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A Note on General Frames for Bivariate Interpolation

TANG Shuo, *ZOU Le*
(Department of Mathematics, Hefei University of Technology, Anhui 230009, China)
(E-mail: ts0610@sina.com)

Abstract Newton interpolation and Thiele-type continued fractions interpolation may be the favoured linear interpolation and nonlinear interpolation, but these two interpolations could not solve all the interpolant problems. In this paper, several general frames are established by introducing multiple parameters and they are extensions and improvements of those for the general frames studied by Tan and Fang. Numerical examples are given to show the effectiveness of the results in this paper.

Keywords continued fractions; blending rational interpolant; unattainable point.

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1. Introduction

In 1999, Tan and Fang studied the general frames for bivariate interpolation in [1], including the classical Newton polynomial interpolation and Thiele-type continued fractions interpolation. But these two interpolants are powerless when inverse differences are nonexistent or meeting unattainable points interpolant problems. It is quite natural to ask one question, namely, whether we can construct new general frames, including the general frames studied by Tan and Fang. This paper gives a positive answer to the question.

As in [1], the general frame for univariate interpolation was given as follows:

Given a set of real points $X_n = \{x_0, x_1, \ldots, x_n\} \subset [a, b] \subset R$ and a function $f(x)$ defined in $[a, b]$. Let

$$\begin{cases} S_n(x;\eta) = u_n(\eta), \\ S_k(x;\eta) = u_k(\eta) + (x - x_k)\left[S_{k+1}(x;\eta)\right]^{\eta}, \ k = n-1, n-2, \ldots, 1, 0, \\ S(x;\eta) = S_0(x;\eta), \end{cases} \tag{1}$$

where η takes the value 1 or -1, $u_i(\eta) = F^{(\eta)}[x_0, x_1, \ldots, x_i]$, which is computed by the following steps:

$$F^{(\eta)}[x_p] = f(x_p), \quad p = 0, 1, \ldots, n,$$

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$$F^{(\eta)}\left[x_0, x_1, \ldots, x_i\right] = \left(\frac{F^{(\eta)}\left[x_0, x_1, \ldots, x_{i-2}, x_i\right] - F^{(\eta)}\left[x_0, x_1, \ldots, x_{i-2}, x_{i-1}\right]}{x_i - x_{i-1}}\right)^{\eta}. \tag{2}$$

Clearly, $S(x;1)$ is the Newton polynomial interpolation and $S(x;-1)$ is Thiele-type continued fractions interpolation.

2. General frames for univariate interpolation

Tan and Zhao considered block-based Newton-like blending interpolation and Thiele-like blending rational interpolation[2,3]. Now we construct a new frame, which includes these two blending rational interpolations and may have other block-based blending rational interpolations via choosing parameters appropriately.

Given a set of real point $X_n = \{x_0, x_1, \ldots, x_n\} \subset [a,b] \subset R$ and a function $f(x)$ defined in $[a,b]$.

Let us consider the blending rational interpolant on sets as follows:

Suppose the set X_n is divided into $u+1$ subsets X_n^s $(s = 0, 1, \ldots, u.)$

$$X_n^s = \{x_{c_s}, x_{c_s+1}, \ldots, x_{d_s}\}, \quad s = 0, 1, \ldots, u.$$

The subsets may be achieved by reordering the interpolation points if necessary. Obviously, one gets

$$\sum_{s=0}^{u}(d_s - c_s + 1) = n + 1.$$

Let us construct the following recursive schemes:

$$\begin{cases} S_u(x;\eta_u) = I_u(x), \\ S_s(x;\eta_s) = I_s(x) + \omega_s(x)\left[S_{s+1}(x;\eta_{s+1})\right]^{\eta_{s+1}}, \ s = u-1, u-2, \ldots, 1, 0, \\ S(x;\eta) = S_0(x;\eta_0), \end{cases} \tag{3}$$

where $|\eta_i| = 1$, and

$$\omega_s(x) = \prod_{i=c_s}^{d_s}(x - x_i),$$

and $I_s(x)$ $(s = 0, 1, \ldots, u)$ are interpolating polynomials or rational interpolating functions on the subsets $\{x_{c_s}, x_{c_s+1}, \ldots, x_{d_s}\}$ $(s = 0, 1, \ldots, u)$ and

$$I_s(x_i) = f_i^{s(\eta_s)}, \quad i = c_s, c_s+1, \ldots, d_s; s = 0, 1, \ldots, u, \tag{4}$$

and for $s = 1, 2, \ldots, u$,

$$f_i^{s(\eta_s)} = \left(\frac{f_i^{s-1(\eta_{s-1})} - I_{s-1}(x_i)}{\omega_{s-1}(x_i)}\right)^{\eta_s}, \quad i = c_s, c_s+1, \ldots, n. \tag{5}$$

