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A Survey of Numerical Method for Solitary Waves (Ⅱ)*

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III Finite Element Methods

We take element $I_j=[jh-h,\ jh],I=\sum_{j=1}^{N}I_j$. Let Φ_h and Ψ_h be the trail function space and test function space respectively with basis $\varphi_j(x)$ and $\psi_j(x)$. We suppose

$$U^h(x,t)=\sum_{j=1}^{N}U_j(t)\varphi_j(x),\quad x\in I,\ t\geqslant 0.$$

The usual Galerkin method is to find $U^h\in L^\infty(0,T;\Phi_h)$ satisfying

$$\begin{cases}\int_0^1\left(\frac{\partial U^h}{\partial t}V-\frac{1}{2}(U^h)^2\frac{\partial V}{\partial x}+\frac{\partial U^h}{\partial x}\ \frac{\partial^2V}{\partial x^2}\right)dx=0,\ \forall\ V\in\Phi_h,\\ U^h(x,0)=U_0^h(x)\in\Phi_h,\Phi_h\subset H^2.\end{cases}\tag{21}$$

In order to improve the stability and convergence, the dissipative finite element scheme is used, i. e. to find $U^h\in L^\infty(0,T;\Phi_h)$ such that

$$\begin{cases}\int_0^1\left(\frac{\partial U^h}{\partial t}+U^h\frac{\partial U^h}{\partial x}+\frac{\partial^3U^h}{\partial x^3}\right)\left(V+\alpha h^3\frac{\partial^3V}{\partial x^3}\right)dx=0,\ \forall\ V\in\Phi_h,\\ U^h(x,0)=U_0^h(x)\in\Phi_h,\Phi_h\subset H^3.\end{cases}\tag{22}$$

Now let μ and k be integers, $\mu>k\geqslant 0$,

$$S_h(\mu,k)=\{V(x)/V(x+1)=V(x),\quad V(x)\in C^k(\mathbb{R})$$
$$V(x)|I_j \text{ is } \mu \text{ degree polynomail},\ 1\leqslant j\leqslant N\}.$$

Wahlbin (1974) adopted scheme (22) with $\alpha=1$, $\Phi_h=S_h(\mu,k)$, $k\geqslant 2$, and obtained the following result:

Theorem 2 If $\Phi_h=S_h(\mu,k)$, $k\geqslant 2$, $\alpha=1$, T is fixed,

$$\int_0^1(U_0-U_0^h)^2dx=O(h^{2\mu+2}),$$

then for all $t\leqslant T$, we have

$$\int_0^1(U(x,t)-U^h(x,t))^2dx=O(h^{2\mu+2}).$$

Alexander & Morris (1979) made numerical experiments for different values of α. Since the computation is too complicated with $\mu\geqslant 3$, hence Winther (1980) rewrote (1) as follows

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$$\frac{\partial U}{\partial t}-\frac{\partial W}{\partial x}=0,$$
$$W=-\left(\frac{\partial^2 U}{\partial x^2}+\frac{U^2}{2}\right), \tag{23}$$

then to find $(U^h, W^h)\in L^\infty(0,T;\Phi_h)\times L^\infty(0,T;\Phi_h)$ such that

$$\frac{\partial U^h}{\partial t}-\frac{\partial W^h}{\partial x}=0,$$
$$\int_0^1 W^h V dx=\int_0^1\left(\frac{\partial U^h}{\partial x}\frac{\partial V}{\partial x}-\frac{(U^h)^2}{2}V\right)dx=0,\quad \forall\ V\in\Phi_h, \tag{24}$$
$$U^h(x,0)=U_0^h(x)\in\Phi_h,\quad \Phi_h\subset H^1.$$

Recently two new techniques were used for saving work.

