Journal of Applied Analysis and Computation
Volume 9, Number 2, April 2019, 601–614
Website:http://jaac.ijournal.cn
DOI:10.11948/2156-907X.20180119

CONTROL AND ADAPTIVE MODIFIED FUNCTION PROJECTIVE SYNCHRONIZATION OF LIU CHAOTIC DYNAMICAL SYSTEM

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Abstract In this work, the feedback control method is proposed to control the behaviour of Liu chaotic dynamical system. The controlled system is stable under some conditions on the parameters of the system determined by Routh-Hurwitz criterion. This paper also presents the adaptive modified function projective synchronization (AMFPS) between two identical Liu chaotic dynamical systems. Based on the Lyapunov stability theorem, adaptive control laws are designed to achieving the AMFPS. Finally, some numerical simulations are obtained to validate the proposed methods.

Keywords Feedback control method, Liu chaotic dynamical system, Routh-Hurwitz criterion, Lyapunov stability, projective synchronization.

MSC(2010) 37D06, 37D45, 46N40.

1. Introduction

Control and synchronization have attracted increasing attention due to their applications in biological networks, chemical reactors, physics and secure communication [34]. Controlling chaos has become a challenging topic in nonlinear dynamical, which consists in adding an input control to attempt stabilizing an unstable equilibrium point. This input control can be set using linear and nonlinear feedback control, adaptive control and active control [1, 6, 8, 17, 19, 25, 31].

Different kind of chaos synchronization have been studied such that complete synchronization, anti-phase synchronization, generalized synchronization and projective synchronization [3, 5, 10–14, 16, 18, 21, 23, 24, 27–29, 33, 35–37]. Projective synchronization has been an active area of research and improved by many authors. In recent years, modified projective synchronization is introduced in [20], where the responses of the synchronized dynamical states synchronize up to a constant scaling matrix [4, 26, 32]. Later, a new synchronization method called function projective synchronization is proposed, see [2, 7, 9, 15, 30, 38]. It means that the drive and

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response systems could be synchronized up to a scaling function. More recently, a new type of synchronization phenomenon is discussed, modified function projective synchronization, where the drive and response systems could be synchronized up to desired scaling function matrix [39, 40]. Particularly, MFPS is the more general definition of MPS and FPS when the scaling function matrix is chosen by a constant matrix and a scaling function, respectively.

The object of this paper is to guide the chaotic trajectories for Liu chaotic dynamical system and to study the modified function project synchronization (MFPS) of two identical Liu chaotic system with known parameters.

The rest of this paper is organized as follows: In Section 2, we provide a description of the Liu chaotic systems. In Section 3, the feedback control of Liu chaotic system is presented. In Section 4, we discuss the modified function projective synchronization of identical Liu chaotic systems. In Section 5, numerical results are given to demonstrate the effectiveness of the proposed methods. Conclusion is obtained in the final section.

2. Liu chaotic system

Liu chaotic system [22] is described by the following system of differential equations:

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = bx - kxz, \\ \dot{z} = -cz + hx^2, \end{cases} \tag{2.1}$$

where the parameters a, b, c, h and k are positive real constants. For example, a chaotic attractor for the parameters $a = 10$, $b = 40$, $c = 2.5$, $k = 1$ and $h = 4$ is shown in Figure 1.

2.1. Equilibrium points and stability

The equilibrium point and stability have been addressed in [22], here we a give a direct analysis of the equilibrium points and stability analysis of the model.

By putting the right side of equation of system (2.1) equal to zero, that is;

$$\begin{cases} a(y - x) = 0, \\ bx - kxz = 0, \\ -cz + hx^2 = 0. \end{cases}$$

This system has three equilibrium points:

$$\mathbf{P}_1 = (0, 0, 0), \quad \mathbf{P}_{2,3} = \Big(\pm \sqrt{cb/hk}, \pm\sqrt{cb/hk}, \frac{b}{k}\Big).$$

The eigenvalues at each equilibrium point can be obtained as shown in Table 1.

From Table 1, we show that all the equilibrium points are unstable, since at least one eigenvalue has positive real part for each equilibrium point.

