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An Equality for Trace of Matrix over a Generalized Quaternion Algebra

CHENG Shi-zhen[1,2], *TIAN Yong-ge*[3]

(1. Dept. of Basic Sci., Beijing Institute of Civil Engineering and Architecture, Beijing 100044, China;
2. School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China;
3. Dept. of Math. and Statistics, Queen's University Kingston, Ontario, Canada K7L 3N6)
(E-mail: shizhen122@sohu.com)

Abstract: The well-known trace equality of similar matrices does not necessarily hold for matrices over non-commutative algebras and rings. An interesting question is to give conditions such that trace equality of similar matrices holds for matrices over a non-commutative algebra or ring. In this note, we show that for any two matrices A and B over a generalized quaternion algebra defined on an arbitrary field $\mathbf{F}$ of characteristic not equal to two, if A and B are similar and the main diagonal elements of A and B are in $\mathbf{F}$, then their traces are equal.

Key words: generalized quaternion algebra; matrix; similarity; trace.
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Let $\mathbf{F}$ be an arbitrary field of characteristic not equal to two, and let u and v be two nonzero elements of $\mathbf{F}$. A generalized quaternion algebra $(\frac{u,v}{\mathbf{F}})$ is a four dimensional vector space over $\mathbf{F}$ with a basis 1, i, j and k satisfying the multiplication rules

$$i^2 = u, \quad j^2 = v, \quad k = ij = -ji, \tag{1}$$

where 1 acts as unity element[1−3]. Element of $(\frac{u,v}{\mathbf{F}})$ is written as

$$a = a_0 + a_1 i + a_2 j + a_3 k, \ \ a_0,\ a_1,\ a_2,\ a_3 \in \mathbf{F}.$$

According to this definition, the generalized quaternion algebra $(\frac{u,v}{\mathbf{F}})$ is a four-dimensional associative algebra with its central field $\mathbf{F}$. In particular, when $\mathbf{F}$ is the real field $\mathbf{R}$ and $u = v = -1$, $(\frac{u,v}{\mathbf{F}})$ is the well-known Hamilton quaternion algebra; when $\mathbf{F} = \mathbf{R}$, $u = 1$ and $v = -1$, $(\frac{u,v}{\mathbf{F}})$ is the real split quaternion algebra; when $\mathbf{F}$ is the complex field $\mathbf{C}$ and $u = v = -1$, $(\frac{u,v}{\mathbf{F}})$ is called the well-known complex quaternion algebra.

As usual, two matrices A and B over $(\frac{u,v}{\mathbf{F}})$ are said to be similar if there is an invertible matrix X over $(\frac{u,v}{\mathbf{F}})$ such $XAX^{-1} = B$, and is denoted by $A \sim B$. The trace of a square matrix $A = (a_{st})$ of order n over $(\frac{u,v}{\mathbf{F}})$ is defined by $\mathrm{tr}(A) = \sum_1^n a_{ss}$. The trace of a square matrix over is one of the simplest concepts in linear algebra, which has various remarkable applications in many fields of mathematics. The calculation of many mathematics quantities can be reduced to the calculation of traces of square matrices.

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Since $(\frac{u,v}{\mathbf{F}})$ is not non-commutative, that is, $ab \neq ba$ for $a,\ b \in \mathbf{F}$ in general, the trace equality $\mathrm{tr}(MN) = \mathrm{tr}(NM)$ does not hold for matrices over $(\frac{u,v}{\mathbf{F}})$. Consequently, $\mathrm{tr}(XAX^{-1}) = \mathrm{tr}(A)$ does not hold for A and X over $(\frac{u,v}{\mathbf{F}})$. A natural problem in the investigation into the trace of matrices over $(\frac{u,v}{\mathbf{F}})$ is to determine necessary and sufficient conditions for $\mathrm{tr}(XAX^{-1}) = \mathrm{tr}(A)$ to hold. A direct motivation for us to consider this topic arises from extending various trace equalities and inequalities for complex matrices to real quaternion matrices.

The main result of this note is:

Theorem 1 *Let $A = (a_{st})$ and $B = (b_{st})$ be a pair of square matrices of order n over $(\frac{u,v}{\mathbf{F}})$. If $A \sim B$ and $a_{ss},\ b_{ss} \in \mathbf{F}$ for $s = 1, \cdots, n$, then $\mathrm{tr}(A) = \mathrm{tr}(B)$ holds.*

$A \sim B$ implies that there is an invertible matrix X over $(\frac{u,v}{\mathbf{F}})$ such that $XAX^{-1} = B$. Thus, $\mathrm{tr}(XAX^{-1}) = \mathrm{tr}(B)$. But $\mathrm{tr}(XAX^{-1}) = \mathrm{tr}(AX^{-1}X)$ is not true in general.

Our method to prove Theorem 1 is to convert matrices over $(\frac{u,v}{\mathbf{F}})$ to matrices over $\mathbf{F}$.

