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# AVERAGING FOR MEASURE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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**Abstract:** In this paper, we study the averaging for measure functional differential equations with infinite delay. By using the averaging theorem for generalized ordinary differential equations, under the measure functional differential equations with infinite delay is equivalent to the generalized ordinary differential equations under some conditions, the periodic and non-periodic averaging theorem for this class of retarded functional differential equations is obtained, which generalizes some related results.

**Keywords:** averaging methods; measure functional differential equations; generalized ordinary differential equations; Kurzweil-Stieltjes integral

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## 1 Introduction

In paper [4], the authors stated very nice stability results of measure functional differential equations with infinite delay, especially proved that measure functional differential equations with infinite delay is equivalent to the generalized ordinary differential equations under some conditions. In [5] and [8], the authors described the averaging methods for generalized ordinary differential equations and there were many sources described the averaging methods for ordinary differential equation, such as [5, 8, 9].

In the present paper, we establish an averaging result for measure functional differential equations with infinite delay. This theorems is based on the averaging method for ordinary differential equations, then we consider the classical averaging theorems for ordinary equations are concerned with the initial-value problem

$$x^{'}(t)=\varepsilon f(t,x(t))+\varepsilon^2 g(t,x(t),\varepsilon),\ \ x(t_0)=x_0,$$

where $\varepsilon>0$ is a small parameter. Assume that $f$ is $T$-periodic in the first argument, then we can obtain an approximate of this initial-value problem by neglecting the $\varepsilon^2$-term and

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taking the averaging of $f$ with respect to $t$, i.e, we consider the equation

$$y'(t) = \varepsilon f(y(t)), \quad y(t_0) = x_0,$$

where

$$f_0(y) = \frac{1}{T} \int_{t_0}^{t_0+T} f(t, y) dt.$$

The proof of periodic averaging theorem, which can be traced back to paper [1, 2] or [3].

Now, we consider the measure differential equations.

Have a system described by ordinary differential equation

$$\frac{dx}{dt} = f(t, x) \tag{1.1}$$

is acted upon by perturbation, the perturbed system is generally given by ordinary differential equation of the form $\frac{dx}{dt} = f(t, x) + G(t, x)$. Assume the perturbation term $G(t(x))$ is continuous or integrable and as such the state of the system changes continuously with respect to time. However some system one cannot expect the perturbations to be well-behaved. Such as the perturbations are impulsive. So we have the following equations

$$Dx = f(t, x) + G(t, x)Du, \tag{1.2}$$

where $Du$ denotes the distributional derivative of function $u$. If $u$ is a function of bounded variation, $Du$ can be identified with a Stieltjes measure and will have the effect of suddenly changing the state of the system at the points of discontinuity of $u$. In [11], equations of form (1.2) are called measure differential equations. Equation (1.2) have the special case when $G$ was considered by Schmaedeke [11]. In order to apply the methods of Riemann-Stieltjes integrals in the subsequent analysis we assumed to be a continuous function of $t$. In [12], the authors introduce the following functional differential equation

$$Dx = f(t, x_t) + G(t, x_t)Du, \tag{1.3}$$

where $x_t$ represents the restriction of the function $x(\cdot)$ on the interval $[m(t), n(t)]$, $m$ and $n$ being functions with the property $m(t) \leq n(t) \leq t$. In this case, the methods of R.S. integrals are unapplicable because of the possibility that $G(t, x_t)$ and $u(t)$ may have common discontinuities, and Lebsgue-Stieltjes integrals are therefor used.

Moreover, in [13], a important theorem which was considered as the main contents is as following.

$x(\cdot)$ is a solution of (1.2) through $(t_0, x_0)$ on an interval $I$ with left end point $t_0$, if and only if $x(\cdot)$ satisfies the following equations

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds + \int_{t_0}^{t} G(s, x(s)) du(s).$$

