in \mathring{P}$, $\lambda_r > 0$ such that $A\xi_r = \lambda_r \xi_r$;

(b) A has a unique fixed point in $\mathring{P}$.

Proof Firstly, $\forall x, y \in \mathring{P}$, we have

$$\theta < m(x, y)y \leq x \leq M(x, y)y.$$

By Lemma 2.1, there exists an equivalent norm $\|\cdot\|_1$ of E, which satisfies the condition: $\|\cdot\|_1$ is monotonic. Thus, for $\|x\|_1 = \|y\|_1$, we can get

$$0 < m(x, y) \leq 1 \leq M(x, y).$$

Moreover, ξ is still the limit point of sequence $\{A^n x_0\}_0^\infty$ in norm $\|\cdot\|_1$.

Secondly, in view of $A(m(x, y)y) \leq Ax \leq A(M(x, y)y)$ and Definition 3.1, we have

$$(m(x, y))^\alpha Ay \leq Ax \leq (M(x, y))^\alpha Ay.$$

Hence

$$m(Ax, Ay) \geq (m(x, y))^\alpha,\ M(Ax, Ay) \leq (M(x, y))^\alpha.$$

Further

$$d(Ax, Ay) = \ln \frac{M(Ax, Ay)}{m(Ax, Ay)} \leq \ln \frac{(M(x, y))^\alpha}{(m(x, y))^\alpha} = \alpha d(x, y).$$

Thus for $\|x\|_1 = \|y\|_1$, we have $d(Ax, Ay) \leq \alpha d(x, y)$. Therefore, for $\|x\|_1 = \|y\|_1$ with $x \neq y$, we have $d(Ax, Ay) < d(x, y)$.

Thirdly, let $A_1 x = \frac{rAx}{\|Ax\|_1}$, $\forall r > 0$. Then $A_1 : \mathring{P} \bigcap S_r^{(1)} \to \mathring{P} \bigcap S_r^{(1)}$, where $S_r^{(1)} = \{x \in E \mid \|x\|_1 = r\}$. Moreover, A_1 satisfies the following conditions:

(1) $\forall x, y \in \mathring{P} \bigcap S_r^{(1)}$ with $x \neq y$,

$$d(A_1 x, A_1 y) = d(\frac{rAx}{\|Ax\|_1}, \frac{rAy}{\|Ay\|_1}) = d(Ax, Ay) < d(x, y).$$

That is, A_1 is contractive in $\mathring{P} \bigcap S_r^{(1)}$.

(2) From inductive method, it is easy to prove that $A_1^n x_0 = r\frac{A^n x_0}{\|A^n x_0\|_1}$, $n = 0, 1, 2, \ldots$.

(3) $\xi_r := \frac{r\xi}{\|\xi\|_1}$ is a limit point of $\{A_1^n x_0\}_{n=0}^\infty$ in Hilbert's projective metric d.

In fact, by (Q), there exists $\{n_k\} \subset \{n\}$ such that $A^{n_k} x_0 \to \xi$ in norm $\|\cdot\|$. So we have $A^{n_k} x_0 \to \xi$ in norm $\|\cdot\|_1$. Further, $\|A^{n_k} x_0\|_1 \to \|\xi\|_1$. Thus,

$$A_1^{n_k} x_0 = \frac{rA^{n_k} x_0}{\|A^{n_k} x_0\|_1} \to \frac{r\xi}{\|\xi\|_1} = \xi_r \in \mathring{P} \bigcap S_r^{(1)}$$

in norm $\|\cdot\|$ and then $A_1^{n_k} x_0 \to \xi_r$ in norm $\|\cdot\|_1$. By Theorem 2.3, $d(A_1^{n_k} x_0, \xi_r) \to 0$ as $k \to \infty$. Since $(\mathring{P} \bigcap S_r^{(1)}, d)$ is complete, it follows from Lemma 3.1 that ξ_r is the unique fixed point of A_1 in $\mathring{P} \bigcap S_r^{(1)}$. That is to say, $A_1 \xi_r = \xi_r = \frac{rA\xi_r}{\|A\xi_r\|_1}$. Let $\lambda_r = \frac{\|A\xi_r\|_1}{r}$. Then $\lambda_r > 0$ and $A\xi_r = \lambda_r \xi_r$. So conclusion (a) holds.

