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Quasidirect Decompositions of Full Rank Hankel Matrices

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Abstract: This paper studies the decomposition of a full rank (rectangular) Hankel matrix into the sum of Hankel matrices whose ranks add up to that of the original matrix.

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1. Introduction

Hankel matrices are an important family of matrices and they have various applications; see, e.g., [1–6]. The quasidirect decompositions of square Hankel matrices and singular (not necessarily square) Hankel matrices are investigated by Fielder[1] and Vavřín[2], respectively. In this paper, we study the quasidirect decomposition of full rank rectangular Hankel matrices, show the existence and the uniqueness of the decomposition, and discuss the conditions under which a full rank Hankel matrix is H-indecomposable. We shall see that not every rectangular Hankel matrix of full rank can be expressed in the form described in Assertion 2.3 in [2]. However, if it does have an expression in the form, then the decomposition is unique. (Therefore the Remark 2.4 in [2] is false.)

Throughout the paper, the matrices to be considered are over the field of complex numbers. As usual, $r(A)$ denotes the rank of matrix A. We shall adopt the notations and terminology that have been used by other authors in the literature; see [1-4].

We say that an $m\times n$ matrix A is singular if it does not have full rank, i.e., $r(A) < \min\{m,n\}$. An $m \times n$ Hankel matrix H is a rectangular matrix of the form

$$\begin{pmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ h_1 & h_2 & \cdots & h_n \\ \cdots & \cdots & \cdots & \cdots \\ h_{m-1} & h_m & \cdots & h_{m+n-2} \end{pmatrix}.$$

When $m \le n$ and $h_m = \cdots = h_{m+n-2} = 0$ (or $m \le n$ and $h_0 = \cdots = h_{n-2} = 0$), we say the Hankel matrix is upper (or lower) triangular. An $m \times n$ lower triangular Hankel matrix H is called degenerate if $r(H) < \min\{m,n\}$.

If $H = (h_{i+j})_{i=0,j=0}^{m-1,n-1}$ is an $m \times n$ Hankel matrix with $m \le n$, we denote by $\check{H}$ the $(m+1)\times(n-1)$ Hankel matrix $\check{H} = (h_{i+j})_{i=0,j=0}^{m,n-2}$ which is obtained from H by rearranging the

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entries of H. H is further said to be *proper* if the leading submatrix of order $r(H)$ is invertible and $r(H)=r(\check{H})$.

The sum of two matrices A and B of the same size is said to be quasidirect if $r(A+B)=r(A)+r(B)$. A Hankel matrix is said to be H-indecomposable if it cannot be expressed as a quasidirect sum of nonzero Hankel matrices.

For positive integers r and n with $r\le n$ and for complex number t, we denote by $P_{rn}(t)$ the $r\times n$ matrix[1]

$$P_{rn}(t)=\left(\begin{pmatrix} j\\ i\end{pmatrix} t^{j-i}\right)_{i=0,j=0}^{r-1,n-1},$$

and write $P_{rn}(\infty)=(0,J_r)$, where J_r is the $r\times r$ backward identity matrix with $(i,r-i+1)$-entry 1 and elsewhere 0, $i=1,2,\dots,r$[7,8].

Let $B(w,p)$ be the Bézout matrix of polynomials $w(x)$ and $p(x)$[4].

2. Preliminary results

Lemma 2.1 *Let $f(x)=f_0+f_1x+\cdots+f_{n-1}x^{n-1}+x^n$ and $g(x)=g_0+g_1x+\cdots+g_{n-1}x^{n-1}$ be two polynomials. Let $B(f,g)$ be the $n\times n$ Bézout matrix of $f(x)$ and $g(x)$. Then $g(x)$ can be written in the form*

$$g(x)=\pi_n(x)B(f,g)e_n$$

where $\pi_n(x)=(1,x,\cdots,x^{n-1})$, and $e_n=(0,\cdots,0,1)^\top$, where $\top$ stands for the transpose.