Theorem 1 *Given $f(x_p)$ at each point $x_p \in X_n$ and (3)–(5) hold and*

$$f_p^{0(\eta_0)}[x_p] = f(x_p), \quad p = 0, 1, \ldots, n,$$

then $S(x;\eta)$ defined in (3) interpolates $f(x_p)$ at point $x_p \in X_n$.

Proof Suppose $c_s \leqslant p \leqslant d_s$, we have

$$S_{s-1}(x_p;\eta_{s-1}) = I_{s-1}(x_p) + \omega_{s-1}(x_p)\left[S_s(x_p;\eta_s)\right]^{\eta_s} = f_p^{s-1(\eta_{s-1})},$$

$$\cdots$$

$$S_1(x_p;\eta_1) = f_p^{1(\eta_1)},$$

$$S_0(x_p;\eta_0) = f_p^{0(\eta_0)} = S(x_p;\eta) = f(x_p).$$

Theorem 1 is proved. □

Obviously, if we let all the values of η_i take 1, then $S(x;\eta)$ is block-based Newton-like blending interpolation[2]; if we let all the values of η_i take -1, then $S(x;\eta)$ is block-based Thiele-like blending rational interpolation[3]. If we set $u=n$, then $S(x;\eta)$ is point based general frame, but it does not amount to the general frames given in [1], and it can be used to deal with the interpolation problems when there occur unattainable points or inverse differences are nonexistent.

3. General frames for bivariate interpolation

Now we will generalize the frames to multivariate cases. Given a set of two-dimensional points in R^2 which satisfies the including property, i.e., if a point belongs to $\prod_{m,n}$, then the rectangular subset of points emanating from the origin with the given point as its furthermost corner, also lies in $\prod_{m,n}$.

Suppose that $f(x,y)$ is defined on $D \supset \prod_{m,n}$. We divide $\prod_{m,n}$ into $(u+1)\times(v+1)$ subsets and it can be written in the following form:

$$\begin{aligned}\prod{}_{m,n} &= \left\{\pi_{m,n}^{s,t} \,|\, t=0,1,\ldots,v_s; s=0,1,\ldots,u\right\} \\ &= \left\{\pi_{m,n}^{s,t} \,|\, s=0,1,\ldots,u_t; t=0,1,\ldots,v\right\},\end{aligned} \tag{6}$$

where

$$\begin{aligned}\pi_{m,n}^{s,t} = \{(x_i,y_j)|c_s \leqslant i \leqslant d_s, h_t \leqslant j \leqslant r_t\}\ (s&=0,1,\ldots,u_t; t=0,1,\ldots,v \\ &\text{or } t=0,1,\ldots,v_s; s=0,1,\ldots,u)\end{aligned} \tag{7}$$

and

$$u_0=u, v_0=v, u_0 \geqslant u_1 \geqslant \cdots \geqslant u_v; v_0 \geqslant v_1 \geqslant \cdots \geqslant v_u.$$

For $s=0,1,\ldots,u$, let

$$\begin{cases} S_{s,v_s}(y;\delta_s,\eta_{s,v_s}) = I_{s,v_s}(\delta_s;\eta_{s,v_s}), \\ S_{s,t}(y;\delta_s,\eta_{s,t}) = I_{s,t}(x,y;\delta_s,\eta_{s,t}) + \varpi_t(y)\left[S_{s,t+1}(y;\delta_s,\eta_{s,t+1})\right]^{\eta_{s,t+1}}, \\ \qquad t = v_s-1, v_s-2,\ldots,1,0, \\ S_s(y;\delta_s,\eta_{s,0}) = S_{s,0}(y;\delta_s,\eta_{s,0}), \\ p_u(x,y;\delta_u,\eta_{u,0}) = S_u(y;\delta_u,\eta_{u,0}), \\ p_s(x,y;\delta_s,\eta_{s,0}) = S_s(y;\delta_s,\eta_{s,0}) + \omega_s(x)\left[p_{s+1}(x,y;\delta_{s+1},\eta_{s+1,0})\right]^{\delta_{s+1}}, \\ \qquad s = u-1,u-2,\ldots,1,0, \end{cases} \tag{8}$$