(i) Petrov-Galerkin method, i. e. $\Phi_h\neq\Psi_h$,

(ii) Product Approximation Method, i. e. we take $\sum_{j=1}^N g(U_j^h(t))\varphi_j(x)$ as the approximation of $g(U)$. For simplicity we denote

$$\dot U_l^h(t)=\frac{dU_l^h(t)}{dt},\qquad \varphi_l'(x)=\frac{\partial\varphi_l(x)}{\partial x},\text{etc}$$

then the product Petrov-Galerkin Method for (1) is as follows:

$$\sum_{l=1}^N\int_0^1\left[\dot U_l(t)\varphi_l(x)\psi_j(x)-\frac{1}{2}U_l^2(t)\varphi_l(x)\psi_j'(x)+U_l(t)\varphi_l'(x)\psi_j''(x)\right]dx=0. \tag{25}$$

In the following discussion, we put $\Phi_h=S_h(1,0)$, i. e.

$$\varphi_j(x)=\varphi\left(\frac{x}{h}-j\right),$$
$$\varphi(x)=\begin{cases}1-|x|, & x<1,\\ 0 & x\geq 1.\end{cases}$$

Sanz-Serna & Christie (1981) took $\Psi_h=S_h(3,\ 2)$ which is the smoothest spline function space, $\psi_j(x)$ has support with length 4h. After computation we get from (25)

$$\begin{aligned}&\frac{1}{120}(\dot U_{j+2}^h+26\dot U_{j+1}^h+66\dot U_j^h+26\dot U_{j-1}^h+\dot U_{j-2}^h)\\&+\frac{1}{48h}[(U_{j+2}^h)^2+10(U_{j+1}^h)^2-10(U_{j-1}^h)^2-(U_{j-2}^h)^2]\\&+\frac{1}{2h^3}(U_{j+2}^h-2U_{j+1}^h+2U_{j-1}^h-U_{j-2}^h)=0.\end{aligned} \tag{26}$$

Schoombie (1980) and Mitchell & Schoombie (1981) took Ψ_h as shift function space, i. e.

$$\psi_j(x)=\varphi_{j-a}(x),\qquad 0<a\leq\frac{1}{2},$$

then it follows from (25) that

$$\sum_{l=1}^N\int_{-\infty}^{\infty}[\dot U_l(t)\varphi_l(x)+\frac{1}{2}U_l^2(t)\varphi_l'(x)+U_l(t)\varphi_l'''(x)]\varphi_{j-a}(x)dx=0.$$

After computation, we get

$$\sum_{i=-2}^{1}[a_i\dot U_{j+i}^h(t)+b_i(U_{j+i}^h(t))^2+c_iU_{j+i}^h(t)]=0,$$

where

$$a_l=\int_{-\infty}^{\infty}\varphi_{j+l}(x)\varphi_{j-a}(x)dx,\quad b_l=\int_{-\infty}^{\infty}\varphi'_{j+l}(x)\varphi_{j-a}(x)dx,\quad c_l=\int_{-\infty}^{\infty}\varphi'''_{j+l}(x)\varphi_{j-a}(x)dx.$$

Now let

$$V*W(x)=\int_{-\infty}^{\infty}V(\xi)W(x-\xi)d\xi,$$

and $B_l(x)$ be B-spline function (See Schoenberg (1966)), i.e.

$$B_0(x)=\begin{cases}1, & |x|<\frac{1}{2},\\ 0, & |x|\geqslant\frac{1}{2},\end{cases}$$

$$B_l(x)=B_0*B_{l-1}(x),\quad l\geqslant1,$$

then

$$a_l=\int_{-\infty}^{\infty}\varphi\left(\frac{x}{h}-j-l\right)\varphi\left(\frac{x}{h}-j+a\right)dx=\int_{-\infty}^{\infty}\varphi(\xi-(a+l))\varphi(\xi)d\xi$$

$$=\varphi*\varphi(a+l)=B_1*B_1(a+l)=B_0*B_0*B_1(a+l)=B_l*B_2(a+l)=B_3(a+l).$$

Similarly

$$b_l=B'_3(a+l),\quad c_l=B'''_3(a+l).$$

The values of a_l, b_l and c_l are given in table 2.

	$l=-2$	$l=-1$	$l=0$	$l=1$
a_l	$\frac{a^3}{6}$	$\frac{1}{6}(1+3a+3a^2-3a^3)$	$\frac{1}{6}(4-6a^2+3a^3)$	$\frac{1}{6}(1-a)^3$
b_l	$\frac{a^2}{2}$	$\frac{1}{2}(1-a)(1+3a)$	$\frac{a}{2}(-4+3a)$	$-\frac{1}{2}(1-a)^2$
c_l	1	-3	3	-1

Table 2

Finite element method was used for R. L. W. equation also, e. g.

$$\begin{cases}\int_0^l\left(\frac{\partial U^h}{\partial t}+a\frac{\partial U^h}{\partial x}+U^h\frac{\partial U^h}{\partial x}-\beta\frac{\partial^3U^h}{\partial x^2\partial t}\right)\left(V+h\frac{\partial V}{\partial x}\right)dx=0,\quad\forall\, V\in\Phi_h,\\ U^h(x,0)=U_0^h(x)\in\Phi_h,\quad \Phi_h\subset H^2.\end{cases}\tag{27}$$

The detail could be found in Alexander & Morris (1979) and Bona, Pritchard & Scott (1980).

