第25卷第1期 数 学 研 究 与 评 论 Vol.25, No.1
2005年2月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION Feb., 2005

Article ID: 1000-341X(2005)01-0031-06 **Document code**: A

Normal Families and Uniqueness Related to Shared Values

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Abstract: Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ. Let a be a non-zero finite value and k be a positive integer. If for every $f \in \mathcal{F}$,

(i) f and $f^{(k)}$ share a ;

(ii) the zeros of $f(z)$ are of multiplicity $\geq k+1$,

then $\mathcal{F}$ is normal on Δ.

We also proved corresponding results on normal functions and a uniqueness theorem of entire functions .

Key words: meromorphic function; normal family; shared values; uniqueness
MSC(2000): 30D35, 30D45
CLC number: O174.52

0. Introduction

Let D be a domain in $\mathbf{C}$. Define for f meromorphic on D and $a, b \in \mathbf{C}$,

$$\bar{E}_f(a) = f^{-1}(\{a\}) \cap D = \{z \in D : f(z) = a\}.$$

If $g(z) = b$ whenever $f(z) = a$, we write $f(z) = a \Rightarrow g(z) = b$.If $f(z) = a \Rightarrow g(z) = b$ and $g(z) = b \Rightarrow f(z) = a$, we write $f(z) = a \Leftrightarrow g(z) = b$.If $f(z) = a \Leftrightarrow g(z) = a$, then we say that f and g share a, i.e.

$$\bar{E}_f(a) = \bar{E}_g(a).$$

If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicities).

A meromorphic function f on $\mathbf{C}$ is called a normal function if there exists a positive number M such that

$$f^\sharp(z) \leq M.$$

Here, as usual, $f^\sharp(z) = \dfrac{|f'(z)|}{1 + |f(z)|^2}$ denotes the spherical derivative of $f(z)$.

Mues and Steinmetz[6] proved

Theorem A *Let f be a non-constant meromorphic function in D, and a_1, a_2 and a_3 be distinct complex numbers.If f and f' share a_1, a_2 and a_3, then $f \equiv f'$.*

Received date: 2002-12-31

W.Schwick seems to have been the first to draw a connection between normality criteria and shared values. He proved the following theorem[7]

Theorem B *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ and a_1, a_2 and a_3 be distinct complex numbers. If f and f' share a_1, a_2 and a_3 for every $f \in \mathcal{F}$, then $\mathcal{F}$ is normal on Δ.*

In this paper, we prove:

Theorem 1 *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ. Let a be a non-zero finite value and k be a positive integer, $k \geq 1$. If for every $f \in \mathcal{F}$,*

(i) f and $f^{(k)}$ share a ;

(ii) the zeros of $f(z)$ are of multiplicity $\geq k+1$,

then $\mathcal{F}$ is normal on Δ.

Remark 1 The following examples show that the assumption on the zeros of $f(z)$ in Theorem 1 is necessary.

For $k=1$, let $\mathcal{F} = \{f_n : f_n = \frac{e^{(n+1)z}-a}{n+1} + a\}$,where a is a non-zero complex number, then f and f' share a, but $\mathcal{F}$ is not normal in Δ.

For $k \geq 2$, let $\mathcal{F}=\{f_n(z) : f_n(z) = n(e^{c_1 z} - e^{c_2 z})\}$ $n = 1,2,3,\cdots$,where $c_1 \neq c_2$ and $c_j^k = 1, j = 1,2$. Obviously, for each $f \in \mathcal{F}$, $f = f^{(k)}$,f and $f^{(k)}$ share every numbers,but $\mathcal{F}$ is not normal in Δ.

From Theorem 1, we can get the result of Fang Mingliang[8]

Corollary 1.1 *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ. Let a be a non-zero finite value and k is a positive integer. If for every $f \in \mathcal{F}$,f and $f^{(k)}$ share a and $f \neq 0$, then $\mathcal{F}$ is normal on Δ.*

Theorem 2 *Let f be a meromorphic function on $\mathbf{C}$ and a be a non-zero finite value. If for every $f \in \mathcal{F}$,*

(i) f and $f^{(k)}$ share a ;

(ii) the zeros of $f(z)$ are of multiplicity $\geq k+1$,

then f is a normal function on $\mathbf{C}$.