Proof Following from the well known Barnett formula

$$B(f,g)=B(f,1)g(C_f),$$

where C_f is the companion matrix of $f(x)$, the last row of $B(f,g)$ is

$$\begin{aligned}(1,0,\cdots,0)\sum_{j=0}^{n-1} g_jC_f^j &= (1,0,\cdots,0)\begin{pmatrix} g_0 & g_1 & \cdots & g_{n-1}\\ & & * & \end{pmatrix}\\ &=(g_0,g_1,\cdots,g_{n-1}).\end{aligned}$$

This implies that the last column of $B(f,g)$ is $(g_0,g_1,\cdots,g_{n-1})^\top$ since $B(f,g)^\top=B(f,g)$. It follows that

$$\pi_n(x)B(f,g)e_n=\pi_n(x)(g_0,g_1,\cdots,g_{n-1})^\top=g(x).$$

Lemma 2.2 *Let $h(x)$ be a polynomial. Let $B(f,g)$ be the $n\times n$ nonsingular Bézout matrix of $f(x)$ and $g(x)$ with $f(x)$ monic and $\deg f(x)=n>\deg g(x)$. Then the Smith canonical form of the matrix*

$$\begin{pmatrix} B(f,g)^{-1} & e_n\\ \pi_n(x) & h(x)\end{pmatrix}$$

is (up to a nonzero constant factor)

$$\begin{pmatrix} I_n & 0\\ 0 & h(x)-g(x)\end{pmatrix}.$$

Proof To prove the assertion, it suffices to notice the identity

$$\begin{pmatrix} B(f,g)^{-1} & e_n \\ \pi_n(x) & h(x) \end{pmatrix}\begin{pmatrix} I_n & -B(f,g)e_n \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} B(f,g)^{-1} & 0 \\ \pi_n(x) & h(x)-\pi_n(x)B(f,g)e_n \end{pmatrix}$$

and use Lemma 2.1. □

Lemma 2.3[1] *Let* $H=(h_{i+j})_{i=0,j=0}^{m-1,n-1}$ *be an* $m\times n$ *Hankel matrix with* $m<n$. *Then there exist numbers* $\lambda_0,\ldots,\lambda_{s-1},\lambda_s=1, 0\le s\le m$, *such that*

$$\begin{pmatrix} h_0 & \cdots & h_{s-1} & h_s \\ \vdots & & \vdots & \vdots \\ h_{m-1} & \cdots & h_{m+s-2} & h_{m+s-1} \end{pmatrix}\begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_{s-1} \\ \lambda_s \end{pmatrix}=0, \tag{2.1}$$

where s *is the maximum one among the orders of nonsingular leading submatrices of* H.

Proof Since there exists a nonnegative integer s, $0\le s\le m$, such that the first s columns of H are linearly independent while the first $s+1$ columns of H are linearly dependent, there are numbers $\lambda_0,\ldots,\lambda_{s-1},\lambda_s=1$, such that (2.1) holds.

From (2.1), we have

$$\begin{pmatrix} 1 & & & & & \\ \vdots & \ddots & & & & \\ 0 & \cdots & 1 & & & \\ \lambda_0 & \cdots & \lambda_{s-1} & \lambda_s & & \\ & \ddots & & \ddots & \ddots & \\ & & \lambda_0 & \cdots & \lambda_{s-1} & \lambda_s \end{pmatrix}\begin{pmatrix} h_0 & \cdots & h_{s-1} & h_s & \cdots & h_{n-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_{s-1} & \cdots & h_{2s-2} & h_{2s-1} & \cdots & h_{s+n-2} \\ h_s & \cdots & h_{2s-1} & h_{2s} & \cdots & h_{s+n-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_{m-1} & \cdots & h_{m+s-2} & h_{m+s-1} & \cdots & h_{m+n-2} \end{pmatrix}\times$$

$$\begin{pmatrix} 1 & \cdots & 0 & \lambda_0 & & \\ & \ddots & \vdots & \vdots & \ddots & \\ & & 1 & \lambda_{s-1} & & \lambda_0 \\ & & & \lambda_s & \ddots & \vdots \\ & & & & \ddots & \lambda_{s-1} \\ & & & & & \lambda_s \end{pmatrix}$$

$$=\begin{pmatrix} h_0 & \cdots & h_{s-1} & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdot \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ h_{s-1} & \cdots & h_{2s-1} & 0 & 0 & \cdots & 0 & h'_{m+s} & \cdots & h'_{s+n-2} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & h'_{m+s} & h'_{m+s+1} & \cdots & h'_{s+n-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & h'_{m+s} & \cdots & \cdot & \cdot & \cdots & h'_{m+n-2} \end{pmatrix},$$

where

$$h'_j=(h_{j-s},\cdots,h_j)\begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_s \end{pmatrix},\quad j=m+s,\ldots,m+n-2. \tag{2.2}$$