where $|\eta_{s,t}| = |\delta_s| = 1$, $\omega_s(x) = \Pi_{i=c_s}^{d_s}(x-x_i)$, $\varpi_t(y) = \Pi_{j=h_t}^{r_t}(y-y_j)$. Then the following theorem holds

Theorem 2 *Given $f_{i,j}$ at each point $(x_i, y_j) \in \prod_{m,n}$. Let*

$$f^{0,0(\delta_0,\eta_{0,0})}(x_i, y_j) = f_{i,j}. \tag{9}$$

$I_{0,t}(x, y)$ $(t = 0, 1, \ldots, v)$ are bivariate polynomial or rational interpolations on the subsets $\pi^{0,t}_{m,n}$, namely,

$$I_{0,t}(x_i, y_j) = f^{0,t(\delta_0,\eta_{0,t})}_{i,j},\ c_0 \leqslant i \leqslant d_0; h_t \leqslant j \leqslant r_t, t = 0, 1, \ldots, v. \tag{10}$$

For $t = 1, 2, \ldots, v$,

$$f^{0,t(\delta_0,\eta_{0,t})}_{i,j} = \left(\frac{f^{0,t-1(\delta_0,\eta_{0,t-1})}_{i,j} - I_{0,t-1}(x_i, y_j)}{\varpi_{t-1}(y_j)}\right)^{\eta_{0,t}}, \tag{11}$$

$$i = 0, 1, \ldots, m;\ j = h_t, h_t + 1, \ldots, n.$$

For $s = 1, 2, \ldots, u$,

$$f^{s,0(\delta_s,\eta_{s,0})}_{i,j} = \left(\frac{f^{s-1,0(\delta_{s-1},\eta_{s-1,0})}_{i,j} - S_{s-1,0}(y_j; \delta_{s-1}, \eta_{s-1,0})}{\omega_{s-1}(x_i)}\right)^{\delta_s}, \tag{12}$$

$$i = c_s,\ c_s + 1, \ldots, m;\ j = 0, 1, \ldots, n.$$

For $s = 1, 2, \ldots, u$ and $t = 1, 2, \ldots, v$,

$$f^{s,t(\delta_s,\eta_{s,t})}_{i,j} = \left(\frac{f^{s,t-1(\delta_s,\eta_{s,t-1})}_{i,j} - I_{s,t-1}(x_i, y_j)}{\varpi_{t-1}(y_j)}\right)^{\eta_{s,t}}, \tag{13}$$

$$i = c_s, c_s + 1, \ldots, m;\ j = h_t, h_t + 1, \ldots, n,$$

where $I_{s,t}(x, y)$ $(s = 1, 2, \ldots, u; t = 0, 1, \ldots, v)$ are bivariate polynomials or rational interpolations on the subsets $\pi^{s,t}_{m,n}$, namely,

$$I_{s,t}(x_i, y_j; \delta_i, \eta_{i,j}) = I_{s,t}(\delta_i, \eta_{i,j}) = I_{s,t}(x_i, y_j) = f^{s,t(\delta_s,\eta_{s,t})}_{i,j} \tag{14}$$

$$c_s \leqslant i \leqslant d_s, h_t \leqslant j \leqslant r_t, s = 0, 1, \ldots, u; t = 0, 1, \ldots, v.$$

If all the above interpolants $I_{s,t}(x, y)$ $(s = 1, 2, \ldots, u; t = 0, 1, \ldots, v)$ satisfying (10) and (14) exist, then $p_0(x, y; \delta_0, \eta_{0,0})$ defined in (8) interpolates $f_{i,j}$ at point $(x_i, y_j) \in \prod_{m,n}$.

