i.e.,

$$\max_{u \in H_2^{\perp} \setminus \{0\}, v \in H_1^{\perp} \setminus \{0\}} \frac{|(u, v)|}{\|u\| \|v\|} \leq \upsilon,$$

i.e.,

$$\max_{u \in H_0^{\perp} \setminus \{0\}} \frac{\|P_{H_1^{\perp}} u\|}{\|u\|} \leq \upsilon$$

Acknowledgement *The author is grateful to Prof. Jiang Erxiong for his help on this research.*

References

[1] P. E. Bjorstard, *Multiplicative and additive schwarz methods*, DDM for PDE (T. F. Chan et al, eds), SIAM, 1989.

[2] J. Mandel and S. McCormick, *Iterative solution of elliptic equation with refinement: the two-level case*, DDM for PDE (T. F. Chan et al, eds), SIAM, 1989.

[3] Z. H. Cao, *Multi-grid method*, Fudan University Press, Shanghai, 1989 (in Chinese).

[4] S. Zhang, *On the convergence rate of additive and multiplicative schwarz method*, Chinese Symposium on CompntationalMathematics, Tianjin, 1991.

[5] J. G. Sun, *Matrix perturbation analysis*, Science Press, Beijing, 1987 (in Chinese).

Schwarz 交替方向法的代数处理

黄 建 国

(上海交大应用数学系, 上海 200240)

摘 要

本文利用代数方法获得二子区域情形带松驰因子的加法型 Schwarz 交替方向法的最优松驰因子, 结果表明代数平均是最优的. 接着通过反例说明该结果不可推广至多子区域情形. 最后, 本文将该代数技巧用于证明一些现有的重要结果[1], [2], [3], 和原有证明相比, 现证简单、直观.

Journal of Mathematical Research & Exposition
Vol. 17, No. 2, 159-164, May 1997

An Algebraic Approach to the Schwarz Alternating Methods*

Huang Jianguo
(Dept. of Appl. Math., Shanghai Jiao Tong University, 200240)

Abstract In this paper, the choice of the optimal parameters for a relaxation additive Schwarz alternating method in two subregions case is obtained by an algebraic method, which shows that the arithmetic average is the best. A counterexample illustrates that the same result is not true for many subregions case. In the last, this technique is applied to demonstrate some well-known results[1],[2],[3] simply and intuitively.

Keywords Schwarz alternating method, space decomposition, convergence rate.

Classification AMS(1991) 65F10, 65F30/CCL O241.6

We consider the following finite-dimensional problem:

$$\begin{cases} u \in V \\ a(u, v) = (f, v), u \in V, \end{cases} \tag{1}$$

which arises from the discretization of regular elliptic equations (systems) by finite element or finite difference methods. Here $a(\cdot, \cdot)$ denotes the inner-product of a finite-dimensional Hilbert space V with induced norm $\|\cdot\|$, $(\cdot, \cdot)$ means the dual form on $V' \times V$, and V' represents the dual space of V, $f \in V'$. Assume that V is split into two subspaces V_1 and V_2, i.e.,

$$V = V_1 + V_2. \tag{2}$$

Then it casts some additive Schwarz alternating methods solving (1) as follows:

ALG 1 Assume $u^0 \in V$ is the initial guess. Then u^n denotes the approximation at the nth step defined by

$$\begin{cases} u^{2n+i} - u^{2n+1+i} \in V_{i+1}, \\ a(u^{2n+i+1}, v) = (f, v), v \in V_{i+1}, i = 0, 1, \end{cases}$$

which goes to the accurate solution u as n tends to infinity.