Thus, s is as desired in the assertion. □

Denote by $\varphi(x)$ the polynomial

$$\varphi(x)=\lambda_0+\lambda_1 x+\cdots+\lambda_{s-1}x^{s-1}+x^s. \tag{2.3}$$

Then $\varphi(x)$ is uniquely determined by H, and furthermore we have

(a) $\varphi(x)=1$ if and only if $(h_0,\cdots,h_{m-1})=0$;

(b) If $\varphi(x)\neq 1$, then

$$\varphi(x)=\det\begin{pmatrix} h_0 & \cdots & h_{s-1} & h_s \\ \vdots & & \vdots & \vdots \\ h_{s-1} & \cdots & h_{2s-2} & h_{2s-1} \\ 1 & \cdots & x^{s-1} & x^s \end{pmatrix} \Big/ \det\begin{pmatrix} h_0 & \cdots & h_{s-1} \\ \vdots & & \vdots \\ h_{s-1} & \cdots & h_{2s-2} \end{pmatrix}.$$

Theorem 2.4 *Let $H=(h_{i+j})_{i=0,j=0}^{m-1,n-1}$ be an $m\times n$ Hankel matrix with $r(H)=m<n$. If the Smith canonical form of the matrix*

$$\begin{pmatrix} H \\ \pi_n(x) \end{pmatrix} \tag{2.4}$$

is

$$\begin{pmatrix} I_m & 0 & 0 \\ 0 & \psi(x) & 0 \end{pmatrix}, \tag{2.5}$$

then we have the following:

(a) If $r(\check{H})=m+1$, then $\psi(x)=1$;

(b) If $r(\check{H})=m$, then $\psi(x)=\varphi(x)$, where $\varphi(x)$ is defined by (2.3).

Proof The similar transformations used in the proof of Lemma 2.3 carry the matrices (2.4) and $\check{H}$ respectively into the forms

$$\begin{pmatrix} h_0 & \cdots & h_{n-1} \\ \vdots & & \vdots \\ h_{m-1} & \cdots & h_{m+n-2} \\ 1 & \cdots & x^{n-1} \end{pmatrix} \begin{pmatrix} 1 & \cdots & 0 & \lambda_0 & & \\ & \ddots & \vdots & \vdots & \ddots & \\ & & 1 & \lambda_{s-1} & & \lambda_0 \\ & & & \lambda_s & \ddots & \vdots \\ & & & & \ddots & \lambda_{s-1} \\ & & & & & \lambda_s \end{pmatrix}$$

$$=\begin{pmatrix} h_0 & \cdots & h_{s-1} & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdot \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ h_{s-1} & \cdots & h_{2s-2} & 0 & 0 & \cdots & 0 & h'_{m+s} & \cdots & h'_{s+n-2} \\ h_s & \cdots & h_{2s-1} & 0 & 0 & \cdots & h'_{m+s} & h'_{m+s+1} & \cdots & h'_{s+n-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ h_{m-1} & \cdots & h_{m+s-2} & 0 & h'_{m+s} & \cdots & \cdot & h'_{2m} & \cdots & h'_{m+n-2} \\ 1 & \cdots & x^{s-1} & \varphi(x) & x\varphi(x) & \cdots & \cdot & x^{m+1-s}\varphi(x) & \cdots & x^{n-1-s}\varphi(x) \end{pmatrix} \tag{2.6}$$

and

$$\begin{pmatrix} h_0 & \cdots & h_{n-2} \\ \vdots & & \vdots \\ h_{m-1} & \cdots & h_{m+n-3} \\ h_m & \cdots & h_{m+n-2} \end{pmatrix} \begin{pmatrix} 1 & \cdots & 0 & \lambda_0 & & \\ & \ddots & \vdots & \vdots & \ddots & \\ & & 1 & \lambda_{s-1} & & \lambda_0 \\ & & & \lambda_s & \ddots & \vdots \\ & & & & \ddots & \lambda_{s-1} \\ & & & & & \lambda_s \end{pmatrix}$$