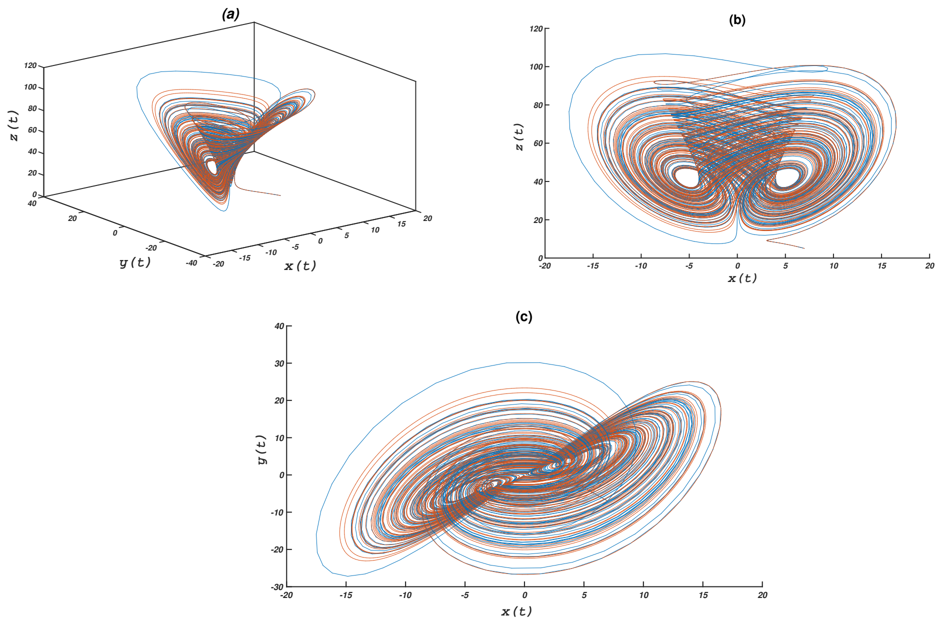

Figure 1. Liu chaotic system in (a) 3-dimensional (b) x-z plane and (c) x-y plane where $a = 10$, $b = 40$, $c = 2.5$, $k = 1$ and $h = 4$.

Table 1. Eigenvalues and stability of equilibrium points

Equilibrium points	Eigenvalues	Stable/Unstable
$\mathbf{P}_1$	$\lambda_1 = 15.615$, $\lambda_2 = -25.615$, $\lambda_3 = -2.5$	Unstable
$\mathbf{P}_2$	$\lambda_1 = -17.561$, $\lambda_2 = 2.53 + 10.367i$, $\lambda_3 = 2.53 - 10.367i$	Unstable
$\mathbf{P}_3$	$\lambda_1 = -17.561$, $\lambda_2 = 2.53 + 10.367i$, $\lambda_3 = 2.53 - 10.367i$	Unstable

3. Controlling Liu System

In order to control the Liu system to the unstable fixed points (x_i, y_i, z_i) for $i = 1, 2, 3$, we introduce the feedback control to guide the chaotic trajectory $(x(t), y(t), z(t))$ to the unstable equilibrium points (x_i, y_i, z_i) for $i = 1, 2, 3$. Let system (2.1) be controlled by the following:

$$\begin{cases} \dot{x} = a(y - x) - k_{i1}(x - x_i), \\ \dot{y} = bx - kxz - k_{i2}(y - y_i), \\ \dot{z} = -cz + hx^2 - k_{i3}(z - z_i), \end{cases} \tag{3.1}$$

where $i = 1, 2, 3$.

3.1. First

For $i = 1$, the controlled system (3.1) has one equilibrium point $(x_1, y_1, z_1) = (0, 0, 0)$. Then, control system (3.1) by a linear feedback control of the form:

$$\begin{cases} \dot{x} = a(y - x) - k_{11}(x - x_1), \\ \dot{y} = bx - kxz - k_{12}(y - y_1), \\ \dot{z} = -cz + hx^2 - k_{13}(z - z_1). \end{cases} \tag{3.2}$$

