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OSCILLATION OF NONLINEAR IMPULSIVE DELAY HYPERBOLIC EQUATION WITH FUNCTIONAL ARGUMENTS VIA RICCATI METHOD

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Abstract: In this paper, we mainly deal with the oscillation problems of nonlinear impulsive hyperbolic equation with functional arguments. By using integral averaging method and a generalized Riccati technique, a sufficient condition for oscillation of the solutions of nonlinear impulsive hyperbolic equation with functional arguments is obtained. We can make better use of some existing conclusions about oscillation of the solutions of impulsive ordinary differential equations with delay.

Keywords: oscillation; impulsive; delay; hyperbolic equation; Riccati inequality

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1 Introduction

The theories of nonlinear partial functional differential equations are applied in many fields. In recent years the research of oscillation to impulsive partial differential systems caught more and more attention. In this paper, we study the oscillation properties of the solutions to impulsive delay hyperbolic equation

$$\begin{aligned}\frac{\partial}{\partial t}((r(t)\frac{\partial}{\partial t}u(x,t)) &= a(t)h(u(x,t))\triangle u(x,t)-\sum_{i=1}^{n}b_i(t)h_i(u(x,\tau_i(t)))\triangle u(x,\tau_i(t))\\ &\quad+\sum_{j=1}^{m}q_j(x,t)\varphi_j(u(x,t)),\ t\neq t_k,(x,t)\in\Omega\equiv G\times(0,+\infty),\quad(1.1)\\ u(x,t_k^+)-u(x,t_k^-) &= \alpha_k u(x,t_k),\quad t=t_k,k=1,2,\cdots,\quad(1.2)\\ u_t(x,t_k^+)-u_t(x,t_k^-) &= \beta_k u_t(x,t_k),\quad t=t_k,k=1,2,\cdots.\quad(1.3)\end{aligned}$$

The following is the boundary condition

$$\frac{\partial u(x,t)}{\partial n}+\gamma(x,t)u(x,t)=0,\ \ (x,t)\in\partial G\times[0,+\infty).\tag{1.4}$$

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where G is a bounded domain of R^n with the smooth boundary ∂G and n is the unit exterior normal vector to ∂G.

Following are the basic hypothesis

(H1) $r(t) \in C([0,+\infty);(0,+\infty))$, $a(t), b_i(t) \in PC([0,+\infty);[0,+\infty))$, $i=1,2,\cdots,n$. $\gamma(x,t) \in C(R_+ \times \partial G, R_+)$. $q_j(x,t) \in C(\bar{\Omega};[0,+\infty))$, $j=1,2,\cdots,m$, where PC denotes the class of functions which are piecewise continuous in t with discontinuities of the first kind only at $t=t_k, k=1,2,\cdots$.

(H2) $\tau_i(t) \in C([0,+\infty);R)$, $\lim\limits_{t\to+\infty} \tau_i(t)=+\infty$, $i=1,2,\cdots,n$.

(H3) $h(u), h_i(u) \in C(R,R)$,$uh(u) \geq 0$,$uh'(u) \geq 0$, $uh_i'(u) \geq 0$, $i=1,2,\cdots,n$; $\varphi_j(s) \in C(R,R)$, $\frac{\varphi_j(s)}{s} \geq C_j = \text{const.} > 0$ for $s \neq 0$. $\alpha_k, \beta_k = \text{const.} > -1$, $0<t_1<t_2<\cdots<t_k<\cdots$, $\lim\limits_{t\to+\infty} t_k = +\infty$, $k=1,2,\cdots$.

We introduce the notations $U(t) = \displaystyle\int_G u(x,t)dx$ and $q_j(t) = \min\limits_{x\in\bar{G}} q_j(x,t)$.

Definition 1.1 The solution $u(x,t)$ of the problems (1.1)–(1.4) is said to be nonoscillatory in domain Ω if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.

Definition 1.2 We say that functions H_i, $i=1,2$, belong to a function class H, if $H_i \in C(D;[0,+\infty))$, $i=1,2$, satisfy

1. $H_i(t,s)=0$, $i=1,2$ for $t=s$,
2. $H_i(t,s)>0$, $i=1,2$ for $t>s$,

where $D=\{(t,s): 0<s\leq t<+\infty\}$. Moreover, the partial derivatives $\partial H_1/\partial s$ and $\partial H_2/\partial s$ exist on D such that

$$\frac{\partial H_1}{\partial s}(t,s) = h_1(t,s)H_1(t,s) \text{ and } \frac{\partial H_2}{\partial s}(t,s) = -h_2(t,s)H_2(t,s),$$

where $h_1, h_2 \in C_{loc}(D;\mathbb{R})$.