ALG 2 Assume that $\omega_1, \omega_2 \in (0, 1)$ are two relaxation parameters, $\omega_1 + \omega_2 = 1$, $u^0 \in V$ is the initial guess. Then u^n is formed by

* Received Jun. 6, 1994. This work was supported by the National Science Foundation of China.

$$\begin{cases} u^n - u^{n,i} \in V_i, \\ a(u^{n,i}, v) = (f, v), v \in V, \quad i = 1, 2, \end{cases}$$

$$u^{n+1} = \omega_1 u^{n,1} + \omega_2 u^{n,2}.$$

In the case of $\omega_1 = \omega_2 = \frac{1}{2}$, ALG 2 is called ALG 2' which is particularly important. The benefit of the above algorithms is the parallelization and the scale reduction of the original problem. As usual, let $\varepsilon^n = u^n - u$, thus the iterative operators for ALG 1, 2, 2' are

$$\varepsilon^{2n+1+i} = P_{V_{i+1}^{\perp}} \varepsilon^{2n+i}, \, i = 0, 1,$$
$$\varepsilon^{n+1} = (\omega_1 P_{V_1^{\perp}} + \omega_2 P_{V_2^{\perp}}) \varepsilon^n,$$
$$\varepsilon^{n+1} = \frac{1}{2} (P_{V_1^{\perp}} + P_{V_2^{\perp}}) \varepsilon^n,$$

respectively, here $V_i^{\perp}$ denotes the orthogonal complementary subspace of V_i in V with respect to the inner-product $a(\cdot, \cdot)$ and the relative orthogonal projection operator is denoted by $P_{V_i^{\perp}}$, $i = 1, 2$. Let $\rho(A)$ denote the spectral radius of a linear operator A which decides the convergence rate of its relative iteration. The interesting and useful questions are finding the relationship between the convergence rates of ALG1 and ALG2', and the choices of the optimal relaxation parameters ω_1, ω_2 to ensure the deepest convergence of ALG2.

The first question was answered by Bjorstard[1] who gave a beautiful identity between $\rho(P_{V_2^{\perp}} P_{V_1^{\perp}})$ and $\rho(\frac{1}{2}(P_{V_1^{\perp}} + P_{V_2^{\perp}}))$, but his proof is tedious. Afterwards, Dr. Zhang Sheng[4] introduced an elegant proof. In this paper, we shall present an algebraic proof for this result, which is more intuitive and essential. This idea can also be utilized to deal with the second question and demonstrate the fundamental lemma of the two level multi-grid method.

In order to derive these results, we first recall some well-known lemmas.

Lemma 1[5] *Assume both of the matrices $X \in R^{n \times r}$, $Y \in R^{n \times s}$ are columns orthogonal with $r \leq s$. Then there exist $Q \in O(n)$ (set of $n \times n$ orthogonal matrices), $U \in O(r)$, $V \in O(s)$ such that*

$$QXU = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}, QYV = \begin{bmatrix} \Gamma & 0 \\ 0 & I \\ \Sigma & 0 \end{bmatrix},$$

where $\Gamma = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r)$, $0 \leq \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_r \leq 1$, $\Sigma^2 + \Gamma^2 = 1$, $\Sigma \geq 0$, I denotes the identity matrix.

Lemma 2[5] *Assume $A \in R^{m \times n}$. Then the orthogonal projection operator from R^m onto the range space $R(A)$ with respect to the conventional Euclidean inner-product is $P_{R(A)} = AA^+$, here A^+ denotes the Moore-Penrose inverse.*

Theorem 1 *As $\omega_1 = \omega_2 = \frac{1}{2}$, the convergence rate of ALGI is optimal.*

Proof Without loss of generality, let $\dim V = n$, $\dim V_1^\perp = r$, $\dim V_2^\perp = s$ with $r \le s$. Let $\{\varphi_i\}_1^n$ denote the orthonormal bases of V. Introduce the following isometric transformation:

$$\begin{aligned} &T: R^n \to V, \\ &x = (x_1, x_2, \cdots x_n)' \mapsto Tx = \sum_{i=1}^n x_i \varphi_i. \end{aligned} \tag{3}$$

