Therefore $P/J(P)$ is also an Artinian S- moudle. On the other hand, $P/J(P)$ is semisimple. So $P/J(P)$ is Noetherian. Thus $P/J(P)$ is an object of mod-S of finite length. Hence $\mathrm{End}_S(P/J(P))$ is semiprimary (see p.37 in [4]). Moreover $\mathrm{End}_S(P/J(P))$ is semilocal. So $\mathrm{End}_S P/J(\mathrm{End}_S P)$ is also Artinian. That is, $R \simeq \mathrm{End}_S P$ is semilocal, as asserted.

Corollary 4 *Let R be a Dedekind domain. Then the following statements are equivalent:*

(1) R is semilocal.

(2) There exists some non-zero ideal whose top is Artinian.

Proof Since R is a Dedekind domain, every non-zero ideal must be invertible. By virtue of Theorem 3, the result follows.

References

[1] N.Jacobson, *The Structure of Rings* , Colloquium Pobl. AMS., 1956.

[2] B.R.McDonald, *Linear Algebra over Commutative Rings* , Marcel Dekker, Inc., New York and Basel, 1984.

[3] P.Menal, *Cancellation modules over regular rings* , Lecture Notes in Math., 1328, Ring Theory, Springer-Verlag, 1988.

[4] C.Faith, *Algebra II,* Springer-Verlag, Berlin, Heidelberg, New York, 1976.

半 局 部 环 的 一 个 特 征

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摘 要

给出了可换半局部环的一个外部特征，证明了 R 为半局部环当且仅当存在可逆 R- 模，其 Top 为 Artin 模.

Journal of Mathematical Research & Exposition
Vol.16, No.2, 190–192, May 1996

A Characterization of Semilocal Rings *

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Abstract In this note, we prove that if R is commutative then R is semilocal if and only if there exists some invertible module whose Top is Artinian.

Keywords semilocal ring, invertible module.

Classification AMS(1991) 13H99/CCL O153.3

At first, we give a lemma.

Lemma 1 *Let* $\phi : R \twoheadrightarrow S$ *be a epimorphic ring homomorphism with* $\ker\phi \subset J(R)$. *Then* $R/J(R) \simeq S/J(S)$.

Proof Since $\phi : R \to S$ is epimorphic, we have an isomorphism: $R/\ker\phi \overset{\phi^*}{\simeq} S$. So $J(R/\ker\phi) \overset{\phi^*}{\simeq} J(S)$. By virtue of the homomorphism theorem, we can obtain ψ : $(R/\ker\phi)/J(R/\ker\phi) \overset{\psi}{\simeq} S/J(S)$.

$$\begin{array}{ccc} R/\ker\phi & \xrightarrow{\phi^*} & S \\ \downarrow & & \downarrow \\ (R/\ker\phi)/J(R/\ker\phi) & \overset{\psi}{\dashrightarrow} & S/J(S) \end{array}$$

On the other hand, $J(R/\ker\phi) \supset J(R)+\ker\phi/\ker\phi = J(R)/\ker\phi$. Since $J(R/\ker\phi) = \cap M \subset \cap(N/\ker\phi) = (\cap N)/\ker\phi = J(R)/\ker\phi$, where M and N are maximal submodules of $R/\ker\phi$ and R respectively, we have $J(R/\ker\phi) = J(R)/\ker\phi$. Thus $S/J(S) \simeq (R/\ker\phi)/(J(R/\ker\phi)) \simeq (R/\ker\phi)/(J(R)/\ker\phi) \simeq R/J(R)$, as required.

As an immediate consequence, we have

Corollary 2 *The following statements are equivalent:*

(1) R *is semilocal.*

(2) $R[[x_1, \cdots, x_n]]$ *is semilocal.*

Now we can derive the following

Theorem 3 *Let* R *be a commutative ring. Then the following statements are equivalent:*

*Received Oct.8, 1993.

(1) R is semilocal.

(2) There exists some invertible R-module whose Top is Artinian.