So according to the above contents we can arrival at a conclusion measure functional differential equations with delay have the form $Dx = G(s, x_s)dg(s)$ is equivalent the following form

$$x(t) = x(t_0) + \int_{t_0}^{t} G(s, x_s)du(s).$$

In this paper, we shall consider the following initial value problem of measure differential equations

$$\begin{cases} x(t) = x(t_0) + \displaystyle\int_{t_0}^{t} f(x_s, s)dg(s), & t \in [t_0, t_0 + \sigma], \\ x_{t_0}(v) = x^0(\lambda), & v \in (-\infty, t_0 + \sigma], \end{cases} \tag{1.4}$$

where $x$ is an unknown function with values in $R^n$ and the symbol $x_s$ denotes the function $x_s(\tau) = x(s + \tau)$ defined on $(-\infty, 0]$, which corresponding to the length of the delay. The integral on the right-hand side of (1.4) is the Kurzweil-Stieltjes integral with respect to a nondecreasing function g, where the function $f : P \times [t_0, t_0 + \sigma] \to R^n$ and a nondecreasing function $g : [t_0, t_0 + \sigma] \to R$, where

$$P = \{x_t : x \in O, t \in [t_0, t_0 + \sigma]\} \subset H_0, \ \ H_0 \subset X((-\infty, 0], R^n)$$

is a Banach space satisfying conditions (H1)–(H6), $t_0 \in R, \sigma > 0, O \subset H_{t_0+\sigma}$ is a space satisfying conditions (1)–(6) of Lemma 2.7, $X((-\infty, 0], R^n)$ be denoted the set of all regulated functions $f : X(-\infty, 0] \to R^n$.

Our candidate for the phase space of a measure function differential equations with infinite delay is a linear space $H_0 \subset X((-\infty, 0], R^n)$ equipped with a norm denoted by $\| \cdot \|_\star$. We assume that this normed linear space $H_0$ satisfies the following conditions

(H1) $H_0$ is complete.

(H2) If $x \in H_0$ and $t < 0$, then $x_t \in H_0$.

(H3) There exist a locally bounded function $k_1 : (-\infty, 0] \to R^+$ such that if $x \in H_0$ and $t \le 0$, then $\| x(t) \| \le k_1(t) \| x \|_\star$.

(H4) There exist a function $k_2 : (0, \infty) \to [1, \infty)$ such that if $\sigma > 0$ and $x \in H_0$ is a function whose support is contained in $[-\sigma, 0]$, then

$$\| x \|_\star \le k_2(\sigma) \sup_{t \in [-\sigma, 0]} \| x(t) \| .$$

(H5) There exist a locally bounded function $k_3 : (-\infty, 0] \to R^+$ such that if $x \in H_0$ and $t \le 0$, then

$$\| x_t \|_\star \le k_3(t) \| x \|_\star .$$

(H6) If $x \in H_0$, then the function $t \mapsto \| x_t \|_\star$ is regulated on $(-\infty, 0]$.

Also, we assume that $f : P \times [t_0, t_0 + \sigma] \to R^n$ satisfies the following conditions

(A) The integral $\displaystyle\int_{t_0}^{t_0+\sigma} f(x_t, g)dg(t)$ exists for every $x \in O$.

(B) There exists a function $M:[t_0,t_o+\sigma]\to R^+$, which is Kurzweil-Stieltjies integrable with respect to $g$, such that

$$\| \int_a^b f(x_t,t)dg(t) \| \leq \int_a^b M(t)dg(t),$$

whenever $x\in O$ and $[a,b]\subseteq[t_0,t_o+\sigma]$.

(C) There exists a function $L:[t_0,t_o+\sigma]\to R^+$, which is Kurzweil-Stieltjies integrable with respect to $g$, such that

$$\| \int_a^b (f(x_t,t)-f(y_t,t))dg(t) \| \leq \int_a^b L(t) \| x_t-y_t \|_\star dg(t),$$

whenever $x,y\in O$ and $[a,b]\subseteq[t_0,t_o+\sigma]$ (we are assuming that the integral on the right-hand side exists).

In this paper, we using measure functional differential equations with infinite delay can translate into generalized oridinary differential equations, this prove is given in paper [4]. According to [5], the first we have a conclusion of periodic averaging theorem for generalized ordinary differential equations. We then show that the classical periodic averaging theorem about measure functional differential equations with infinite delay. The next part, according to [8] we have a conclusion of Non-periodic averaging theorem about measure functional differential equations with infinite delay.

## 2 Generalized Ordinary Differential Equations

We start this section with a short summary of Kurzweil integral, which plays a crucial role in the theory of generalized ordinary differential equations.