In recent years, there was much research activity concerning the oscillation theory of nonlinear hyperbolic equations with functional arguments by employing Riccati technique. Riccati techniques were used to obtain various oscillation results. Recently, Shoukaku and Yoshida [2] derived oscillation criteria by using oscillation criteria of Riccati inequality. In this work, we study the hyperbolic equation with impulsive.

2 Main Results

Theorem 2.1 If for each $T \geq 0$, there exist $(H_1,H_2) \in H$ and $a,b,c \in \mathbb{R}$ such that $T \leq a < c < b$ and

$$\begin{aligned}&\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\prod_{t_1\leq t_k<s}(\frac{1+\beta_k}{1+\alpha_k})^{-1}(C_l q_l(s)-\frac{1}{4}r(s)\lambda_1^2(s,a))\psi(s)ds\\&+\frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\prod_{t_1\leq t_k<s}(\frac{1+\beta_k}{1+\alpha_k})^{-1}(C_l q_l(s)-\frac{1}{4}r(s)\lambda_2^2(b,s))\psi(s)ds>0,\end{aligned} \tag{2.1}$$

then every solution of the problems (1.1)–(1.4) oscillates in Ω, where

$$\psi(t) \in C^1((T_0, +\infty); (0, +\infty))$$

for some $t_1 > 0$ and

$$\lambda_1(s,t) = \frac{\psi'(s)}{\psi(s)} + h_1(s,t), \ \ \lambda_2(t,s) = \frac{\psi'(s)}{\psi(s)} - h_2(t,s).$$

Proof Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problems (1.1)–(1.4). Without loss of generality we may assume that $u(x,t) > 0$ in $G \times [t_0, +\infty)$ for some $t_0 > 0$ because the case where $u(x,t) < 0$ can be treated similarly. Since (H2) holds, we see that $u(x, \tau_i(t)) > 0$ $(i = 1, 2, \cdots n)$ in $G \times [t_1, +\infty)$ for some $t_1 \geq t_0$.

(1) For $t \geq t_1,\ t \neq t_k,\ k = 1, 2, \cdots$, integrating (1) with respect to x over G, we obtain

$$\begin{aligned}\frac{d}{dt}(r(t)\int_G \frac{\partial}{\partial t} u(x,t)dx) \ &= \ a(t)\int_G h(u(x,t))\triangle u(x,t)dx - \sum_{j=1}^{m}\int_G q_j(x,t)\varphi_j(u(x,t)) \\ &\quad + \sum_{i=1}^{n} b_i(t)\int_G h_i(u(x,\tau_i(t)))\triangle u(x,\tau_i(t))dx.\end{aligned}$$

By Green's formula and the boundary condition, we have

$$\begin{aligned}&\int_G h(u(x,t))\Delta u(x,t)dx = \int_{\partial G} h(u(x,t))\frac{\partial u(x,t)}{\partial n}ds - \int_G h'(u)|gradu|^2 dx \\ = \ &-\int_G \gamma(x,t)u(x,t)h(u(x,t))ds - \int_G h'(u)|gradu|^2 dx \leq 0, \\ &\int_G h_i(u(x,\tau_i(t)))\Delta u(x,\tau_i(t))dx \leq 0.\end{aligned}$$

For condition (H3) we can easily obtain

$$\int_G q_j(x,t)\varphi_j(u(x,t))dx \geq C_j q_j(t)\int_G u(x,t)dx,$$

then $U(t) > 0$, and it follows that

$$(r(t)U'(t))' + \sum_{i=1}^{m} C_i(t)q_i(t)U(t) \leq 0.$$

For some $l \in \{1, 2, \cdots, m\}$, we can get

$$(r(t)U'(t))' + C_l q_l(t)U(t) \leq 0, \ \ t \geq t_1, t \neq t_k.$$

(2) For $t = t_k$, $k = 1, 2, \cdots$. From (1.2)–(1.3), we have that

$$\begin{aligned}&\int_G u(x,t_k^+)dx - \int_G u(x,t_k^-)dx = \alpha_k\int_G u(x,t_k), \\ &\int_G u_t(x,t_k^+)dx - \int_G u_t(x,t_k^-)dx = \beta_k\int_G u_t(x,t_k),\end{aligned}$$

that is

$$U(t_k^+) = (1+\alpha_k)U(t_k),\ U'(t_k^+) = (1+\beta_k)U'(t_k).$$

Thus we obtain that the functions $U(t)$ is a eventually positive solution of the impulsive differential inequality

$$\begin{aligned} &(r(t)y'(t))' + C_l q_l(t)y(t) \le 0,\\ &y(t_k^+) = (1+\alpha_k)y(t_k),\\ &y'(t_k^+) = (1+\beta_k)y'(t_k). \end{aligned} \tag{2.2}$$