We shall show that T is an orthogonal transformation. That is, let T' denote the transpose of T. Then

$$T'T = I_{R^n}, TT' = I_V, \tag{4}$$

where T' is defined by

$$\begin{cases} T': V \to R^n, \\ a(Tx, v) = [x, T'v], x \in R^n, v \in V, \end{cases} \tag{5}$$

here $[\cdot, \cdot]$ is the inner-product of the Euclidean space R^n, $[x, y] = \sum_{i=1}^n x_i y_i$. We only prove the second identity in (4), proof of the first one is similar. For any $u, v \in V$,

$$\begin{aligned} a(TT'u, v) &= [T'u, T'v] \quad \text{(from (5))} \\ &= \frac{1}{4}\{[T'(u+v), T'(u+v)] - [T'(u-v), T'(u-v)]\} \\ &= \frac{1}{4}\{a(TT'(u+v), TT'(u+v)) - a(TT'(u-v), TT'(u-v))\} \\ &= a(TT'u, TT'v) \end{aligned}$$

which shows TT' is the identity operator on V, since TT' is surjective.

Let $P_1 = T'P_{V_1^\perp}T$, $P_2 = T'P_{V_2^\perp}T$. It follows from (4) that P_1 and P_2 are both orthogonal projection operators on R^n, and

$$\omega_1 P_{V_1^\perp} + \omega_2 P_{V_2^\perp} = T(\omega_1 P_1 + \omega_2 P_2)T'.$$

Suppose that $\{a_i\}_1^r$ and $\{b_i\}_1^s$ are the orthonormal bases of the range space of P_k, respectively, $k = 1, 2$. Define

$$X = [a_1, a_2, \cdots a_r] \in R^{n \times r}, \quad Y = [b_1, b_2, \cdots b_s] \in R^{n \times s}.$$

Then from Lemma 2, it is easy to know that

$$P_1 = XX', P_2 = YY'.$$

From Lemma 1, there exist $Q \in O(n)$, $U \in O(r)$, and $W \in O(s)$ such that

$$X = Q'\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} U', \quad Y = Q'\begin{bmatrix} \Gamma & 0 \\ 0 & I \\ \Sigma & 0 \end{bmatrix} W',$$

here $\Gamma = diag(\sigma_1, \sigma_2, \cdots \sigma_r)$, $0 \le \sigma_1 \le \sigma_2 \le \cdots \le \sigma_r < 1$ (the last inequality comes from (2)), $\Sigma^2 + \Gamma^2 = I$, $\Sigma \ge 0$. Hence,

$$\omega_1 P_1 + \omega_2 P_2 = Q \begin{bmatrix} \omega_1 I + \omega_2 \Gamma^2 & 0 & \omega_2 \Gamma\Sigma \\ 0 & \omega_2 I & 0 \\ \omega_2 \Gamma\Sigma & 0 & \omega_2 \Sigma^2 \end{bmatrix} Q',$$

with the eigenvalues

$$\lambda = \frac{1}{2}(1 \pm (1 - 4(1 - \sigma_i^2)\omega_2(1 - \omega_2))^{1/2}), \ or\ for\ \omega_2, i = 1, 2, \cdots r.$$

It is trivial that

$$\omega_2 \le \frac{1}{2}(1 + (1 - 4(1 - \sigma_r^2)\omega_2(1 - \omega_2))^{1/2}), 0 < \omega_2 < 1.$$

Therefore,

$$\rho(\omega_1 P_1 + \omega_2 P_2) = \frac{1}{2}(1 + (1 - 4(1 - \sigma_r^2)\omega_2(1 - \omega_2))^{1/2}).$$

Observing that $\omega_2(1 - \omega_2) \le 1/4$, we see

$$\rho(\omega_1 P_1 + \omega_2 P_2) \ge (1 + \sigma_r)/2,$$

and the equality is arrived only if $\omega_1 = \omega_2 = 1/2$. Because of the spectred in invariance of the orthogonal transformation. Theorem 1 then follows in the last. □

Remark If the space V is split into three or more subspace, we can design similarly an additive Schwarz alternating iterative algorithm with iterative operator

$$A = \sum_{i=1}^{m} \omega_i P_{V_i^\perp}, \sum_{i=1}^{m} \omega_i = 1, and\ \omega_i > 0.$$

It can be shown that the optimal relaxation parameters for $m \ge 3$ are not $\omega_1 = \omega_2 = \cdots = \omega_m = 1/m$.