A function $\delta:[a,b]\to R^+$. A partition of interval $[a,b]$ with division points $a=\alpha_0\leq\alpha_1\leq\cdots\leq\alpha_k=b$ and tags $\tau_i\in[\alpha_{i-1},\alpha_i]$ is called $\delta$-fine if $[\alpha_{i-1},\alpha_i]\subset[\tau_i-\delta(\tau_i),\tau_i+\delta(\tau_i)]$, $i=1,\ 2,\ \cdots,\ k$.

A matrix-valued function $U:[a,b]\times[a,b]\to R^{n\times m}$ is called Kurzweil integrable on $[a,b]$, if there is a matrix $I\in R^{n\times m}$ such that for every $\varepsilon>0$, there is a gauge $\delta$ on $[a,b]$ such that

$$\Big\| \sum_{i=1}^k (U(\tau_i,\alpha_i)-U(\tau_i,\alpha_{i-1}))-I \Big\| \leq \varepsilon \tag{2.1}$$

for every $\delta$-fine partition $D$. In this case, we define $\displaystyle\int_a^b D_tU(\tau,t)=I$.

An important special case is the Kurzweil-Stieltjes integral of a function $f:[a,b]\to R^n$ with respect to a function $g:[a,b]\to R$, which corresponds to the choice $U(\tau,t)=f(\tau)g(t)$ and will be denoted by $\displaystyle\int_a^b f(t)dg(t)$.

Consider a set $G\subset R^n\times R$, $(x,t)\in G$. A function $x:[a,b]\to B$ is called a solution of the generalized ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t}=D_tF(x,t), \tag{2.2}$$

whenever

$$x(s) = x(a) + \int_a^s D_t F(x,t), \qquad s \in [a,b].$$

Some basic knowledge in the theory of generalized ordinary differential equations is the book [6]. It is known that an ordinary differential equation $x^{'}(t) = f(x(t),t)$ is equivalent to the generalized equation

$$\frac{dx}{d\tau} = DF(x,t),$$

where $F(x,t) = \displaystyle\int_{t_0}^t f(x,s)ds$. However, generalized equation include many other types of equation such as measure functional differential equation.

**Definition 2.1** [6] Let $X$ be a Banach space. Consider a set $O \subset X$, a function $F : O \times [t_0, t_0 + \sigma] \to X$ belongs to the class $\mathcal{F}(O \times [t_0, t_0 + \sigma],\ h,\ k)$, if the following conditions is satisfied.

(F1) there exists a nondecreasing function $h : [t_0, t_0 + \sigma] \to R$ such that $F : O \times [t_0, t_0 + \sigma] \to X$ satisfies

$$\|F(x, s_2) - F(x, s_1)\| \leq |h(s_2) - h(s_1)| \tag{2.3}$$

for every $x \in O$ and $s_1, s_2 \in [t_0, t_0 + \sigma]$,

$$h(t) = k_2(\sigma) \int_{t_0}^t M(s)dg(s), \quad t \in [t_0, t_0 + \sigma];$$

(F2) there exists a nondecreasing function $k : [t_0, t_0 + \sigma] \to R$ such that $F : O \times [t_0, t_0 + \sigma] \to X$ satisfies

$$\|F(x, s_2) - F(x, s_1) - F(y, s_2) + F(y, s_1)\| \leq \|x - y\| \cdot |k(s_2) - k(s_1)| \tag{2.4}$$

for every $x, y \in O$ and $s_1, s_2 \in [t_0, t_0 + \sigma]$,

$$k(t) = k_2(\sigma)(\sup_{s \in [-\sigma, 0]} k_3(s)) \int_{t_0}^t L(s)dg(s), \quad t \in [t_0, t_0 + \sigma].$$