$$= \begin{pmatrix} h_0 & \cdots & h_{s-1} & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdot \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ h_{s-1} & \cdots & h_{2s-2} & 0 & 0 & \cdots & 0 & h'_{m+s} & \cdots & h'_{s+n-3} \\ h_s & \cdots & h_{2s-1} & 0 & 0 & \cdots & h'_{m+s} & \cdot & \cdots & h'_{s+n-2} \\ h_{s+1} & \cdots & h_{2s} & 0 & 0 & \cdots & \cdot & \cdot & \cdots & h'_{s+n-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ h_{m-1} & \cdots & h_{m+s-2} & 0 & h'_{m+s} & \cdots & \cdot & \cdot & \cdots & h'_{m+n-3} \\ h_m & \cdots & h_{m+s-1} & h'_{m+s} & \cdot & \cdots & \cdot & \cdot & \cdots & h'_{m+n-2} \end{pmatrix}, \tag{2.7}$$

where h'_j are the same as in (2.2).

By (2.6), $r(H) = m$ implies

$$(h'_{m+s}, \cdots, h'_{s+n-1}) \neq 0. \tag{2.8}$$

By (2.7), we obtain $r(\check{H}) \geq m$, and $r(\check{H}) = m+1$ if and only if

$$(h'_{m+s}, \cdots, h'_{s+n-2}) \neq 0.$$

If $(h'_{m+s}, \cdots, h'_{s+n-2}) \neq 0$, without loss of generality, we may assume h'_{m+s} to be the first nonzero number in the sequence $h'_{m+s}, \ldots, h'_{s+n-2}$. Denote $H_s = (h_{i+j})_{i,j=0}^{s-1}$. Then H_s is nonsingular and compatible with $\varphi(x)$ (see (2.1)). It is well known that in this case there exists a polynomial $\gamma(x)$ with degree less than s and relatively prime with $\varphi(x)$ such that $H_s^{-1} = B(\varphi, \gamma)$, the $s \times s$ Bézout matrix of $\varphi(x)$ and $\gamma(x)$. By Lemma 2.2, the matrix

$$\begin{pmatrix} H_s & 0 & h'_{m+s} e_s \\ \pi_s(x) & \varphi(x) & x^{m+1-s}\varphi(x) \end{pmatrix}$$

is equivalent to

$$\begin{pmatrix} H_s & 0 & 0 \\ \pi_s(x) & \varphi(x) & x^{m+1-s}\varphi(x) - h'_{m+s}\gamma(x) \end{pmatrix},$$

and therefore, equivalent to

$$\begin{pmatrix} I_s & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

because $\varphi(x)$ and $\gamma(x)$ are coprime. Thus (2.6) is equivalent to (2.5) with $\psi(x) = 1$.

Now let $(h'_{m+s}, \cdots, h'_{s+n-2}) = 0$, equivalently $r(\check{H}) = m$. In the case of $s = m$, (2.6) is obviously equivalent to (2.5) with $\psi(x) = \varphi(x)$. In the case of $s < m$, by (2.8), we have $h'_{s+n-1} \neq 0$. Then (2.6) is equivalent to (2.5) with $\psi(x) = \varphi(x)$ as well. □

The polynomial $\psi(x)$ introduced in Theorem 2.4 is referred to as the H-polynomial of the full rank Hankel matrix H.

Remark 1 In case where H is singular, we have $(h'_{m+s},\cdots,h'_{s+n-1})=0$ by (2.6). Therefore (2.6) is equivalent to (2.5) with $\psi(x)=\varphi(x)$. Then $\varphi(x)$ is the H-polynomial of H due to Theorem 2.12 in [1] and Assertion 2.11 in [2].

3. Main results - The decompositions and uniqueness

Theorem 3.1 *Let $H=(h_{i+j})_{i=0,j=0}^{m-1,n-1}$ be an $m\times n$ Hankel matrix with $r(H)=m<n$. If $r(\check{H})=m$, H can be uniquely expressed as a quasidirect sum of a proper Hankel matrix H_p and a lower triangular Hankel matrix H_d with $r(H_p)=\deg\psi(x)$, where $\psi(x)$ is the H-polynomial of H.*

Proof Since $r(\check{H})=m$, in light of Theorem 2.4, $\psi(x)=\varphi(x)$ and $(h'_{m+s},\cdots,h'_{s+n-2})=0$. If $s=0$, H itself is a lower triangular Hankel matrix. If $s=m$, H is already a proper Hankel matrix by definition.