Finally, we prove that $x^* = \lambda_r^{\frac{1}{1-\alpha}} \xi_r$ is the unique fixed point of A in $\mathring{P}$. In fact,

$$Ax^* = A(\lambda_r^{\frac{1}{1-\alpha}} \xi_r) = \lambda_r^{\frac{\alpha}{1-\alpha}} A\xi_r = \lambda_r^{\frac{\alpha}{1-\alpha}} \lambda_r \xi_r = \lambda_r^{\frac{1}{1-\alpha}} \xi_r = x^*.$$

Suppose there exists $y^* \in \mathring{P}$ such that $Ay^* = y^*$. Let

$$x_1 = \frac{rx^*}{\|x^*\|_1}, \quad y_1 = \frac{ry^*}{\|y^*\|_1}.$$

Then $x_1, y_1 \in \mathring{P} \bigcap S_r^{(1)}$ and

$$Ax_1 = A(\frac{rx^*}{\|x^*\|_1}) = (\frac{r}{\|x^*\|_1})^\alpha Ax^*, \ Ay_1 = A(\frac{ry^*}{\|y^*\|_1}) = (\frac{r}{\|y^*\|_1})^\alpha Ay^*.$$

Hence,

$$\begin{aligned} d(x^*, y^*) = d(Ax^*, Ay^*) &= d((\frac{\|x^*\|_1}{r})^\alpha Ax_1, (\frac{\|y^*\|_1}{r})^\alpha Ay_1) \\ &= d(\frac{rAx_1}{\|Ax_1\|_1}, \frac{rAy_1}{\|Ay_1\|_1}). \end{aligned}$$

Thus, for $\frac{x^*}{\|x^*\|_1} \neq \frac{y^*}{\|y^*\|_1}$, i.e., $x^* \neq \lambda y^*$ $(\lambda > 0)$, we have from (1)

$$d(x^*, y^*) = d(\frac{rAx_1}{\|Ax_1\|_1}, \frac{rAy_1}{\|Ay_1\|_1}) < d(x_1, y_1) = d(\frac{rx^*}{\|x^*\|_1}, \frac{ry^*}{\|y^*\|_1}) = d(x^*, y^*).$$

This is a contradiction. So $x^* = \lambda y^*$ and

$$x^* = Ax^* = A(\lambda y^*) = \lambda^\alpha Ay^* = \lambda^\alpha y^* = \lambda y^*.$$

Then we obtain $\lambda = 1$ and hence $x^* = y^*$. Then conclusion (b) also holds. □

Remark 3.2 Let $E = C[0,1]$, $P = \{x \in E | x(t) \geq 0,\ t \in [0,1]\}$. Then P is a normal and solid cone, $\mathring{P} = \{x \in E | x(t) > 0,\ t \in [0,1]\}$. Consider a simple operator $Ax(t) = x^{\frac{1}{2}}(t)$, $x \in \mathring{P}$. So we have $A : \mathring{P} \to \mathring{P}$ is increasing and positive homogeneous of degree $\frac{1}{2}$. Take $x_0 = 2$, the sequence $\{A^n x_0\}_0^\infty = \{2^{\frac{1}{2^n}}\}_0^\infty$ has a limit point $1 \in \mathring{P}$. Hence, all the conditions of Theorem 3.1 are satisfied. Therefore, we have

(a) For any given $r > 0$, there exist $\xi_r \in \mathring{P}$, $\lambda_r > 0$ such that $A\xi_r = \lambda_r \xi_r$.

(b) A has a unique fixed point in $\mathring{P}$.

In fact, for any given $r > 0$, let $\xi_r = r$ and $\lambda_r = r^{-\frac{1}{2}}$. Then $A\xi_r = r^{\frac{1}{2}} = \lambda_r \xi_r$. Moreover, $x^* = \lambda_r^{\frac{1}{1-\alpha}} \xi_r = r^{-1} r = 1$ is the unique fixed point of A in $\mathring{P}$.

Next we consider the nonlinear algebraic system of the form

$$x^m = Tx^{m-1}, \tag{3.1}$$

where $m > 1$ and x denotes the column vector $\mathrm{col}(x_1, x_2, \ldots, x_n)$, $T = (t_{ij})_{n\times n}$ is an $n \times n$ matrix and all its entries are nonnegative numbers.