**Theorem 2.2** [6] If $f : [a,b] \to R^n$ is a regulated function and $g : [a,b] \to R$ is a nondecreasing function, then the integral $\displaystyle\int_a^b f(s)dg(s)$ exists. Moreover,

$$\| \int_a^b f(s)dg(s) \| \leq \int_a^b \| f(s) \| \, dg(s).$$

**Lemma 2.3** [6] Let $B \subset R^n, \Omega = B \times [a,b]$. Assume that $F : \Omega \to R^n$ belong to the class $\mathcal{F}(\Omega, h)$. If $x, y : [a,b] \to B$ is regulated functions, then the integral $\displaystyle\int_a^b DF(x(t),t)$ exists and

$$\| \int_a^b DF(x(\tau),t) \| \leq h(b) - h(a).$$

**Lemma 2.4** [6] Let $B \subset R^n, \Omega = B \times [a,b]$. Assume that $F : \Omega \to R^n$ belong to the class $\mathcal{F}(\Omega, h)$. Then every solution $x : [\alpha, \beta] \to B$ of the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DF(x,t)$$

is a regulated function.

**Lemma 2.5** [5] Let $B \subset R^n, \Omega = B \times [a,b]$. Assume that $F : \Omega \to R^n$ belong to the class $\mathcal{F}(\Omega, h)$. If $x, y : [a,b] \to B$ are regulated functions, then

$$\| \int_a^b D[F(x(\tau),t) - F(y(\tau),t)] \| \leq \int_a^b \| x(t) - y(t) \| \, dh(t).$$

This lemma was proved in [5] Lemma 5.

**Lemma 2.6** [6] Let $h : [a,b] \to [0,+\infty)$ be a nondecreasing left-continuous function, $k > 0,\ l \geq 0$. Assume that $\psi : [a,b] \to [0,+\infty)$ is bounded and satisfies

$$\psi(\xi) \leq k + l \int_a^\xi \psi(\tau) \mathrm{d}h(\tau),\ \xi \in [a,b],$$

then $\psi(\xi) \leq k e^{l(h(\xi)-h(a))}$ for every $\xi \in [a,b]$.

The next Theorem is very important for prove periodic averaging of measure functional differential equation with infinite delay. This theorem was proved in [5].

**Theorem 2.7** [5] Let $B \subset R^n, \Omega = B \times [0,\infty], \varepsilon_0 > 0, L > 0$. Consider functions $F : \Omega \to R^n$ and $G : \Omega \times (0, \varepsilon_0] \to R^n$ which satisfy the following conditions

(1) there exist nondecreasing left-continuous functions $h_1, h_2 : [0,\infty) \to [0,\infty)$ such that $F$ belongs to the class $\mathcal{F}(\Omega, h_1)$, and for every fixed $\varepsilon \in (0, \varepsilon_0]$, the function $(x,t) \to G(x,t,\varepsilon)$ belongs to the class $\mathcal{F}(\Omega, h_2)$;

(2) $F(x,0) = 0$ and $G(x,0,\varepsilon) = 0$ for every $x \in B, \varepsilon \in (0, \varepsilon_0]$;

(3) there exist a number $T > 0$ and a bounded Lipschitz-continuous function $M : B \to R^n$ such that $F(x, t+T) - F(x,t) = M(x)$ for every $x \in B$ and $t \in [0,\infty)$;

(4) there exist a constant $\alpha > 0$ such that $h_1(iT) - h_1((i-1)T) \leq \alpha$ for every $i \in N$;

(5) there exist a constant $\beta > 0$ such that $| \frac{h_2(t)}{t} | \leq \beta$ for every $t \geq \frac{L}{\varepsilon_0}$. Let

$$F_0 = \frac{F(x,T)}{T}, \quad x \in B.$$

Suppose that for every $\varepsilon \in (0, \varepsilon_0]$. The initial-value problems

$$\begin{aligned} &\frac{dx}{d\tau} = D[\varepsilon F(x,t) + \varepsilon^2 G(x,t,\varepsilon)], \quad x(0) = x_0(\varepsilon), \\ &y'(t) = \varepsilon F_0(y(t)), \quad y(0) = y_0(\varepsilon) \end{aligned}$$

have solution $x_\varepsilon, y_\varepsilon : [0, \frac{L}{\varepsilon}] \to B$. If there is a constant $J > 0$ such that $\| x_0(\varepsilon) - y_0(\varepsilon) \| \leq J\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0)$, then there exist a constant $K > 0$ such that $\| x_\varepsilon(t) - y_\varepsilon(t) \| \leq K\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0]$ and $t \in [0, \frac{L}{\varepsilon}]$.