Set $w(t) = \frac{r(t)U'(t)}{U(t)}$ for $t \ge t_1$. From (2.2), we obtain that

$$w'(t) + \frac{1}{r(t)}w^2(t) \le -C_l q_l(t),\ \ w(t_k^+) = \frac{1+\beta_k}{1+\alpha_k}w(t_k).$$

Define $v(t) = (\prod_{t_1\le t_k<t}(\frac{1+\beta_k}{1+\alpha_k})^{-1})w(t)$. In fact, $w(t)$ is continuous on each interval $(t_k, t_{k+1}]$, and in view of $w(t_k^+) = (\frac{1+\beta_k}{1+\alpha_k})w(t_k)$, it follows that for $t \ge t_1$,

$$v(t_k^+) = \prod_{t_1\le t_j\le t_k}(\frac{1+\beta_k}{1+\alpha_k})^{-1}w(t_k^+) = \prod_{t_1\le t_j<t_k}(\frac{1+\beta_k}{1+\alpha_k})^{-1}w(t_k) = v(t_k),$$

and for all $t \ge t_1$,

$$v(t_k^-) = \prod_{t_1\le t_j\le t_{k-1}}(\frac{1+\beta_k}{1+\alpha_k})^{-1}w(t_k^-) = \prod_{t_1\le t_j<t_k}(\frac{1+\beta_k}{1+\alpha_k})^{-1}w(t_k) = v(t_k),$$

which implies that $v(t)$ is continuous on $[t_1, +\infty)$,

$$\begin{aligned} & v'(t) + (\prod_{t_1\le t_k<t}\frac{1+\beta_k}{1+\alpha_k})\frac{1}{r(t)}v^2(t) + (\prod_{t_1\le t_k<t}(\frac{1+\beta_k}{1+\alpha_k})^{-1})C_l q_l(t)\\ = \ & (\prod_{t_1\le t_k<t}(\frac{1+\beta_k}{1+\alpha_k})^{-1})w'(t) + \frac{1}{r(t)}(\prod_{t_1\le t_k<t}\frac{1+\beta_k}{1+\alpha_k})(\prod_{t_1\le t_k<t}(\frac{1+\beta_k}{1+\alpha_k})^{-1})^2w^2(t)\\ & +(\prod_{t_1\le t_k<t}(\frac{1+\beta_k}{1+\alpha_k})^{-1})C_l q_l(t)\\ = \ & \prod_{t_1\le t_k<t}(\frac{1+\beta_k}{1+\alpha_k})^{-1}[w'(t) + \frac{1}{r(t)}w^2(t) + C_l q_l(t)] \le 0. \end{aligned}$$

That is to say

$$v'(t) + (\prod_{t_1\le t_k\le t}\frac{1+\beta_k}{1+\alpha_k})\frac{1}{r(t)}v^2(t) \le -(\prod_{t_1\le t_k<t}(\frac{1+\beta_k}{1+\alpha_k})^{-1})C_l q_l(t). \tag{2.3}$$

Multiplying (2.3) by $\psi(s)$, we obtain

$$(\prod_{t_1\le t_k<s}(\frac{1+\beta_k}{1+\alpha_k})^{-1})\psi(s)C_l q_l(s) \le -\psi(s)v'(s) - (\prod_{t_1\le t_k<s}\frac{1+\beta_k}{1+\alpha_k})\frac{\psi(s)}{r(s)}v^2(s). \tag{2.4}$$

Multiplying (2.4) by $H_2(t,s)$ and integrating over $[c,t]$ for $t\in[c,b)$, we have

$$\begin{aligned}
&\int_c^t (\prod_{t_1\le t_k<s}(\frac{1+\beta_k}{1+\alpha_k})^{-1})H_2(t,s)\psi(s)C_lq_l(s)ds\\
\le\ & -\int_c^t H_2(t,s)\psi(s)v'(s)ds-\int_c^t H_2(t,s)(\prod_{t_1\le t_k<s}\frac{1+\beta_k}{1+\alpha_k})\frac{\psi(s)}{r(s)}v^2(s)ds\\
=\ & H_2(t,c)v(c)\psi(c)-\int_c^t H_2(t,s)(\sqrt{\frac{\prod_{t_1\le t_k<s}\frac{1+\beta_k}{1+\alpha_k}}{r(s)}}v(s)-\frac{1}{2}\lambda_2(t,s)\sqrt{\frac{r(s)}{\prod_{t_1\le t_k<s}(\frac{1+\beta_k}{1+\alpha_k}}})^2\psi(s)ds\\
&+\frac{1}{4}\int_c^t\prod_{t_1\le t_k<s}(\frac{1+\beta_k}{1+\alpha_k})H_2(t,s)r(s)\psi(s)\lambda_2^2(t,s)ds\\
\le\ & H_2(t,c)v(c)\psi(c)+\frac{1}{4}\int_c^t\prod_{t_1\le t_k<s}(\frac{1+\beta_k}{1+\alpha_k})H_2(t,s)r(s)\psi(s)\lambda_2^2(t,s)ds,
\end{aligned}$$