Theorem 3 *Let f be a non-constant entire function and k be a positive integer. If f and $f^{(k)}$ share $a \neq 0$ CM, and the zeros of $f(z)$ are of multiplicity $\geq k+1$, then $f \equiv f^{(k)}$.*

1. Some lemmas

Lemma 1[3] *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ, of which all the zeros have multiplicity at least k. Suppose there exists $A \geq 1$, such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in \mathcal{F}$, then if $\mathcal{F}$ is not normal,there exist, for each $0 \leq \alpha \leq k$,*

(a) a number r, $0 < r < 1$;

(b) points z_n, $|z_n| < r$;

(c) functions $f_n \in \mathcal{F}$;

(d) positive numbers $\varrho_n \to 0$,

such that

$$\frac{f_n(z_n+\varrho_n\xi)}{\varrho_n^{\alpha}} \to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a meromorphic function on $\mathbf{C}$ such that $g^{\sharp}(\xi) \le g^{\sharp}(0) = kA+1$.

Lemma 2[4] *Let $R(z)$ be a non-constant rational function, let k be a positive integer and let b be a non-zero complex number.If the zeros of $R(z)$ are of multiplicity at least $k+1$, and $R^{(k)}(z) \ne b$, then*

$$R(z) = \frac{(\gamma z+\delta)^{k+1}}{\alpha z+\beta},$$

where $\alpha, \beta, \gamma, \delta$ are constants such that $\alpha\gamma \ne 0$, $|\beta|+|\delta| \ne 0$.

Lemma 3[1] *Let $f(z)$ be a transcendental meromorphic function of finite order, and k be a positive integer , let a be a non-zero finite complex number. If all zeros of $f(z)$ are of multiplicity at least $k+1$, then $f^{(k)}(z)$ assumes a infinitely often.*

Lemma 4[2] *A normal meromorphic function has order at most 2, a normal holomorphic function has order at most 1.*

Lemma 5[5] *Let $f(z)$ be a noncontant entire function of finite order, let $a \ne 0$ be a finite constant, and let k be a positive integer. If $f(z)$ and $f^{(k)}(z)$ share the value a CM, then we have $\dfrac{f^{(k)}-a}{f-a} \equiv c$, for some non-zero constant c.*

2. Proofs of the theorems

Proof of Theorem 1 Suppose that $\mathcal{F}$ is not normal on Δ, then by lemma 1 , we have $f_n \in \mathcal{F}$, $z_n \in \Delta$ and $\varrho_n \to 0^+$ such that

$$\frac{f_n(z_n+\varrho_n\xi)}{\varrho_n^{k}} \to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a non-constant meromorphic function satisfying $g^{\sharp}(\xi) \le g^{\sharp}(0) = k(|a|+1)+1$. By Lemma 4, we have that g is a meromorphic function of finite order.

We claim that:

(i) the zeros of g have multiplicity $\ge k+1$;

(ii) $g^{(k)}(\xi) \ne a$;

(iii) the poles of $g(\xi)$ are multiple.

Indeed, suppose $g(\xi_0)=0$, then since g is not a constant, there exists $\xi_n, \xi_n \to \xi_0$, such that

$$g_n(\xi_n) = \frac{f_n(z_n+\varrho_n\xi_n)}{\varrho_n^{k}} = 0 \quad (n \text{ large enough}).$$

Thus

$$f_n(z_n+\varrho_n\xi_n)=0.$$

Since the zeros of f have multiplicity $\geq k+1$, we have

$$f_n^{(j)}(z_n+\varrho_n\xi_n)=0, \quad j=0,1\cdots,k.$$

$$g_n^{(j)}(\xi_n)=\frac{f_n^{(j)}(z_n+\varrho_n\xi_n)}{\varrho_n^{k-j}}=0, \quad j=0,1,\cdots,k.$$

Since

$$g^{(j)}(\xi_0)=\lim_{n\to\infty}g_n^{(j)}(\xi_n), \quad j=0,1,\cdots,k,$$

it follows that the zeros of $g(\xi)$ have multiplicity at least $k+1$, so (i) is proved.

Now, suppose there exists ξ_0 such that $g^{(k)}(\xi_0)=a$, then we have $g^{(k)}(\xi)\not\equiv a$, since otherwise g is a polynomial whose degree is exactly k, which contradicts (i).

Since $g^{(k)}(\xi)=a$, but $g^{(k)}(\xi)\not\equiv a$, there exists $\xi_n, \xi_n\to\xi_0$ such that , for n sufficiently large,

$$g_n^{(k)}(\xi_n)=f_n^{(k)}(z_n+\varrho_n\xi_n)=a.$$

It now follows from $\bar{E}_{f_n}(a)=\bar{E}_{f_n^{(k)}}(a)$ that

$$f_n(z_n+\varrho_n\xi_n)=a.$$

Thus

$$g_n(\xi_n)=\frac{f_n(z_n+\varrho_n\xi_n)}{\varrho_n^k}=\frac{a}{\varrho_n^k}.$$

Letting $n\to\infty$, we have $g(\xi_0)=\infty$, which contradicts $g^{(k)}(\xi_0)=a$, thus $g^{(k)}(\xi)\neq a$.