The controlled system (3.2) has one equilibrium point (x_1, y_1, z_1). We linearize (3.2) about this equilibrium point. Then, the linearized system is given by:

$$\begin{cases} \dot{X} = -(a + k_{11})X + aY, \\ \dot{Y} = -k_{12}Y + (b - kz_1)X - kx_1 Z, \\ \dot{Z} = -(c + k_{13})Z + 2hx_1 X, \end{cases} \tag{3.3}$$

where $(x_1, y_1, z_1) = (0, 0, 0)$, that is;

$$\begin{cases} \dot{X} = -(a + k_{11})X + aY, \\ \dot{Y} = -k_{12}Y + bX, \\ \dot{Z} = -(c + k_{13})Z. \end{cases} \tag{3.4}$$

Lemma 3.1. *The zero solution of the linearized system (3.4) is asymptotically stable whenever the conditions on the gained matrix are $k_{11} = k_{13} = 0$ and $k_{12} > b$.*

Proof. The Jacobian matrix of (3.4) is given by:

$$J = \begin{bmatrix} -a & a & 0 \\ b & -k_{12} & 0 \\ 0 & 0 & -c \end{bmatrix},$$

the eigenvalues of the Jacobian matrix satisfy the equation:

$$(-c - \lambda)\Big[(-a - \lambda)(-k_{12} - \lambda) - ab\Big] = 0,$$

then

$$(-c - \lambda)\Big(\lambda^2 + (a + k_{12})\lambda + ak_{12} - ab\Big) = 0,$$

we then obtain

$$\lambda_1 = -c \text{ and } \lambda_{2,3} = \frac{-(a + k_{12}) \pm \sqrt{(a + k_{12})^2 - 4(ak_{12} - ab)}}{2}$$

It holds that $\lambda_1 < 0$ and $\lambda_{2,3} < 0$. This implies that the eigenvalues have negative real parts and therefore the zero solution of (3.4) is asymptotically stable. $\square$

3.2. Second

For $i = 2$, the controlled system (3.1) has one equilibrium point $(x_2, y_2, z_2) = \left(\sqrt{cb/hk}, \sqrt{cb/hk}, \frac{b}{k}\right)$. Let system (3.1) be controlled by a linear feedback control of the form:

$$\begin{cases} \dot{x} = a(y - x) - k_{21}(x - x_2), \\ \dot{y} = bx - kxz - k_{22}(y - y_2), \\ \dot{z} = -cz + hx^2 - k_{23}(z - z_2). \end{cases} \tag{3.5}$$

The controlled system (3.5) has one equilibrium point (x_2, y_2, z_2). We linearize (3.5) about this equilibrium point. Then the linearized system is given by:

$$\begin{cases} \dot{X} = -(a + k_{21})X + aY, \\ \dot{Y} = -k_{22}Y + (b - kz_2)X - kx_2Z, \\ \dot{Z} = -(c + k_{23})Z + 2hx_2X, \end{cases} \tag{3.6}$$

where $(x_2, y_2, z_2) = \left(\sqrt{cb/hk}, \sqrt{cb/hk}, \frac{b}{k}\right)$, that is;

$$\begin{cases} \dot{X} = -(a + k_{21})X + aY, \\ \dot{Y} = -k_{22}Y - k\sqrt{cb/hk}Z, \\ \dot{Z} = -(c + k_{23})Z + 2h\sqrt{cb/hk}X. \end{cases} \tag{3.7}$$

Lemma 3.2. *The zero solution of the linearized system (3.7) is asymptotically stable whenever the conditions on the gain matrix are* $k_{21} = k_{23} = 0$ *and* $k_{22} > 0$.

Proof. The proof of this lemma depends on the condition of Routh-Hurwitz. The Jacobian matrix of (3.7) is given by:

$$J = \begin{bmatrix} -a & a & 0 \\ 0 & -k_{22} & -k\sqrt{cb/hk} \\ 2h\sqrt{cb/hk} & 0 & -c \end{bmatrix},$$

the eigenvalues of the Jacobian matrix satisfy the equation:

$$\lambda^3 + (a + c + k_{22})\lambda^2 + (ac + ck_{22} + ak_{22})\lambda + ack_{22} + 2abc = 0,$$

where

$$\begin{aligned} a_1 &= a + c + k_{22}, \\ a_2 &= ac + ck_{22} + ak_{22}, \\ a_3 &= ack_{22} + 2abc. \end{aligned}$$