and so

$$\frac{1}{H_2(t,c)}\int_c^t H_2(t,s)\prod_{t_1\le t_k<s}(\frac{1+\beta_k}{1+\alpha_k})^{-1}(C_lq_l(s)-\frac{1}{4}r(s)\lambda_2^2(t,s))\psi(s)ds\le v(c)\psi(c).$$

Let $t\to b^-$ in the above, we obtain

$$\frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\prod_{t_1\le t_k<s}(\frac{1+\beta_k}{1+\alpha_k})^{-1}(C_lq_l(s)-\frac{1}{4}r(s)\lambda_2^2(b,s))\psi(s)ds\le v(c)\psi(c). \tag{2.5}$$

On the other hand, multiplying (2.4) by $H_1(s,t)$ and integrating over $[t,c]$ for $t\in(a,c]$, we obtain

$$\begin{aligned}
&\int_t^c (\prod_{t_1\le t_k<s}(\frac{1+\beta_k}{1+\alpha_k})^{-1})H_1(s,t)\psi(s)C_lq_l(s)ds\\
\le\ & -\int_t^c H_1(s,t)\psi(s)v'(s)ds-\int_t^c H_1(s,t)(\prod_{t_1\le t_k<s}\frac{1+\beta_k}{1+\alpha_k})\frac{\psi(s)}{r(s)}v^2(s)ds\\
=\ & -H_1(c,t)v(c)\psi(c)-\int_t^c H_1(s,t)(\sqrt{\frac{\prod_{t_1\le t_k<s}\frac{1+\beta_k}{1+\alpha_k}}{r(s)}}v(s)-\frac{1}{2}\lambda_1(s,t)\sqrt{\frac{r(s)}{\prod_{t_1\le t_k<s}\frac{1+\beta_k}{1+\alpha_k}}})^2\psi(s)ds\\
&+\frac{1}{4}\int_t^c\prod_{t_1\le t_k<s}(\frac{1+\beta_k}{1+\alpha_k})H_1(s,t)r(s)\psi(s)\lambda_1^2(s,t)ds\\
\le\ & -H_1(c,t)v(c)\psi(c)+\frac{1}{4}\int_t^c\prod_{t_1\le t_k<s}(\frac{1+\beta_k}{1+\alpha_k})H_1(s,t)r(s)\psi(s)\lambda_1^2(s,t)ds,
\end{aligned}$$

and so

$$\frac{1}{H_1(c,t)}\int_t^c H_1(s,t)\prod_{t_1\le t_k<s}(\frac{1+\beta_k}{1+\alpha_k})^{-1}(C_lq_l(s)-\frac{1}{4}r(s)\lambda_1^2(s,t))\psi(s)ds\le -v(c)\psi(c).$$

Let $t \to a^+$ in the above, we get

$$\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\prod_{t_1\le t_k<s}(\frac{1+\beta_k}{1+\alpha_k})^{-1}(C_l q_l(s)-\frac{1}{4}r(s)\lambda_1^2(s,a))\psi(s)ds \le -v(c)\psi(c). \quad (2.6)$$

Adding (2.5) and (2.6), we easily obtain the following

$$\begin{aligned}&\frac{1}{H_1(c,a)}\int_a^c H_1(s,a)\prod_{t_1\le t_k<s}(\frac{1+\beta_k}{1+\alpha_k})^{-1}(C_l q_l(s)-\frac{1}{4}r(s)\lambda_1^2(s,a))\psi(s)ds\\ +\quad&\frac{1}{H_2(b,c)}\int_c^b H_2(b,s)\prod_{t_1\le t_k<s}(\frac{1+\beta_k}{1+\alpha_k})^{-1}(C_l q_l(s)-\frac{1}{4}r(s)\lambda_2^2(b,s))\psi(s)ds \le 0,\end{aligned}$$

which contradicts condition (2.1).

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里卡蒂方法研究带泛函参数的非线性脉冲时滞双曲方程的振动性

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摘要: 本文研究了带泛函参数的非线性脉冲时滞双曲方程的振动性问题. 利用积分平均法和里卡蒂方法得到了这类方程解的振动性的一个充分条件, 对非线性时滞双曲方程解的震动性进行了推广, 能更好地利用一些现有的脉冲时滞常微分方程解的振动性的结论.

关键词: 振动; 脉冲; 时滞; 双曲方程; Riccati不等式

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