For nonlinear Klein-Gordon equation, Kuo Pen-yu (1982) proposed the scheme as follows

$$\begin{cases}\int_0^1\left[\frac{\partial^2U^h}{\partial t^2}V+\frac{\partial U^h}{\partial x}\frac{\partial V}{\partial x}-U^hV+(U^h)^3V\right]dx=0,\quad\forall\, V\in\Phi^h,\\ \left.\frac{\partial U^h(x,t)}{\partial t}\right|_{t=0}=U_1^h(x)\in\Phi_h,\\ U^h(x,0)=U_0^h(x)\in\Phi_h,\quad \Phi_h\subset H^1\cap L^4.\end{cases}\tag{28}$$

By using the technique given in Kuo Pen-yu (1982), we can obtain the strict error estimation.

Manoranjan (1980) used finite element method for Sine-Gordon equation. i.e. to find $U^h\in L^\infty(0,T;\Phi_h)$ such that

$$\begin{cases} \int_0^1 \left(\frac{\partial^2 U^h}{\partial t^2} V + \frac{\partial U^h}{\partial x} \frac{\partial V}{\partial x} + \sin U^h \cdot V \right) dx = 0, \quad \forall V \in \Phi_h, \\ \left. \frac{\partial U^h(x,t)}{\partial t} \right|_{t=0} = U_1^h(x) \in \Phi_h, \\ U^h(x,0) = U_0^h(x) \in \Phi_h, \quad \Phi_h \subset H^1. \end{cases} \tag{29}$$

Clearly we can use high degree finite element to get the scheme with high order accuracy. But they are always implicit. So we must pay more work by using finite element than by using difference method. On the other hand, the numerical results showed that the finite difference scheme is better sometimes than finite element even though the high degree element is applied.

IV Spectral methods

A numerical procedure competitive with the finite difference methods and finite element methods is the Fourier expansion method.

The hybrid methods where the Fourier expansion is partially employed have been proposed by Gazdag (1973), Tappert (1974), Canosa & Gazdag (1977) and others. Canosa & Gazdag (1977) considered the following equation.

$$\begin{cases} \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} - \gamma \frac{\partial^2 U}{\partial x^2} + \delta \frac{\partial^3 U}{\partial x^3} = 0, & \gamma > 0, \delta > 0, x \in \mathbb{R}, \ t > 0, \\ \lim_{x \to -\infty} U(x,t) = 1, & t \geq 0, \\ \lim_{x \to +\infty} U(x,t) = 0, & t \geq 0, \\ U(x,0) = U_0(x), & x \in \mathbb{R}. \end{cases} \tag{30}$$

They put

$$U^h(x,t+\tau) = U^h(x,t) + \tau \frac{\partial U^h(x,t)}{\partial t} + \frac{\tau^2}{2} \frac{\partial^2 U^h(x,t)}{\partial t^2} + \frac{\tau^3}{6} \frac{\partial^3 U^h(x,t)}{\partial t^3}, \tag{31}$$

and derived

$$\frac{\partial U^h}{\partial t} = -U^h \frac{\partial U^h}{\partial x} + \gamma \frac{\partial^2 U^h}{\partial x^2} - \delta \frac{\partial^3 U^h}{\partial x^3}, \tag{32}$$

$$\frac{\partial^2 U^h}{\partial t^2} = -\frac{\partial U^h}{\partial t} \frac{\partial U^h}{\partial x} - U^h \frac{\partial}{\partial x}\left(\frac{\partial U^h}{\partial t}\right) + \gamma \frac{\partial^2}{\partial x^2}\left(\frac{\partial U^h}{\partial t}\right) - \delta \frac{\partial^3}{\partial x^3}\left(\frac{\partial U^h}{\partial t}\right), \tag{33}$$