According to the Routh-Hurwitz condition, it holds that $a_1 > 0$, $a_1a_2 > a_3$ and $a_3 > 0$. This implies that the eigenvalues have negative real parts and therefore the zero solution of (3.7) is asymptotically stable. □

3.3. Third

For $i = 3$, the controlled system (3.1) has one equilibrium point $(x_3, y_3, z_3) = \left(-\sqrt{cb/hk}, -\sqrt{cb/hk}, \frac{b}{k}\right)$. Let system (3.1) be controlled by a linear feedback control of the form:

$$\begin{cases} \dot{x} = a(y-x) - k_{31}(x - x_3), \\ \dot{y} = bx - kxz - k_{32}(y - y_3), \\ \dot{z} = -cz + hx^2 - k_{33}(z - z_3). \end{cases} \tag{3.8}$$

The controlled system (3.8) has one equilibrium point (x_3, y_3, z_3). We linearize (3.8) about this equilibrium point. Then, the linearized system is given by:

$$\begin{cases} \dot{X} = -(a + k_{31})X + aY, \\ \dot{Y} = -k_{32}Y + (b - kz_3)X - kx_3 Z, \\ \dot{Z} = -(c + k_{33})Z + 2hx_3 X, \end{cases} \tag{3.9}$$

where $(x_3, y_3, z_3) = \left(-\sqrt{cb/hk}, -\sqrt{cb/hk}, \frac{b}{k}\right)$, that is;

$$\begin{cases} \dot{X} = -(a + k_{31})X + aY, \\ \dot{Y} = -k_{32}Y + k\sqrt{cb/hk}Z, \\ \dot{Z} = -(c + k_{33})Z - 2h\sqrt{cb/hk}X. \end{cases} \tag{3.10}$$

Lemma 3.3. *The zero solution of the linearized system (3.10) is asymptotically stable whenever the conditions on the gain matrix are* $k_{31} = k_{33} = 0$ *and* $k_{32} > 0$

Proof. The proof of this lemma depends on the condition of Routh-Hurwitz. The Jacobian matrix of (3.7) is given by:

$$J = \begin{bmatrix} -a & a & 0 \\ 0 & -k_{32} & +k\sqrt{cb/hk} \\ -2h\sqrt{cb/hk} & 0 & -c \end{bmatrix},$$

the eigenvalues of the Jacobian matrix satisfy the equation:

$$\lambda^3 + (a + c + k_{32})\lambda^2 + (ac + ck_{32} + ak_{32})\lambda + ack_{22} - 2abc = 0,$$

where

$$\begin{aligned} a_1 &= a + c + k_{32}, \\ a_2 &= ac + ck_{32} + ak_{32}, \\ a_3 &= ack_{32} + 2abc. \end{aligned}$$

According to the Routh-Hurwitz condition, it holds that $a_1 > 0$, $a_1 a_2 > a_3$ and $a_3 > 0$. This implies that the eigenvalues have negative real parts and therefore the zero solution of (3.7) is asymptotically stable. □

4. The adaptive modified function projective synchronization of chaotic systems

The drive system and the response system are defined as:

$$\begin{cases} \dot{x} = f(x), \\ \dot{y} = g(y) + U(t, x, y), \end{cases} \tag{4.1}$$

where $x, y \in \mathbf{R}^n$ are the state vectors, $f, g : R^n \to R^n$ are differentiable vector functions, $U(t, x, y)$ is a control function. Let the vector error state be:

$$e = y - \Lambda(t)x, \tag{4.2}$$

where $\Lambda(t)$ is an n^{th}-order diagonal matrix, *i.e.* $\Lambda(t) = diag\{\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)\}$ and $\alpha_i(t)$ are continuously differentiable functions, $\alpha_i(t) \neq 0$ for all t.