To establish the correspondence between measure functional differential equations and generalized ordinary differential equations, we also need a suitable space $H_a$ of regulated functions defined on $(-\infty, a]$, where $a \in R$, the next lemma shows that the spaces $H_a$ inherit all important properties of $H_0$.

**Lemma 2.8** [5] If $H_0 \subset G((-\infty, 0], R^n)$ is a space satisfying conditions (H1)–(H6), then the following statements are true for every $a \in R$,

(1) $H_a$ is complete.

(2) If $x \in H_a$ and $t \leq a$, then $x_t \in H_0$.

(3) If $t \leq a$ and $x \in H_a$, then $\| x(t) \| \leq k_1(t-a) \| x \|_\star$.

(4) If $\sigma > 0$ and $x \in H_{a+\sigma}$ is a function whose support is contained in $[a, a+\sigma]$, then

$$\| x \|_\star \leq k_2(\sigma) \sup_{t \in [a, a+\sigma]} \| x(t) \| .$$

(5) If $x \in H_{a+\sigma}$ and $t \leq a+\sigma$, then$\| x_t \|_\star \leq k_3(t-a-\sigma) \| x \|_\star$.

(6) If $x \in H_{a+\sigma}$, then the function $t \mapsto \| x_t \|_\star$ is regulated on $(-\infty, a+\sigma]$.

## 3 Periodic Averaging

In this section, we use Theorem 2.7 to derive a periodic averaging theorem for measure functional differential equations with infinite delay.

**Theorem 3.1** Given a set $H_0 \subset G((-\infty, 0], R^n)$ be a Banach space satisfying conditions (H1)–(H6) $t_0 \in R, \sigma > 0, O \subset H_{t_0+\sigma}$ and $P = \{y_t : y \in O, t \in [t_0, t_0+\sigma]\} \subset H_0$ Consider a nondecreasing function $u : [t_0, t_0+\sigma] \to R^n$ and a function $f : P \times [t_0, t_0+\sigma] \to R^n$, assume that $f$ is $T$-periodic and Lipschitz continuous in this argument. Then the measure functional differential equation of the form

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_s, s) du(s), \quad t \in [t_0, t_0+\sigma]$$

is equivalent to a generalized ordinary differential equation of the form

$$\frac{dx}{d\tau} = DF(x, t), \quad t \in [t_0, t_0+\sigma],$$

where $x$ takes values in $O$, and $f : O \times [t_0, t_0+\sigma] \to G[-\infty, t_0+\sigma] \to R^n$ is given by

$$F(x,t)(v) = \begin{cases} 0, & -\infty < v \leq t_0, \\ \displaystyle\int_{t_0}^{v} f(x_s, s) dg(s), & t_0 \leq v \leq t \leq t_0+\sigma, \\ \displaystyle\int_{t_0}^{t} f(x_s, s) dg(s), & t \leq v \leq t_0+\sigma \end{cases}$$

for every $x \in O$ and $t \in [t_0, t_0+\sigma]$. It will turn out that between the solution $x$ and the solution $y$ is described by

$$x(t)(v) = \begin{cases} y(v), & v \in (-\infty, t], \\ y(t), & v \in [t, t_0+\sigma], \end{cases}$$

where $t \in [t_0, t_0 + \sigma]$.

This theorem was proved in [4, Theorem 3.6].

**Theorem 3.2** Assume that $B \subset R^n$, we use the symbol $X([a,b],B)$ to denote the set of all regulated functions $f:[a,b] \to B$. Let $\varepsilon_0 > 0, L > 0, \Omega = X((-\infty,0),B) \times [t_0,\infty)$ consider a pair of bounded Lipschitz-continuous $f:\Omega \to R^n, g:\Omega \times (0,\varepsilon_0] \to R^n$. Assume that $f$ is $T$-periodic in the second argument. Define $f_0 : X \to R^n$ by

$$f_0(y_s) = \frac{1}{T}\int_{t_0}^{t_0+T} f(y_s,s)du(s), \quad y_s \in X.$$

Suppose that for every $\varepsilon \in (0,\varepsilon_0]$, the measure equation

$$y^{'}(t) = \varepsilon f(y_s,s) + \varepsilon^2 g(y_s,s,\varepsilon), \;\; y(t_0) = y_0(\varepsilon),$$