Proof (1)⇒ (2) Since R is semilocal, $R/J(R)$ is an Artinian ring. Let M be R-submodule of $R/J(R)$,

$$(N = \{r \in R | r + J(R) \in M\}.$$

It is easy to verify $N \lhd R$. As R-modules, we have R-module homomorphism $\phi : N \twoheadrightarrow M, r \mapsto r + J(R)$. So $M \simeq N/\ker\phi = N/J(R)$.

Suppose that $R/J(R) \supset N_1/J(R) \supset \cdots$ is a descending sequence of R-modules, where $N_i \lhd R, i \geq 1$. We define $\bar{r} \cdot \bar{n} = \overline{rn}$ for any $\bar{r} \in R/J(R), \bar{n} \in N_i/J(R), i \geq 1$. Since $\bar{r}_1 = \bar{r}, \bar{n}_1 = \bar{n}$ implies that there exist some $a, b \in J(R)$ such that

$$r_1 = r + a, n_1 = n + b,$$

we can prove

$$r_1 n_1 = rn + na + rb + ab \in rn + J(R).$$

Thus $\overline{r_1 n_1} = \overline{rn}$. So the definition is well defined. Since $R/J(R)$ is an artinian $R/J(R)$-module, we can find some $s > 0$ such that

$$N_s/J(R) = N_{s+1}/J(R) = \cdots,$$

So $R/J(R)$ is also an Artinian R-module. Hence R is an invertible R-module whose Top is Artinian.

(2)⇒ (1) Assume that P is an invertible R-module whose Top is Artinian. So P is a finitely generated projective module of rank one.

Let $A = \mathrm{Ann}P$. For any $Q \in \mathrm{Spec}R$, we have

$$A_Q = A \otimes R_Q \simeq AR_Q \simeq AP_Q \simeq A(R_Q \otimes P) = 0.$$

So $A = 0$. That is, P is a faithful module. From Theorem III.11 in [2], we know P is a generator. Using Theorem III.12 in [2], we can prove that the ring R is isomorphic as a ring to the ring of S-modules of P under $R \to \mathrm{End}_S P$ by $r \mapsto \sigma_r, \sigma_r(p) = rp$, where $S = \mathrm{End}_R P$. Moreover, we can prove ${}_S P$ is a finitely generated projective S-module.

By virtue of Lemma III.29 and Theorem III.28 in [2], we know $\ker\Phi = J(\mathrm{End}_S P)$, where $\Phi : \mathrm{End}_S P \to \mathrm{End}_S(P/J(P))$. Using Lemma 1, we have

$$\mathrm{End}_S P/J(\mathrm{End}_S P) \simeq \mathrm{End}_S(P/J(P))/J(\mathrm{End}_S(P/J(P))).$$

Since the top of ${}_R P$ is Artinian, it suffices to show that ${}_S(P/J(P))$ is also an Artinian S-module.

Suppose that $P/J(P) \supset P_1/J(P) \supset \cdots$ is a descending sequence of S-modules. We define $\sigma_r : P_i \to P_i, p \mapsto rp$. Thus we can define $r \cdot \bar{p} = \overline{\sigma_r(p)}$ for any $r \in R, \bar{p} \in P_i/J(P)$. Hence $P_i/J(P)$ is also an R-module. Since $P/J(P)$ is an Artinian R-module, we can find some $n > 0$ such that

$$P_n/J(P) = P_{n+1}/J(P) = \cdots.$$

Therefore $P/J(P)$ is also an Artinian S- moudle. On the other hand, $P/J(P)$ is semisimple. So $P/J(P)$ is Noetherian. Thus $P/J(P)$ is an object of mod-S of finite length. Hence $\mathrm{End}_S(P/J(P))$ is semiprimary (see p.37 in [4]). Moreover $\mathrm{End}_S(P/J(P))$ is semilocal. So $\mathrm{End}_S P/J(\mathrm{End}_S P)$ is also Artinian. That is, $R \simeq \mathrm{End}_S P$ is semilocal, as asserted.

Corollary 4 *Let R be a Dedekind domain. Then the following statements are equivalent:*

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Proof Since R is a Dedekind domain, every non-zero ideal must be invertible. By virtue of Theorem 3, the result follows.

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