Definition 4.1. For the drive system (4.1) and the response system (4.2), it is said that system (4.1) and system (4.2) are modified function projective synchronization (MFPS), if there exists a scaling function matrix $\Lambda(t)$, such that

$$\lim_{t \to +\infty} \|e(t)\| = \lim_{t \to +\infty} \|y - \Lambda(t)x\| = 0.$$

For AMFPS of Liu chaotic system (2.1), the drive (or master) and response (or slave) systems are defined below, respectively,

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1), \\ \dot{y}_1 = bx_1 - kx_1 z_1, \\ \dot{z}_1 = -cz_1 + hx_1^2, \end{cases} \tag{4.3}$$

and Liu system as the response system is given by

$$\begin{cases} \dot{x}_2 = a(y_2 - x_2) + u_1, \\ \dot{y}_2 = bx_2 - kx_2 z_2 + u_2, \\ \dot{z}_2 = -cz_2 + hx_2^2 + u_3, \end{cases} \tag{4.4}$$

where u_1, u_2 and u_3 are the nonlinear controllers such that two chaotic systems can be synchronized in the sense of MFPS, i.e.;

$$\begin{cases} \lim\limits_{t \to +\infty} \|x_2 - (\alpha_{11}x_1 + \alpha_{12})x_1\| = 0, \\ \lim\limits_{t \to +\infty} \|y_2 - (\alpha_{21}y_1 + \alpha_{22})y_1\| = 0, \\ \lim\limits_{t \to +\infty} \|z_2 - (\alpha_{31}z_1 + \alpha_{32})z_1\| = 0. \end{cases} \tag{4.5}$$

The error dynamical system between (4.3) and (4.4) is

$$\begin{cases} \dot{e}_1 = -ae_1 + ay_2 - 2\alpha_{11}ax_1y_1 + \alpha_{11}ax_1^2 - \alpha_{12}ay_1 + u_1, \\ \dot{e}_2 = bx_2 - kx_2z_2 - 2\alpha_{21}bx_1y_1 + 2\alpha_{21}kx_1y_1z_1 - \alpha_{22}bx_1 \\ \qquad +\alpha_{22}kx_1z_1 + u_2, \\ \dot{e}_3 = -ce_3 + hx_2^2 + \alpha_{31}cz_1^2 - 2\alpha_{31}hz_1x_1^2 - \alpha_{32}hx_1^2 + u_3, \end{cases} \tag{4.6}$$

by defining state errors $e_1(t) = x_2 - (\alpha_{11}x_1 + \alpha_{12})x_1$, $e_2(t) = y_2 - (\alpha_{21}y_1 + \alpha_{22})y_1$, $e_3(t) = z_2 - (\alpha_{31}z_1 + \alpha_{32})z_1$.

The object is to find a control law $u_i (i = 1, 2, 3)$ for stabilizing the error variables of the system (4.4). For this, we propose the following control law:

$$\begin{cases} u_1 = -ay_2 + 2a\alpha_{11}x_1y_1 - a\alpha_{11}x_1^2 + a\alpha_{12}y_1, \\ u_2 = -bx_2 + kx_2z_2 + 2\alpha_{21}bx_1y_1 - 2\alpha_{21}kx_1y_1z_1 + \alpha_{22}bx_1 - \alpha_{22}kx_1z_1 \\ \qquad -5y_2 + 5\alpha_{21}y_1^2 + 5\alpha_{22}y_1, \\ u_3 = -hx_2^2 - \alpha_{31}cz_1^2 + 2\alpha_{31}hz_1x_1^2 + \alpha_{32}hx_1^2. \end{cases} \tag{4.7}$$

Then, we have the following theorem.

Theorem 4.1. *For given nonzero scalars $\alpha_i (i = 1, 2, 3)$, AMFPS between the two systems (4.3) and (4.4) will occur by the control input (4.7).*

Proof. Define a Lyapunov function:

$$V = \frac{1}{2}\Big(e_1^2 + e_2^2 + e_3^2\Big). \tag{4.8}$$

The differential of the Lyapunov function along the trajectory of error system (4.6) is:

$$\begin{aligned} \frac{dV}{dt} =& \dot{e}_1e_1 + \dot{e}_2e_2 + \dot{e}_3e_3 \\ =& e_1\Big(- ae_1 + ay_2 - 2\alpha_{11}ax_1y_1 + \alpha_{11}ax_1^2 - \alpha_{12}ay_1 + u_1\Big) \\ & + e_2\Big(bx_2 - kx_2z_2 - 2\alpha_{21}bx_1y_1 + 2\alpha_{21}kx_1y_1z_1 - \alpha_{22}bx_1 + \alpha_{22}kx_1z_1 + u_2\Big) \\ & + e_3\Big(- ce_3 + hx_2^2 + \alpha_{31}cz_1^2 - 2\alpha_{31}hz_1x_1^2 - \alpha_{32}hx_1^2 + u_3\Big). \end{aligned} \tag{4.9}$$