and the ordinary differential equation

$$x^{'}(t) = \varepsilon f_0(x(t)), \quad x(t_0) = x_0(\varepsilon)$$

have solution $y^\varepsilon, x^\varepsilon : [t_0, t_0+\frac{L}{\varepsilon}] \to X$. If there is a constant $C > 0$, such that $\| y^0(\varepsilon) - x^0(\varepsilon) \| \leq C\varepsilon$, for every $\varepsilon \in (0,\varepsilon_0]$, then there exist a constant $A > 0$, such that

$$\| y^\varepsilon(t) - x^\varepsilon(t) \| \leq A\varepsilon$$

for every $\varepsilon \in (0,\varepsilon_0], t \in [t_0, t_0 + \frac{L}{\varepsilon}]$.

**Proof** According to the assumptions, there exist constants $m, l > 0$, such that

$$\begin{aligned}
&\| f(y_s,s) \| \leq m, \quad \| g(y_s,s,\varepsilon) \| \leq m, \\
&\| f(y_s,s) - f(x_s,s) \| \leq l \| y_s - x_s \|, \\
&\| g(y_s,s,\varepsilon) - g(x_s,s,\varepsilon) \| \leq l \| y_s - x_s \|
\end{aligned}$$

for every $y_s, x_s \in X, t \in \; [t_0,\infty), \varepsilon \in (0,\varepsilon_0]$. The function $h_1(t) = h_2(t) : [0,\infty) \to R$ given by

$$h_1(t) = h_2(t) = (m+l)u(t)$$

is left-continuous and nondecreasing

$$\begin{aligned}
F(y,t) &= \int_{t_0}^{t} f(y_s,s)du(s), \;\; y \in X, t \in \; [t_0,\infty), \\
G(y,t,\varepsilon) &= \int_{t_0}^{t} g(y_s,s,\varepsilon)du(s), \;\; y \in X, t \in \; [t_0,\infty).
\end{aligned}$$

If $0 \leq s_1 \leq s_2$ and $y_s, x_s \in X$. Then

$$\| F(y,s_2) - F(y,s_1) \| = \| \int_{s_1}^{s_2} f(y_s,s)du(s) \| \leq m(u(s_2) - u(s_1)) \leq h_1(s_2) - h_1(s_1),$$

$$\begin{aligned}
&\| F(y,s_2) - F(y,s_1) - F(x,s_2) + F(x,s_1) \| = \| \int_{s_1}^{s_2} (f(y_s,s) - f(x_s,s))du(s) \| \\
\leq \;\; & l \| y_s - x_s \| (u(s_2) - u(s_1)) \leq \| y_s - x_s \| (h_1(s_2) - h_1(s_1)).
\end{aligned}$$

It follows that $F$ belongs to the class $\mathcal{F}(X,\ h_1)$. Similarly, if $0 \le s_1 \le s_2$ and $y_s, x_s \in X$ that

$$\| G(y,s_2,\varepsilon) - G(y,s_1,\varepsilon) \| = \| \int_{s_1}^{s_2} g(y_s,s,\varepsilon)du(s) \| \le m(u(s_2)-u(s_1)) \le h_2(s_2) - h_2(s_1),$$

$$\begin{aligned} & \| G(y,s_2,\varepsilon) - G(y,s_1,\varepsilon) - G(x,s_2,\varepsilon) + G(x,s_1,\varepsilon) \| = \| \int_{s_1}^{s_2} g(y_s,s,\varepsilon) - g(x_s,s,\varepsilon)du(s) \| \\ \le\ & l \| y_s - x_s \| (u(s_2)-u(s_1)) \le \| y_s - x_s \| (h_2(s_2)-h_2(s_1)). \end{aligned}$$