Let $E = R^n$, $P = \{\mathrm{col}(x_1, x_2, \ldots, x_n) | x_i \geq 0,\ i = 1, 2, \ldots, n\}$. Then P is a normal and solid cone in R^n, $\mathring{P} = \{\mathrm{col}(x_1, x_2, \ldots, x_n) | x_i > 0,\ i = 1, 2, \ldots, n\}$. For $x = \mathrm{col}(x_1, x_2 \ldots, x_n) \in P$ and $l > 0$, we let $x^l = \mathrm{col}({x_1}^l, {x_2}^l, \ldots, {x_n}^l)$. Note that if $0 \leq x \leq y$, then $\|x\| \leq \|y\|$ and $x^l \leq y^l$. A column vector $x = \mathrm{col}(x_1, x_2 \ldots, x_n) \in R^n$ is said to be a positive solution of (3.1) if $x_k > 0$ for $k \in \{1, 2, \ldots, n\}$ and substitution x into (3.1) renders it an identity.

Theorem 3.2 *Assume that (i) For all* $i \in \{1, 2, \ldots, n\}$, $\mathrm{col}(t_{i1}, t_{i2}, \ldots, t_{in}) \neq 0$ *(here 0 denotes zero vector); (ii) There exists* $x_0 \in \mathring{P}$ *such that the sequence* $\{T(u_k)^{\frac{m-1}{m}}\}_{k=0}^\infty$ *(denote* $u_0 = T(x_0)^{\frac{m-1}{m}}$*) has a limit point in* $\mathring{P}$.
Then (a) For any given $r > 0$, *there exist* $\xi_r \in \mathring{P}$, $\lambda_r > 0$ *such that* $\lambda_r {\xi_r}^m = T{\xi_r}^{m-1}$.

(b) There is a unique $x^* \in \mathring{P}$ *such that* ${x^*}^m = T{x^*}^{m-1}$.

Proof Define an operator $A : P \to E$ by $Ay = T(y)^{\frac{m-1}{m}}$. It follows from the definition of P and condition (i) that $A : \mathring{P} \to \mathring{P}$ is increasing. Further, we can obtain

(1) For $\lambda > 0$ and $y \in \mathring{P}$, $A(\lambda y) = \lambda^{\frac{m-1}{m}} T(y)^{\frac{m-1}{m}} = \lambda^{1-\frac{1}{m}} Ay$, i.e., A is positive homogeneous of degree $1 - \frac{1}{m}$;

(2) The sequence $\{A^k x_0\}_{k=0}^\infty = \{x_0, T(x_0)^{\frac{m-1}{m}}, T(u_k)^{\frac{m-1}{m}}\}_{k=0}^\infty$ has a limit point in $\mathring{P}$.

Thus, an application of Theorem 3.1 implies that (A) For any given $r > 0$, there exist $x_r \in \mathring{P}$, $\lambda_r > 0$ such that $Ax_r = \lambda_r x_r$; (B) There exists a unique $z \in \mathring{P}$ such that $Az = z$. Set $\xi_r = {x_r}^{\frac{1}{m}}$, $x^* = z^{\frac{1}{m}}$, then $\lambda_r {\xi_r}^m = A{\xi_r}^m = T{\xi_r}^{m-1}$, ${x^*}^m = A(x^*)^m = T(x^*)^{m-1}$. The proof is completed. □

Remark 3.3 Let $T = (t_{ij})_{n\times n}$, where $t_{ii} > 0$ for $i = 1, 2, \ldots, n$ and $t_{ij} = 0$ for $i \neq j$. Consider the following equation

$$x^2 = Tx. \tag{3.2}$$

Take $x_0 = \mathrm{col}(x_1, x_2, \ldots, x_n)$, $x_i > 0$ $(i = 1, 2, \ldots, n)$ and set $u_0 = T{x_0}^{\frac{1}{2}}$, then the sequence

$$\{u_0, T(u_k)^{\frac{1}{2}}\}_{k=0}^{\infty} = \{\mathrm{col}((t_{11})^{2-\frac{1}{2^{k-1}}}{x_1}^{\frac{1}{2^k}}, (t_{22})^{2-\frac{1}{2^{k-1}}}{x_2}^{\frac{1}{2^k}}, \ldots, (t_{nn})^{2-\frac{1}{2^{k-1}}}{x_n}^{\frac{1}{2^k}})\}_{k=1}^{\infty}$$

has a limit point $\mathrm{col}({t_{11}}^2, {t_{22}}^2, \ldots, {t_{nn}}^2) \in \mathring{P}$. By Theorem 3.2, the equation (3.2) has a unique positive solution x^* in $\mathring{P}$. It is easy to see that $x^* = \mathrm{col}(t_{11}, t_{22}, \ldots, t_{nn})$.

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