By substituting the the control input(4.7), it gives that:

$$\begin{aligned} \frac{dV}{dt} =& e_1\Big(- ae_1 + ay_2 - 2\alpha_{11}ax_1y_1 + \alpha_{11}ax_1^2 - \alpha_{12}ay_1 - ay_2 + 2a\alpha_{11}x_1y_1 \\ & - a\alpha_{11}x_1^2 + a\alpha_{12}y_1\Big) + e_2\Big(bx_2 - kx_2z_2 - 2\alpha_{21}bx_1y_1 + 2\alpha_{21}kx_1y_1z_1 \\ & - \alpha_{22}bx_1 + \alpha_{22}kx_1z_1 - bx_2 + kx_2z_2 + 2\alpha_{21}bx_1y_1 - 2\alpha_{21}kx_1y_1z_1 \\ & + \alpha_{22}bx_1 - \alpha_{22}kx_1z_1 - 5e_2\Big) + e_3\Big(- ce_3 + hx_2^2 + \alpha_{31}cz_1^2 - 2\alpha_{31}hz_1x_1^2 \\ & - \alpha_{32}hx_1^2 - hx_2^2 - \alpha_{31}cz_1^2 + 2\alpha_{31}hz_1x_1^2 + \alpha_{32}hx_1^2\Big) \\ \Rightarrow \frac{dV}{dt} =& - ae_1^2 - 5e_2^2 - ce_3^2. \end{aligned} \tag{4.10}$$

Then we have,

$$\frac{dV}{dt} = -e^T Pe, \tag{4.11}$$

where

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \quad P = \begin{bmatrix} a & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}.$$

Since $\dot{V}$ is negative definite therefore, the response system synchronize the drive system in the sense of AMFPS by the controller (4.7). This completes the proof. $\square$

5. Numerical results

5.1. Control for Liu chaotic system

In this section, some numerical simulation results are presented to verify the previous analytical results where $a = 10,\ b = 40,\ c = 2.5,\ k = 1$ and $h = 4$. Figure 2 shows the convergence of the trajectory of the controlled system to the unstable equilibrium point $(x_1, y_1, z_1) = (0, 0, 0)$ of the uncontrolled system (2.1). While Figure 3 shows the convergence of the trajectory of the controlled system to the unstable equilibrium point $(x_2, y_2, z_2) = \left(\sqrt{cb/hk}, \sqrt{cb/hk}, \frac{b}{k}\right)$ of the uncontrolled system (2.1). Figure 4 shows the convergence of the trajectory of the controlled system to the unstable equilibrium point $(x_3, y_3, z_3) = \left(-\sqrt{cb/hk}, -\sqrt{cb/hk}, \frac{b}{k}\right)$ of the uncontrolled system (2.1).

5.2. Synchronization for Liu chaotic system

In this section, some numerical simulation results are presented to verify the previous analytical results where $a = 10, b = 40, c = 2.5, k = 1$ and $h = 4$. The initial values of the drive system and response system are taken as:

$$\Big(x_1(0), y_1(0), z_1(0)\Big)^T = (7, -7, 5)^T, \quad \Big(x_2(0), y_2(0), z_2(0)\Big)^T = (20, -3, 16)^T.$$

We choose the scaling functions as:

$$\alpha_1 = 2x_1 + 1,\ \alpha_2 = y_1 + 2 \ \text{ and } \ \alpha_3 = 2z_1 + 3.$$

Figure 5 shows the AMFPS between two identical Liu systems. When the scaling functions are simplified as $\alpha_1 = 2x_1 + 2,\ \alpha_2 = 2y_1 + 2$ and $\alpha_3 = 2z_1 + 2$ the MFPS between two identical Liu systems are shown in Figure 6. Figure 7 shows the AMPS between two identical Liu systems when the scaling factors are simplified as $\alpha_1 = 1,\ \ \alpha_2 = 2$ and $\alpha_3 = 3$. When $\alpha_i = 2$ for $(i = 1, 2, 3)$ the MPS between two identical Liu systems are shown in Figure 8. Furthermore, when the scaling factors are simplified as $\alpha_i = 1$ for $(i = 1, 2, 3)$, the complete synchronization between two identical Liu systems are shown in Figure 9. Finally, when the scaling factors are simplified as $\alpha_i = -1$ for $(i = 1, 2, 3)$ the anti synchronization between two identical Liu systems are shown in Figure 10.