In the case of $0<s<m$, since $(h'_{m+s},\cdots,h'_{s+n-2})=0$ and according to (2.2), (2.1) can be extended to

$$\begin{pmatrix} h_0 & \cdots & h_{s-1} & h_s \\ \vdots & & \vdots & \vdots \\ h_{n-2} & \cdots & h_{n+s-3} & h_{n+s-2} \end{pmatrix}\begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_{s-1} \\ \lambda_s \end{pmatrix}=0.$$

Define numbers $h''_{n+s-1},\ldots,h''_{m+n-2}$ recurrently such that

$$\begin{pmatrix} h_0 & \cdots & h_{s-1} & h_s \\ \vdots & & \vdots & \vdots \\ h_{n-2} & \cdots & h_{n+s-3} & h_{n+s-2} \\ h_{n-1} & \cdots & h_{n+s-2} & h''_{n+s-1} \\ h_n & \cdots & h''_{n+s-1} & h''_{n+s} \\ \vdots & & \vdots & \vdots \\ \cdot & \cdots & h''_{m+n-3} & h''_{m+n-2} \end{pmatrix}\begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_{s-1} \\ \lambda_s \end{pmatrix}=0.$$

Then, since the rank of the matrix $(h_{i+j})_{i=0,j=0}^{s-1}$ equals s, the matrix

$$H_p=\begin{pmatrix} h_0 & \cdots & \cdot & \cdots & h_{n-1} \\ \vdots & & \vdots & & \vdots \\ \cdot & \cdots & \cdot & \cdots & h''_{n+s-1} \\ \vdots & & \vdots & & \vdots \\ h_{m-1} & \cdots & h''_{n+s-1} & \cdots & h''_{m+n-2} \end{pmatrix}$$

is a proper Hankel matrix with $r(H_p)=s$ and having $\psi(x)$ as its H–polynomial. Let

$$H_d=H-H_p$$

$$= \begin{pmatrix} 0 & \cdots & \cdot & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ \cdot & \cdots & \cdot & \cdots & h_{n+s-1} - h''_{n+s-1} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & h_{n+s-1} - h''_{n+s-1} & \cdots & h_{m+n-2} - h''_{m+n-2} \end{pmatrix}.$$

Notice that

$$0 = (h_{n-1}, \cdots, h_{n+s-2}, h''_{n+s-1}) \begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_{s-1} \\ \lambda_s \end{pmatrix}$$

$$= (h_{n-1}, \cdots, h_{n+s-2}, h_{n+s-1}) \begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_{s-1} \\ \lambda_s \end{pmatrix} + (0, \cdots, 0, h''_{n+s-1} - h_{n+s-1}) \begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_{s-1} \\ \lambda_s \end{pmatrix}$$

$$= h'_{n+s-1} + (h''_{n+s-1} - h_{n+s-1}),$$

where h'_{n+s-1} are as given in (2.2). We have $h_{n+s-1} - h''_{n+s-1} = h'_{n+s-1} \neq 0$. This means H_d is a nonzero degenerate Hankel matrix with $r(H_d) = m - s$. Thus the sum $H = H_p + H_d$ is quasidirect.

To show the uniqueness, suppose that $H = \tilde{H}_p + \tilde{H}_d$ is a quasidirect sum, where $\tilde{H}_p$ is proper and $\tilde{H}_d$ is lower triangular. Let $r = r(\tilde{H}_p)$. Then $r(\tilde{H}_d) = m - r$ implies that the first $n-(m-r)$ columns of $\tilde{H}_p$ are identical to those of H. Since $\tilde{H}_p$ is proper and $r+(n-m) \geq r+1$, we see that the first r columns of H are linearly independent and the first $r+1$ columns of H are linearly dependent. So $s = r$, thus $\tilde{H}_p = H_p$. The uniqueness is proved. □

Theorem 3.2 *Let H be an $m \times n$ Hankel matrix with $r(H) = m < n$ and $\psi(x)$ the H-polynomial of H. If $r(\check{H}) = m$ and if*

$$\psi(x) = \prod_{i=1}^{k} (x - t_i)^{r_i},$$

where t_i are distinct, $k \geq 0, r_i > 0$, then H can be written in the form

$$H = \sum_{i=1}^{k} P_{r_i m}(t_i)^{\top} S_i P_{r_i n}(t_i) + P_{r_\infty m}(\infty)^{\top} S_\infty P_{r_\infty n}(\infty), \tag{3.9}$$

where S_i, S_∞ are nonsingular upper triangular Hankel matrices of orders r_i, r_∞ with $r(H) = \sum_{i=1}^{k} r_i + r_\infty$. This decomposition is unique up to the order of the components.