Therefore for every fixed $\varepsilon \in (0,\varepsilon_0]$, the function $(y,s) \mapsto G(y,s,\varepsilon)$ belongs to the class $\mathcal{F}(X,\ h_2)$. It is clear that $F(y,0)=0$ and $G(y,0,\varepsilon)=0$, since $u$ is $T$-periodic. The function $f$ is $T$-period in this argument and it follows that difference

$$F(y,t+T) - F(y,t) = \int_t^{t+T} f(y_s,s)du(s) = \int_0^T f(y_s,s)du(s)$$

does not depend on $t$, we can define $M(x) = F(y,t+T) - F(y,t)$. The following calculations show that $M$ is bounded and Lipschitz-continuous

$$\begin{aligned} & \| M(y) \| = \| F(y,T) - F(y,0) \| = \| \int_0^T f(y_s,s)du(s) \| \le m(u(T)-u(0)) = mT, \\ & \| M(y) - M(x) \| = \| F(y,T) - F(y,0) - F(x,T) + F(x,0) \| \\ = & \ \| \int_0^T f(y_s,s) - f(x_s,s)du(s) \| \le l \| y_s - x_s \| (u(T)-u(0)) \le l \| y_s - x_s \| T. \end{aligned}$$

For every $j \in N$, we have

$$h_1(jT) - h_1((j-1)T) = (m+l)(u(jT) - u(j-1)T) = (m+l)(jT-(j-1)T) = (m+l)T.$$

If $t \ge \frac{L}{\varepsilon_0}$, then

$$\left| \frac{h_2(t)}{t} \right| = (m+l)\frac{u(t)}{t} \le (m+l)\frac{t+T}{t} = (m+l)(1+\frac{T}{t}) \le (m+l)(1+\frac{T\varepsilon_0}{L}).$$

Thus we have checked that all assumptions of Theorem 2.7 are satisfied. To conclude the proof, it is now sufficient to define

$$F_0(y) = \frac{F(y,T)}{T} = \frac{1}{T}\int_0^T f(y_s,s)du(s) = f_0(y_s).$$

By Theorem 3.1, for every $\varepsilon \in (0,\varepsilon_0]$, the function $y^\varepsilon : [t_0, t_0+\frac{L}{\varepsilon}] \to X$ satisfies

$$\frac{dy^\varepsilon}{d\tau} = D[\varepsilon F(y^\varepsilon,s) + \varepsilon^2 G(y^\varepsilon,s,\varepsilon),\ \ y^\varepsilon(0) = y^0(\varepsilon).$$

According to Theorem 2.7, there exists a constant $A > 0$ such that $\| y^\varepsilon(t) - x^\varepsilon(t) \| \le A\varepsilon$ for every $s \in [t_0, t_0+\frac{L}{\varepsilon}]$. which proves the theorem.

## 4 Non-Periodic Averaging

Now we derive non-periodic averaging for measure functional differential equations with infinite delay.

**Theorem 4.1** Consider a number $r>0$ and a function $F: X\times[0,\infty]\to R^n$ such that the following conditions are satisfied.

1. there exists a nondecreasing function $h:[0,\infty]\to R$ such that

$$\| F(x,s_2)-F(x,s_1)\| \leq |h(s_2)-h(s_1)$$

for every $x\in X$ and $s_1,s_2\in[0,\infty]$;

2. there exists a continuous increasing function $\omega:[0,\infty]\to R$ such that $\omega(0)=0$ and

$$\| F(x,s_2)-F(x,s_1)-F(y,s_2)+F(y,s_1)\| \leq \omega \| x-y \| \cdot | h(s_2)-h(s_1) |$$

for every $x,y\in X$ and $s_1,s_2\in[0,\infty)$;

3. there exists a number $C\in R$ such that for every $a\in[0,\infty]$,

$$\limsup_{r\to\infty}\frac{h(a+r)-h(a)}{r}\leq C;$$

4. there exists a function $F_0: X\to R^n$ such that

$$\limsup_{r\to\infty}\frac{F(x,r)}{r}=F_0(x),\ \ x\in X.$$

Assume that the equation $y^{'}(t)=F_0(y(t)),\ y(0)=x_0$ has a unique solution $y:[0,\infty)\to R^n$, which is contained in an interior subset of $X$. Then for every $\mu>0$ and $d>0$ there is an $\varepsilon_0>0$ such that for every $\varepsilon\in(0,\varepsilon_0)$, the generalized ordinary differential equation $\frac{dx}{d\tau}=D(\varepsilon F(x,t)),\ x(0)=x_0$ have a solution$x_\varepsilon:[0,\frac{d}{\varepsilon}]\to R^n$, and the ordinary differential equation

$$y^{'}(t)=\varepsilon F_0(y(t)),\quad y(0)=x_0$$

have a solution $y_\varepsilon:[0,\frac{d}{\varepsilon}]\to R^n$, and $\| x_\varepsilon(t)-y_\varepsilon(t)\| <\mu$ for every $t\in[0,\frac{d}{\varepsilon}]$. This theorem was proved in [10].