Lemma 2[4,6] *Let $A = A_0 + A_1 i + A_2 j + A_3 k$ be an $m \times n$ matrix over $(\frac{u,v}{\mathbf{F}})$, where $A_0, \cdots, A_3$ are $m \times n$ matrices over $\mathbf{F}$. Then the diagonal block matrix $\mathrm{diag}(\,A,\ A,\ A,\ A\,)$ satisfies the following universal factorization equality*

$$J_{4m}\mathrm{diag}(\,A,\ A,\ A,\ A\,)J_{4n} = \begin{bmatrix} A_0 & uA_1 & vA_2 & -uvA_3 \\ A_1 & A_0 & vA_3 & -vA_2 \\ A_2 & -uA_3 & A_0 & uA_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} \stackrel{\mathrm{def}}{=} \Phi(A), \tag{2}$$

where

$$J_{4t} = J_{4t}^{-1} = \frac{1}{2}\begin{bmatrix} I_t & iI_t & jI_t & kI_t \\ u^{-1}iI_t & I_t & -u^{-1}kI_t & -jI_t \\ v^{-1}jI_t & v^{-1}kI_t & I_t & iI_t \\ -(uv)^{-1}kI_t & -v^{-1}jI_t & u^{-1}iI_t & I_t \end{bmatrix}, \quad t = m,\ n.$$

In particular, when $m = n$, (2) becomes a universal similarity factorization equality over $(\frac{u,v}{\mathbf{F}})$.

Lemma 3[4] *Let A and B be $m \times n$ matrices over $(\frac{u,v}{\mathbf{F}})$, C be an $n \times p$ matrix over $(\frac{u,v}{\mathbf{F}})$ and $\lambda \in \mathbf{F}$. Then*

(a) $A = B \Leftrightarrow \Phi(A) = \Phi(B)$.

(b) $\Phi(A + B) = \Phi(A) + \Phi(B)$.

(c) $\Phi(AC) = \Phi(A)\Phi(C)$.

(d) $\Phi(\lambda A) = \Phi(A\lambda) = \lambda\Phi(A)$.

(e) $\Phi(I_m) = I_{4m}$.

(f) If A is invertible, then $\Psi(A^{-1}) = \Psi^{-1}(A)$.

The matrix $\Phi(A)$ in (2) is often called the central representation of A in the literature. If $A \sim B$, i.e., there is an X such that $XAX^{-1} = B$, then applying Lemma 3 (a), (c) and (f) to its both sides yields

$$\Phi(X)\Phi(A)\Phi^{-1}(X) = \Phi(B), \tag{3}$$

here $\Phi(A),\ \Phi(B)$ and $\Phi(X)$ are matrices over $\mathbf{F}$. Hence

$$\mathrm{tr}[\Phi(A)] = \mathrm{tr}[\Phi(B)] \tag{4}$$

holds. If

$$A = A_0 + A_1 i + A_2 j + A_3 k$$

and

$$B = B_0 + B_1 i + B_2 j + B_3 k,$$

then we see from (2) that (4) is equivalent to

$$\mathrm{tr}(A_0) = \mathrm{tr}(B_0). \tag{5}$$

From the above results, we can prove Theorem 1.

Proof of Theorem 1 Suppose

$$A = A_0 + A_1 i + A_2 j + A_3 k$$

and

$$B = B_0 + B_1 i + B_2 j + B_3 k.$$

Note that the main diagonal elements of A and B are all in $\mathbf{F}$ and the main diagonal elements of A_0 and B_0 are all in $\mathbf{F}$. Hence, the main diagonal elements of $A_1,\ A_2,\ A_3$ and $B_1,\ B_2,\ B_3$ must all be zero. In this case,

$$\mathrm{tr}(A) = \mathrm{tr}(A_0), \quad \text{and} \quad \mathrm{tr}(B) = \mathrm{tr}(B_0). \tag{6}$$

On the other hand, since $A \sim B$, we also have (5). The combination of (5) and (6) yields $\mathrm{tr}(A) = \mathrm{tr}(B)$, the desired result. $\square$

Remark 4 From the given condition in Theorem 1 we can only derive that $\mathrm{tr}(A) = \mathrm{tr}(B)$, but cannot derive that $\mathrm{tr}(A^l) = \mathrm{tr}(B^l)$ for $l \geq 2$. In fact, although we know from $XAX^{-1} = B$ that $XA^lX^{-1} = B^l$, the main diagonal elements of A^l and B^l may not be in $\mathbf{F}$. In this case, we can say nothing about the relationship between $\mathrm{tr}(A^l)$ and $\mathrm{tr}(B^l)$ for $l \geq 2$.

When matrices are considered over the real quaternion algebra $\mathbf{H}$, it is well known that they have Jordan decompositions[7]. That is to say, for any $A \in \mathbf{H^{n\times n}}$, there is an invertible $X \in \mathbf{H^{n\times n}}$ such that $XAX^{-1} = J$, where J is a complex Jordan matrix, the diagonal entries of J are a set of complex right eigenvalues $\lambda_1, \ldots, \lambda_n$ of A. Applying Theorem 1 to real quaternion matrices and their Jordan decompositions, one can derive some more explicit conclusions for traces of real quaternion matrices and their right eigenvalues. In such cases, various well-known equalities and inequalities for traces of complex matrices can be extended to real quaternion matrices. For more details, see [5].

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关于广义四元数代数上矩阵的迹的一个等式

程士珍 [1,2], 田永革 [3]
(1. 北京建筑工程学院基础部，北京 100044; 2. 北京师范大学数学科学学院，北京 100875;
3. 加拿大皇后大学数学与统计系 K7L3N6)

摘要: 众所周知, 相似矩阵的迹相等对于非交换代数和环上的矩阵不一定成立. 有趣的问题是给定一个条件使得相似矩阵的迹相等对于非交换代数或非交换环上的矩阵成立. 本文对于特征不是 2 的任意域 **F** 上定义的广义四元数代数上的两个矩阵 A 和 B, 给出如果 A 和 B 相似并且它们的主对角线上的元素在 **F** 中, 那么它们的迹相等.

关键词: 广义四元数代数；矩阵；相似；迹.