Proof By making use of Theorem 3.1, we can uniquely express H into the form

$$H = H_p + H_d,$$

where H_p is proper and H_d is lower triangular. It is obvious that H_d has the form

$$H_d = P_{r_\infty m}(\infty)^{\top} S_\infty P_{r_\infty n}(\infty), \quad r_\infty = r(H_d).$$

So we need only to prove that H_p has the unique decomposition

$$H_p = \sum_{i=1}^{k} P_{r_i m}(t_i)^\top S_i P_{r_i n}(t_i). \tag{3.10}$$

If $0 \le r(H_p) < m$, H_p is a singular proper Hankel matrix with $\psi(x)$ as its H-polynomial. By Assertion 2.3 in [2], H_p can uniquely be written into the form (3.10).

If $r(H_p) = m$, $H = H_p$, we have (2.1) with $s = m$ and $(h'_{m+m}, \cdots, h'_{m+n-2}) = 0$, i.e.,

$$(\lambda_0, \cdots, \lambda_{m-1}, \lambda_m) \begin{pmatrix} h_0 & \cdots & h_{n-2} \\ \vdots & & \vdots \\ h_{m-1} & \cdots & h_{m+n-3} \\ h_m & \cdots & h_{m+n-2} \end{pmatrix} = 0 \tag{3.11}$$

and $r[(h_{i+j})_{i,j=0}^{m-1}] = m$. Let h_{m+n-1} be a number such that

$$(\lambda_0, \cdots, \lambda_{m-1}, \lambda_m) \begin{pmatrix} h_{n-1} \\ \vdots \\ h_{m+n-2} \\ h_{m+n-1} \end{pmatrix} = 0.$$

Then $G = (h_{i+j})_{i=0,j=0}^{m,n-1}$ is an $(m+1) \times n$ singular proper Hankel matrix still having $\psi(x)$ as its H-polynomial. By Assertion 2.3 in [2] again, we have

$$G = \sum_{i=1}^{k} P_{r_i m+1}(t_i)^\top S_i P_{r_i n}(t_i),$$

where S_i is nonsingular upper triangular Hankel matrix of order $r_i, m = r(G) = \sum_{i=1}^{k} r_i$. Consequently, H has the form (3.10).

In what follows we prove the uniqueness of the form (3.10) with $m = \sum\limits_{i=1}^{k} r_i$.

Suppose $H = H_p$ has a expression

$$H = \sum_{j=1}^{l} P_{u_j m}(v_j)^\top \tilde{S}_j P_{u_j n}(v_j), \tag{3.12}$$

where $u_j > 0, m = \sum_{j=1}^{l} u_j, v_j$ are distinct, and $\tilde{S}_j$ are $u_j \times u_j$ nonsingular upper triangular Hankel matrices. Let

$$V_m^\top = (P_{u_1 m}(v_1)^\top, \cdots, P_{u_l m}(v_l)^\top),$$
$$V_n^\top = (P_{u_1 n}(v_1)^\top, \cdots, P_{u_l n}(v_l)^\top),$$
$$\tilde{S} = \mathrm{diag}\{\tilde{S}_1, \cdots, \tilde{S}_l\}.$$

Then $\tilde{S}$ is nonsingular. By the properties of the generalized Vandermonde matrices [3], we see that V_m is nonsingular. Denote by $\tilde{\psi}(x)$ the polynomial

$$\tilde{\psi}(x) = \prod_{j=1}^{l} (x - v_j)^{u_j} = \mu_0 + \cdots + \mu_m x^m \quad (\mu_m = 1).$$

We then have

$$V_n \begin{pmatrix} 1 & \cdots & 0 & \mu_0 & & \\ & \ddots & \vdots & \vdots & \ddots & \\ & & 1 & \mu_{m-1} & & \mu_0 \\ & & & \mu_m & \ddots & \vdots \\ & & & & \ddots & \mu_{m-1} \\ & & & & & \mu_m \end{pmatrix} = (V_m, 0)\,.$$