Conclusions

In this paper, we have presented the feedback control to the Liu chaotic dynamical system. The controlling conditions are derived from Routh-Hurwitz criterion. In addition, we have investigated the AMFPS between two identical Liu chaotic dynamical system. The adaptive control laws are attained for stability of the error

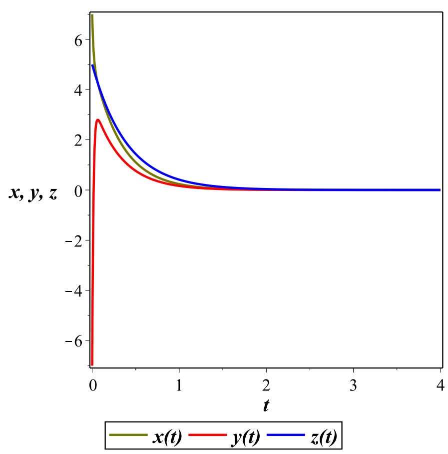

Figure 2. The time responses for the states of the controlled Liu system to a fixed point (x_1, y_1, z_1).

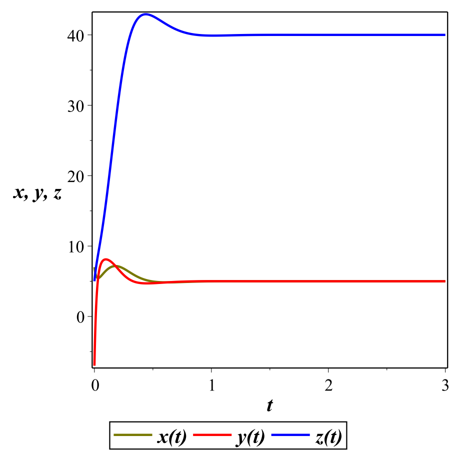

Figure 3. The time responses for the states of the controlled Liu system to a fixed point (x_2, y_2, z_2).

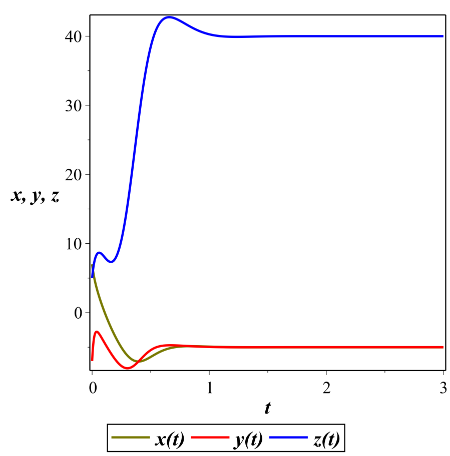

Figure 4. The time responses for the states of the controlled Liu system to a fixed point (x_3, y_3, z_3).

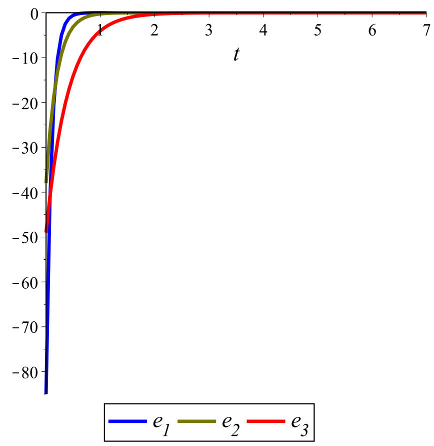

Figure 5. The behaviour of the trajectories $e_1,\ e_1$ *and* e_1 of the error system tends to zero for AMFPS.