Proof Suppose $c_s \leqslant i \leqslant d_s$ and $h_t \leqslant j \leqslant r_t$, from (8) to (14), we have

$$\begin{aligned}
S_{s,t-1}(y_j; \delta_s, \eta_{s,t-1}) &= I_{s,t-1}(x_i, y_j; \delta_s, \eta_{s,t-1}) + \varpi_{t-1}(y_j)\left[S_{s,t}(y_j; \delta_s, \eta_{s,t-1})\right]^{\eta_{s,t}} \\
&= f^{s,t-1(\delta_s,\eta_{s,t-1})}_{i,j}, \\
S_{s,t-2}(y_j; \delta_s, \eta_{s,t-2}) &= f^{s,t-2(\delta_s,\eta_{s,t-2})}_{i,j}, \\
&\cdots \\
S_{s,1}(y_j; \delta_s, \eta_{s,1}) &= f^{s,1(\delta_s,\eta_{s,1})}_{i,j}, \\
S_{s,0}(y_j; \delta_s, \eta_{s,0}) &= f^{s,0(\delta_s,\eta_{s,0})}_{i,j}
\end{aligned}$$

and

$$p_s(x_i, y_j; \delta_s, \eta_{s,0}) = S_{s,0}(y_j; \delta_s, \eta_{s,0}) = f^{s,0(\delta_s,\eta_{s,0})}_{i,j},$$

$$p_{s-1}(x_i, y_j; \delta_{s-1}, \eta_{s-1,0}) = f_{i,j}^{s-1,0(\delta_{s-1},\eta_{s-1,0})},$$
$$\cdots$$
$$p_0(x_i, y_j; \delta_0, \eta_{0,0}) = f_{i,j}^{0,0(\delta_0,\eta_{0,0})} = f_{i,j}.$$

The proof is completed. □

In the general frames as shown above, $p_0(x, y; 1, 1)$ is block-based Newton-like bivariate blending interpolation[2]. As we know, the classical Newton polynomial interpolation and Thiele-type branched continued fractions interpolation are special cases of the block-based bivariate Newton-like blending rational interpolation. $p_0(x, y; -1, -1)$ is block-based Thiele-like bivariate blending rational interpolation[3]. It also includes many well-known interpolant formulae. Just like point based interpolation[6], block-based interpolation also has Newton-Thiele-like and Thiele-Newton-like blending rational interpolations[6]. One can find easily, if all the values of δ_s $(s = 1, 2, \ldots, u)$ take 1 (or -1) and all the values of $\eta_{s,t}$ $(s = 1, 2, \ldots, u; t = 1, 2, \ldots, v)$ take -1 (or 1), then $p_0(x, y; \delta_0, \eta_{0,0})$ are these two schemes. If we let u take m and v take n, then $p_0(x, y; \delta_0, \eta_{0,0})$ is point-based general frames for bivariate interpolation, but it is not simple extension of the results in [1]. It could be used to deal with the interpolation problems where inverse differences are nonexistent or unattainable points occur. So the conclusions in this paper are extension and improvement of the results in [1].

Remark 1 We can also construct the general frames of block-based blending rational interpolation for higher dimensions[5] and the dual interpolation schemes of the Theorem 2 as showed in this paper.

Remark 2 We can generalize the results of this paper to vector-valued or matrix-valued case[9,10].

Remark 3 If conditions are given for the osculatory interpolation, the results in this paper may also include many well-known osculatory interpolant schemes by modifying the conditions of the results appropriately.

4. Numerical examples

In this section, we take simple examples to show the effectiveness of the result in this paper. Example 1 has only one unattainable point. With similar procedure, one can deal with more unattainable points and multivariate interpolant problems involving one or more unattainable points, but the conclusions in [1] could not deal with the problem. Example 2 is given to solve the interpolation problem where inverse differences are nonexistent.