Now we prove (iii).

Let ξ_0 be a pole of $g(\xi)$. If ξ_0 is not a multiple pole of $g(\xi)$, then there exists a disk $K=\{\xi:|\xi-\xi_0|<\delta\}$ (for n sufficiently large) on which there exists and only exists $\xi_n, \xi_n\to\xi_0$ and ξ_n is a simple pole of $g_n(\xi)$. Hence we can deduce that ξ_n is a pole of $g_n^{(k)}(\xi)$ with multiplicity $k+1$ and ξ_0 is a pole of $g^{(k)}(\xi)$ with multiplicity $k+1$. And

$$g_n(\xi)\to g(\xi),$$

$$g_n^{(k)}(\xi)\to g^{(k)}(\xi)$$

locally uniformly on K.

Thus

$$\lim_{\xi_n\to\xi_0}g_n^{(k)}(\xi_n)=\infty. \tag{1}$$

Since

$$\frac{1}{g_n^{(k)}(\xi)}-\frac{\varrho_n^k}{a}\to\frac{1}{g^{(k)}(\xi)}$$

uniformly on K and $\frac{1}{g^{(k)}(\xi)}$ is not a constant, there exist $\xi_n, \xi_n \to \xi_0$, such that (for n large enough)

$$\frac{1}{g_n^{(k)}(\xi_n)} - \frac{\varrho_n^k}{a} = 0,$$

i.e.

$$f_n(z_n + \varrho_n \xi_n) = a.$$

Thus

$$g_n^{(k)}(\xi_n) = f_n^{(k)}(z_n + \varrho_n \xi_n) = a.$$

Letting $n \to \infty$, we get a contradiction by (1).

Thus the poles of $g(\xi)$ are multiple.

Since g has finite order , the zeros of g have multiplicity $\geq k+1$ and $g^{(k)}(\xi) \neq a$. Hence by Lemma 3 we can deduce that $g(\xi)$ is a non-constant rational function. Then by Lemma 2 f has a simple pole, which contradicts (iii).

So the proof of Theorem 1 is completed.

Proof of Theorem 2 Suppose f is not a normal function. Then there exist $z_n \to \infty$ such that $\lim_{n\to\infty} f^\sharp(z_n) = \infty$. Write $f_n(z) = f(z + z_n)$ and set $\mathcal{F} = \{f_n\}$. Then by Marty's criterion, $\mathcal{F}$ is not normal on the unit disc. On the other hand, since $\bar{E}_{f_n}(a) = \bar{E}_{f_n^{(k)}}(a)$, the zeros of $f_n(z)$ are of multiplicity $\geq k+1$. Theorem 1 implies that $\mathcal{F}$ is normal. The contradiction proves the theorem.

Proof of Theorem 3 From Theorem 2 and Lemma 4 we deduce that $f(z)$ has finite order. By Lemma 5 we have

$$\frac{f^{(k)} - a}{f - a} \equiv B, \quad B \neq 0. \tag{2}$$

If $N(r, \frac{1}{f}) \not\equiv 0$,clearly from above we have $f \equiv f^{(k)}$.

If $N(r, \frac{1}{f}) \equiv 0$, then

$$f = e^{\alpha}, \tag{3}$$

where α is a nonconstant entire function.

From (2) we can obtain

$$f(z) = c_1 e^{n_1 z} + \cdots + c_k e^{n_k z} + a(1 - \frac{1}{B}), \tag{4}$$

where $n_j^k = B, j = 1, 2 \cdots, k$.

From (3) and (4) we have $f(z) = Ae^{nz}$, where A and n are nonzero constant.Since f and $f^{(k)}$ share a CM ,we have $n^k = 1$, so $f \equiv f^{(k)}$.Theorem 3 is proved.

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关于分担值的正规族和唯一性定理

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摘要: 设 $\mathcal{F}$ 是单位圆盘 Δ 上的亚纯函数族， a 是一个非零的有穷复数， k 是正整数，如果 $\forall f \in \mathcal{F}$, 满足

1) f 的零点重级 $\geq k+1$;

2) f 和 $f^{(k)}$ IM 分担 a,

则 $\mathcal{F}$ 在 Δ 上正规.

此外，还证明了相应于正规函数以及整函数的唯一性定理方面的的结果.

关键词: 亚纯函数; 正规族; 分担值; 唯一性.