(Notice that here $m < n$ is necessary.) We now can reduce the matrix (2.4) to

$$\begin{aligned} &\begin{pmatrix} H \\ \pi_n(x) \end{pmatrix} \begin{pmatrix} 1 & \cdots & 0 & \mu_0 & & \\ & \ddots & \vdots & \vdots & \ddots & \\ & & 1 & \mu_{m-1} & & \mu_0 \\ & & & \mu_m & \ddots & \vdots \\ & & & & \ddots & \mu_{m-1} \\ & & & & & \mu_m \end{pmatrix} \\ &= \begin{pmatrix} V_m^{\top}\tilde{S} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V_n \\ \pi_n(x) \end{pmatrix} \begin{pmatrix} 1 & \cdots & 0 & \mu_0 & & \\ & \ddots & \vdots & \vdots & \ddots & \\ & & 1 & \mu_{m-1} & & \mu_0 \\ & & & \mu_m & \ddots & \vdots \\ & & & & \ddots & \mu_{m-1} \\ & & & & & \mu_m \end{pmatrix} \\ &= \begin{pmatrix} V_m^{\top}\tilde{S} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V_m & 0 \\ \pi_m(x) & \tilde{\psi}(x)\pi_{n-m}(x) \end{pmatrix}. \end{aligned}$$

Therefore, (2.4) is equivalent to

$$\begin{pmatrix} I_m & 0 & 0 \\ 0 & \tilde{\psi}(x) & 0 \end{pmatrix}.$$

That means $\tilde{\psi}(x) = \psi(x)$; so the decomposition (3.12) is the same as (3.10) with $\sum_{i=1}^{k} = m$, except for the order of the components. □

Theorem 3.3 *Let H be an $m \times n$ Hankel matrix with $r(H) = m < n$. If H is H-decomposable, then $r(\check{H}) = m$.*

Proof Suppose $H = A + B$ is a quasidirect sum of nonzero Hankel matrices A and B. Then we have $0 < r(A),\ r(B) < m$. By Assertion 2.3 in [2], both A and B can be expressed as quasidirect sums $A = A_p + A_d$, $B = B_p + B_d$, where A_p, B_p are proper and A_d, B_d degenerate. Since $0 < r(A),\ r(B) < m < n$, we obtain $r(G) = r(\check{G})$, where G is any one of A_p, A_d, B_p, B_d.

Thus we conclude

$$\begin{aligned} m \le r(\check{H}) &= r(\check{A}+\check{B}) \le r(\check{A})+r(\check{B}) \\ &\le r(\check{A}_p)+r(\check{A}_d)+r(\check{B}_p)+r(\check{B}_d) \\ &= r(A_p)+r(A_d)+r(B_p)+r(B_d) \\ &= r(A)+r(B)=r(H)=m. \end{aligned}$$

Therefore $r(\check{H})=m$. □

4. Consequences

The following results are consequences of Theorem 3.2 and Theorem 3.3.

Corollary 4.1 *Let H be an $m\times n$ Hankel matrix with $r(H)=m<n$. Then H can be decomposed into the form (3.9) with $r(H)=\sum_{i=1}^k r_i + r_\infty$ if and only if $r(\check{H})=m$.*

The proof of Corollary 4.1 is immediate from Theorem 3.2 and Theorem 3.3. For the next corollary, we need a lemma.

Lemma 4.2 *If H is a lower triangular Hankel matrix with $r(H)=m<n$, then H is H-indecomposable.*

Proof Suppose, otherwise, that H could be written as the form $H=A+B$ for some nonzero Hankel matrices A and B such that $r(H)=r(A)+r(B)$. Then A and B are singular and, by Assertion 2.3 in [2], they can be decomposed into the forms

$$A=\sum_{i=1}^k P_{r_i m}(u_i)^\top S_i P_{r_i n}(u_i)+A_d,$$

where u_i are distinct, $k\ge 0, r_i>0$, A_d is degenerate, and

$$B=\sum_{j=1}^l P_{s_j m}(v_j)^\top \tilde{S}_j P_{s_j n}(v_j)+B_d,$$

where v_j are distinct, $l\ge 0, s_j>0$, B_d is degenerate. Since $A+B$ is a quasidirect sum, by the Lemma in [2, p.194], we see that the sets $\{u_i\}$ and $\{v_j\}$ are disjoint, and $A_d=0$ or $B_d=0$. Therefore

$$\sum_{i=1}^k P_{r_i m}(u_i)^\top S_i P_{r_i n}(u_i)+\sum_{j=1}^l P_{s_j m}(v_j)^\top \tilde{S}_j P_{s_j n}(v_j)$$

is quasidirect and, consequently, is proper. According to the uniqueness in Theorem 3.1, we have that $H=A_d+B_d$ is degenerate, a contradiction to $r(H)=m$. □