Example 1 Suppose the interpolating points and the prescribed values of $f(x)$ at the support abscissa x_i are given as follows:

$$(x_0, f_0) = (-2, -\frac{3}{5}),\ (x_1, f_1) = (-1, -\frac{1}{3}),\ (x_2, f_2) = (0, 0),$$
$$(x_3, f_3) = (1, 3),\ (x_4, f_4) = (2, \frac{7}{3}),\ (x_5, f_5) = (3, \frac{13}{5}).$$

We can find (0,0) is unique unattainable point by methods in [8]. We could construct the interpolation formulae by the conclusions in this paper, but we could not construct them with results in [1]. Now we construct three different interpolant formulae as follows

Method 1 Change the order of the elements in the data set and let (0,0) be located at the end of the data set. We can get

$$(x_0, f_0) = (-2, -\frac{3}{5}),\ (x_1, f_1) = (-1, -\frac{1}{3}),\ (x_2, f_2) = (1, 3),$$
$$(x_3, f_3) = (2, \frac{7}{3}),\ (x_4, f_4) = (3, \frac{13}{5}),\ (x_5, f_5) = (0, 0).$$

Using the frame (3), now $u = 5$, we take $\eta_1 = \eta_2 = \eta_3 = -1$, $\eta_4 = \eta_5 = 1$ and can get the interpolant function:

$$T_1(x) = -\frac{3}{5} + \frac{x+2}{\lceil \frac{15}{4}} + \frac{x+1}{\lceil -\frac{24}{35}} + \frac{x-1}{\lceil -\frac{7}{4} + \frac{7}{18}(x-2)(x-3)}$$

and it is easy to verify

$$T_1(x_i) = f_i, \quad i = 0, 1, \ldots, 5.$$

Method 2 We do not reorder the nodes. Using the frame (3), now $u = 5$, we take $\eta_1 = \eta_2 = \eta_3 = -1$, $\eta_4 = \eta_5 = 1$ and can get the following interpolant function

$$T_2(x) = -\frac{3}{5} + \frac{x+2}{\lceil \frac{15}{4}} + \frac{x+1}{\lceil -\frac{12}{5}} + \frac{x}{\lceil \frac{7}{12} + \frac{7}{6}(x-1) + \frac{7}{6}(x-1)(x-2)}$$

and it is easy to verify

$$T_2(x_i) = f_i, \quad i = 0, 1, \ldots, 5.$$

Method 3 We divide the set into two subsets $X_1 = \{-2, -1, 2, 3\}, X_2 = \{0, 1\}$ using the frame (3), we take $\eta_1 = \eta_2 = 1$ and can get the following interpolant function:

$$T_3(x) = -\frac{3}{5} + \frac{x+2}{\lceil \frac{15}{4}} + \frac{x+1}{\lceil -\frac{44}{35}} + \frac{x-2}{\lceil -\frac{7}{4}} + \frac{1}{12}(1-x)(x+2)(x+1)(x-2)(x-3)$$

and it is easy to verify

$$T_3(x_i) = f_i, \quad i = 0, 1, \ldots, 5.$$

Example 2 Let (x_i, y_j) and $f(x_i, y_j)$ be given in the table.

	$x_0 = 0$	$x_1 = 1$	$x_2 = 2$
$y_0 = 0$	1	3	4
$y_1 = 1$	0	1	1
$y_2 = 2$	-2	-3	-1

Table 1 Interpolation datas

Newton-Thiele blending rational interpolation fails in this case, since calculating inverse differences leads to that one of denominator is zero. Instead of the frames in [1], we use the frame

(8). Take $u=v=2$, $\delta_0=\delta_1=\delta_2=1$, $\eta_{0,0}=1, \eta_{0,1}=\eta_{0,2}=-1$, $\eta_{1,0}=1, \eta_{1,1}=\eta_{1,2}=-1$, $\eta_{2,0}=\eta_{2,1}=\eta_{2,2}=1$, we can get

$$S_1 = 1 + \frac{y|}{|-1+\frac{y-1}{3}},$$

$$S_2 = 2 + \frac{y|}{|-1+\frac{y-1}{3}},$$

$$S_3 = \frac{-1}{2} + y(y-1)$$

and we can get the following interpolant function:

$$\begin{aligned} R_{2,2}(x,y) &= S_1 + xS_2 + x(x-1)S_3 \\ &= \frac{8y-8+3xy-20x+7x^2y+4x^2+2x^2y^3-10x^2y^2-2xy^3+10xy^2}{2y-8}. \end{aligned}$$

It is easy to verify

$$R_{2,2}(x_i,y_j) = f(x_i,y_j), \quad i=0,1,2;\ j=0,1,2.$$

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