Example $m = 3$, let $V = V_1 + V_2, V_3 = V_2$. Thus, $V = V_1 + V_2 + V_3$, the iterative matrix

$$A = \sum_{i=1}^{3} \omega_i P_{V_i^\perp} = \omega_1 P_{V_1^\perp} + (\omega_2 + \omega_3) P_{V_2^\perp}.$$

So the optimal parameters should be

$$\omega_1 = \omega_2 + \omega_3 = 1/2, \ and\ not\ \omega_1 = \omega_2 = \omega_3 = 1/3.$$

Now we give algebraic proof for the following results:

Theorem 2[1] *Let ρ, σ denote the spectral radii of $\frac{1}{2}(P_{V_1^\perp} + P_{V_2^\perp})$ and $P_{V_2^\perp} P_{V_1^\perp}$ respectively. Then the following identity holds:*

$$\sigma = (2\rho - 1)^2.$$

Proof In terms of the above argument, we need to consider only the operators

$$\frac{1}{2}(P_2 + P_1) = \frac{1}{2}Q'\begin{bmatrix} I + \Gamma^2 & 0 & \Gamma\Sigma \\ 0 & I & 0 \\ \Sigma\Gamma & 0 & \Sigma^2 \end{bmatrix}Q,$$

$$P_2P_1 = Q'\begin{bmatrix} \Gamma^2 & 0 & 0 \\ 0 & 0 & 0 \\ \Sigma\Gamma & 0 & 0 \end{bmatrix}Q,$$

with $\Gamma = \mathrm{diag}(\sigma_1, \sigma_2, \cdots \sigma_r)$, $0 \le \sigma_1 \le \sigma_2 \le \cdots \le \sigma_r < 1$. Thus $\sigma = \sigma_r^2$, and the eigenvalues of

$$\begin{bmatrix} \Gamma^2 + I & 0 & \Gamma\Sigma \\ 0 & I & 0 \\ \Sigma\Gamma & 0 & \Sigma^2 \end{bmatrix}$$

satisfy that

$$\lambda^2 - 2\lambda + 1 - \sigma_i^2 = 0, \lambda = 1 \pm \sigma_i, i = 1, 2, \cdots r.$$

Therefore,

$$\rho = \frac{1}{2}(1 + \sigma_r),$$

i. e. ,

$$\sigma = (2\rho - 1)^2. \square$$

Theorem 3[2] If $H = H_1 + H_2$, then

$$\cos(H_1^\perp, H_2^\perp) = \cos(H_1, H_2).$$

Proof Obviously,

$$\cos(H_1, H_2) = \rho(P_{H_2}P_{H_1}), \cos(H_1^\perp, H_2^\perp) = \rho(P_{H_2^\perp}P_{H_1^\perp}).$$

Let $\dim H = n$, $\dim H_1 = r$, $\dim H_2 = s$ with $r \le s$. By the same argument above, define $P_i = T' P_{H_i} T$, $i = 1, 2$. Then

$$\rho(P_{H_2}P_{H_1}) = \rho(P_2P_1) = \sigma_r^2,$$

and

$$\rho(P_{H_2^\perp}P_{H_1^\perp}) = \rho((I - P_2)(I - P_1)) = \rho\left(\begin{bmatrix} 0 & 0 & -\Gamma\Sigma \\ 0 & 0 & 0 \\ 0 & 0 & I - \Sigma^2 \end{bmatrix}\right) = \sigma_r^2.$$

The result then follows. $\square$

Theorem 3 leads to the following result which is widely used in multigrid method:

Theorem 4[4] *Let $H = H_1 + H_2$. If there exists $\upsilon \in n(0, 1)$ satisfying*

$$|(v, w)| \le \upsilon \|v\| \|w\|, v \in H_1, w \in H_2, \tag{6}$$

and u is optimal on $H_1 + H_2$, i. e. , $P_{H_2}u = 0$. Then

$$\|u - P_{H_1}u\| \le \upsilon \|u\|.$$

Proof *It follows from Theorem 3 and the conditon (6) that*

$$\cos(H_2^\perp, H_1^\perp) \le \upsilon,$$