$$\begin{aligned} \frac{\partial^3 U^h}{\partial t^3} = & \frac{\partial^2 U^h}{\partial t^2} \frac{\partial U^h}{\partial x} - 2 \frac{\partial U^h}{\partial t} \frac{\partial}{\partial x}\left(\frac{\partial U^h}{\partial t}\right) - U^h \frac{\partial}{\partial x}\left(\frac{\partial^2 U^h}{\partial t^2}\right) \\ & + \gamma \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 U^h}{\partial t^2}\right) - \delta \frac{\partial^3}{\partial x^3}\left(\frac{\partial^2 U^h}{\partial t^2}\right), \end{aligned} \tag{34}$$

where the x-derivative terms in (32)-(34) are computed by Fourier methods, i. e. Let $U^h(n,t)$ be the finite Fourier transform of $U^h(x,t)$ defined over the computational domain. The l'th-order derivative of $U^h(x,t)$ is then given by

$$\frac{\partial^l U^h}{\partial x^l} = \sum_n (in)^l U^h(n,t) e^{inx},$$

where $i^2 = -1$, and the summation is carried out for all wave numbers n which can be represented over the computational mesh. This scheme gave more accurate result than usual difference scheme.

Schamel & Elsässer (1976), Watanabe, Ohishi, Tanaca (1977) devoted other methods. In these methods the unknown function is expanded in terms of the Fourier series and the original partial differential equation is reduced to a set of ordinary differential equations for the Fourier coefficients.

Abe & Inoue (1980) considered the equation (1) with $U_0(x)=\cos 2\pi x, 0\leqslant x\leqslant 1$. Let

$$U^h(x,t)=\sum_{l=-\infty}^{\infty} a_l(t)e^{2i\pi lx},$$

then

$$\frac{da_l(t)}{dt}=-2\pi i\sum_{l=-\infty}^{\infty} la_{l-m}(t)a_m(t)+8i\pi^3l^3a_l(t)$$
$$=-\pi li\sum_{m=-\infty}^{\infty} a_{l-m}(t)a_m(t)+8i\pi^3l^3a_l(t).$$

In order to make truncation, we require

$$a_l(t)=0, \qquad |l|\geqslant l_1.$$

Recently Kuo Pen-yu (1980) proposed an new scheme for periodic solution of problem (1) (assume the Period is 1). He defined

$$J(V(x,t_k),W(x,t_k))=\frac{1}{3}W(x,\ t_k)\frac{\partial V}{\partial x}(x,t_k)\ +\frac{1}{3}\frac{\partial}{\partial x}(W(x,t_k)V(x,t_k)),$$

and let

$$U^{(n)}(x,\ t_k)=\frac{a_0^{(n)}(t_k)}{2}+\sum_{l=1}^{n}a_l^{(n)}(t_k)\cos lx+b_l^{(n)}(t_k)\sin lx,$$

satisfying

$$\begin{cases}\displaystyle\int_0^1\Big[U_t^{(n)}(t_k)\varphi_l(x)+J(U^{(n)}(t_k)+\sigma\tau U_t^{(n)}(t_k),U^{(n)}(t_k))\varphi_l(x)+\gamma\frac{\partial}{\partial x}(U^{(n)}(t_k)\\ \quad +\sigma\tau U_t^{(n)}(t_k))\varphi_l'(x)-\delta\frac{\partial^2}{\partial x^2}(U^{(n)}(t_k)+\sigma\tau U_t^{(n)}(t_k))\varphi_l'(x)\Big]dx=0,\\ U^{(n)}(x,0)=\dfrac{A_0}{2}+\displaystyle\sum_{l=1}^{n}A_l\cos lx+B_l\sin lx,\end{cases}$$

where $\varphi_l=\sin lx$ or $\cos lx$, A_l and B_l are the coefficients of Fourier transform of initial value $U_0(x)$. Kuo Pen-yu gave a series of strict error estimation, from which the convergences followed.

Another method is given by Fornberg & Whitham (1978).

Spectral method has the following advantages:

(i) We get always explicit scheme because of orthogonality of basis functions.

(ii) If we use finite difference method, then the accuracy of approximate solution is limited for the fixed scheme, even though the solution of partial differential equation is very smooth. If we use spectral method, then the more smooth the solution, the more accurate the approximate solution.

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