Corollary 4.3 *Let H be an $m\times n$ Hankel matrix with $r(H)=m<n$ and $\psi(x)$ the H-polynomial of H. If $\psi(x)=(x-t)^m$ for some number t, then H is H-indecomposable.*

Proof In this case, by Theorem 2.4 and Theorem 3.2, we conclude that $r(\check{H})=m$. Then H

can be expressed into the form

$$H = P_{mm}(t)^\top S_m P_{mn}(t), \tag{4.13}$$

where S_m is an $m \times m$ nonsingular upper triangular Hankel matrix. Just as in [1], we can transform H into a lower triangular Hankel matrix by means of

$$J_m P_{mm}(-t)^\top H P_{nn}(-t) J_n = (0, J_m S_m J_m),$$

where J_m is as given before. Lemma 4.2 implies the H-indecomposability of H. □

Theorem 4.4 *Let H be an $m \times n$ Hankel matrix with $r(H) = m < n$. Then H is H-indecomposable if and only if either $\psi(x) = 1$ or $\psi(x) = (x-t)^m$ for some number t, where $\psi(x)$ is the H-polynomial of H.*

Proof Let H be H-indecomposable. If $r(\check{H}) = m+1$, then $\psi(x) = 1$ by Lemma 2.4. If $r(\check{H}) = m$, by Theorem 3.2, H has the form (3.9). Therefore we obtain $k = 0$, or $k = 1$ and $r_\infty = 0$. That is, either $\psi(x) = 1$ or $\psi(x) = (x-t)^m$.

Conversely, if $r(\check{H}) = m+1$, then H is H-indecomposable by Theorem 3.3. If $r(\check{H}) = m$, by Theorem 3.2, H is either a lower triangular Hankel matrix or a Hankel matrix in the form (4.13). So H is H-indecomposable by Lemma 4.2 and Corollary 4.3. □

Remark 2 From the above investigation, we can conclude that not every $m \times n$ ($m < n$) Hankel matrix H with full rank can be expressed in the form (3.9), since, otherwise, Theorem 3.3 gives $r(\check{H}) = m$; and on the other hand, Theorem 3.2 assures the uniqueness of the decomposition.

References:

[1] FIEDLER M. *Quasidirect decompositions of Hankel and Toeplitz matrices* [J]. Linear Algebra Appl., 1984, **61**: 155–174.
[2] VAVŘÍN Z. *A unified approach to Loewner and Hankel matrices* [J]. Linear Algebra Appl., 1991, **143**: 171–222.
[3] FIEDLER M. *Polynomials and Hankel matrices* [J]. Linear Algebra Appl., 1985, **66**: 235–248.
[4] PTÁK V, VAVŘÍN Z. *ZBézout, Hankel, and Loewner matrices* [J]. Linear Algebra Appl., 1993, **184**: 13–36.
[5] FUHRMANN P A. *Remarks on the inversion of Hankel matrices* [J]. Linear Algebra Appl., 1986, **81**: 89–104.
[6] HEINIG G, JUNGNICKEL U. *Hankel matrices generated by the Markov parameters of rational functions* [J]. Linear Algebra Appl., 1986, **76**: 121–135.
[7] HORN R A, JOHNSON C R. *Matrix Analysis* [M]. Cambridge University Press, New York, 1985.
[8] ZHANG F. *Matrix Theory: Basic Results and Techniques* [M]. Springer-Verlag, New York, 1999.

满秩 Hankel 矩阵的拟直分解

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摘要: 本文研究行满秩 Hankel 矩阵分解为一个真正的 (proper)Hankel 矩阵与一个退化的 (degenerate)Hankel 矩阵之拟直和的存在性及唯一性问题.

关键词: Hankel 矩阵; H- 可分解性; 拟直分解.