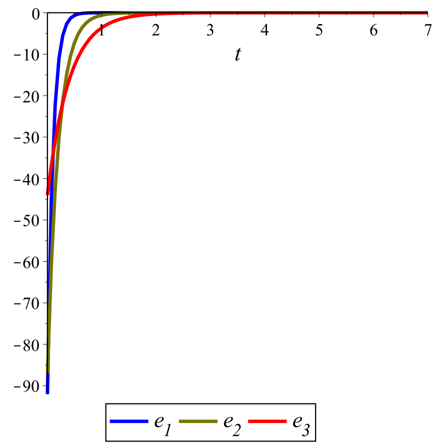

Figure 6. The behaviour of the trajectories e_1, e_1 and $e_1$of the error system tends to zero for MFPS.

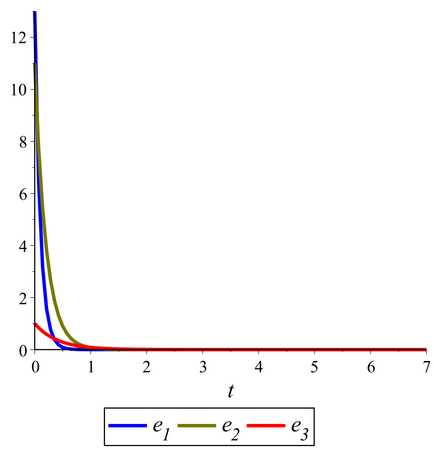

Figure 7. The behaviour of the trajectories e_1, e_1 and e_1 of the error system tends to zero for AMPS.

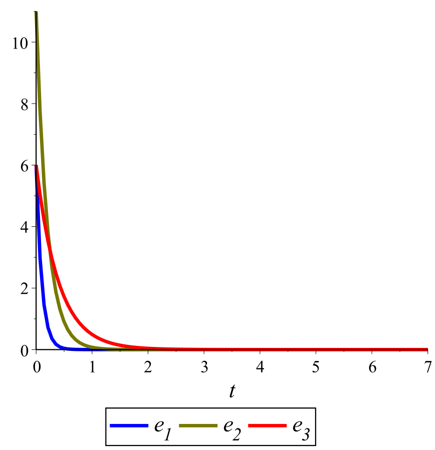

Figure 8. The behaviour of the trajectories e_1, e_1 and e_1 of the error system tends to zero for MPS.

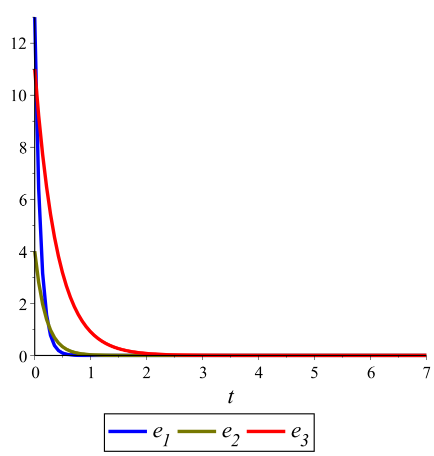

Figure 9. The behaviour of the trajectories $e_1,\ e_1$ and e_1 of the error system tends to zero for complete synchronization.

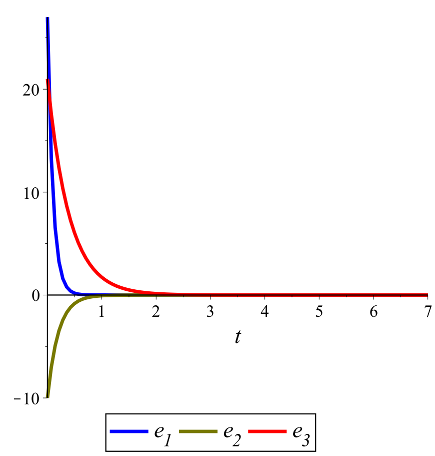

Figure 10. The behaviour of the trajectories $e_1,\ e_1$ and e_1 of the error system tends to zero for anti phase synchronization.

dynamical system by using Lyapunov stability theory. The results are verified by numerical simulations.

Acknowledgements. The authors are grateful to the anonymous referees for their useful suggestions which improve the contents of this article.

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