**Theorem 4.2** Assume that $B\subset R^n$. Let $\varepsilon_0>0, T>0, L>0, r>0$,

$$\Omega=X((-\infty,0),B)\times[t_0,\infty),\limsup_{r\to\infty}\frac{u(t)}{t}<\infty,$$

consider a bounded Lipschitz-continuous $f:\Omega\to R^n$. Assume that $f$ is $T$-periodic in this argument. Define $f_0: X\to R^n$ by

$$f_0(y_s)=\limsup_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T}f(y_s,s)du(s),\quad y_s\in X.$$

Suppose that for every $\varepsilon\in(0,\varepsilon_0]$, the measure equation

$$y^{'}(s)=\varepsilon f_0(y_s,s),\ \ y(t_0)=y_0(\varepsilon),$$

and the ordinary differential equation

$$x^{'}(t) = \varepsilon f_0(x(t)),\ \ x(t_0) = x_0(\varepsilon)$$

have solution $y^\varepsilon, x^\varepsilon : [t_0, t_0+\frac{L}{\varepsilon}] \to X$ for every $\varepsilon \in (0, \varepsilon_0], \xi > 0$, such that $\| y^\varepsilon(t) - x^\varepsilon(t) \| < \xi$ for every $t \in [t_0, t_0 + \frac{L}{\varepsilon}]$.

**Proof** According to the assume of Theorem 3.2, there exist constants $m, l > 0$ such that

$$\begin{aligned}&\| f(y_s, s) \| \leq m,\ \ y_s \in X,\\ &\| f(y_s, s) - f(x_s, s) \| \leq l \| y_s - x_s \|,\ \ y_s, x_s \in X.\end{aligned}$$

Let $h(s) = m \cdot u(s)$ for $t \in [t_0, \infty), \omega(r) = \frac{l}{m} \cdot r$ for every $r \in [0, \infty)$, and

$$F(y, t) = \int_{t_0}^{t} f(y_s, s) du(s),\ \ y_s \in X, t \in\ [t_0, \infty),$$

when $0 \leq s_1 \leq s_2$ and $y \in X$. We have

$$\begin{aligned}&\| F(y, s_2) - F(y, s_1) \| = \| \int_{s_1}^{s_2} f(y_s, s) du(s) \| \leq m(u(s_2) - u(s_1)) = h(s_2) - h(s_1),\\ &\| F(y, s_2) - F(y, s_1) - F(x, s_2) + F(x, s_1) \| = \| \int_{s_1}^{s_2} f(y_s, s) - f(x_s, s) du(s) \|\\ \leq\ \ & l \| y_s - x_s \| (u(s_2) - u(s_1)) = l \| y_s - x_s \| \cdot \frac{h(s_2) - h(s_1)}{m} \omega \| y_s - x_s \| h(s_2 - h(s_1)).\end{aligned}$$

Since $\limsup\limits_{r\to\infty} \frac{u(t)}{t} < \infty$, there exist number $N > 0$ such that $\frac{u(t)}{t} \leq N$ for every $t \in [t_0, \infty)$, then

$$\limsup_{r\to\infty} \frac{h(a+r) - h(a)}{r} < \limsup_{r\to\infty} \frac{m \cdot (u(a+r) - u(a))}{r} < m \cdot N.$$

Moreover,

$$\limsup_{r\to\infty} \frac{F(y, r)}{r} = f_0(y),\ \ y \in X$$

and thus we see that $F$ satisfies all four assumptions of Theorem 4.1. According to this theorem, given a $\xi > 0$ this is an $\varepsilon_0 >$ such that for every $\varepsilon \in (0, \varepsilon_0)$, the generalized ordinary differential equation $\frac{dy}{d\tau} = D(\varepsilon F(y, t))$ have a solution $y^\varepsilon : [t_0, t_0 + \frac{L}{\varepsilon}] \to R^n$, the ordinary differential equation $x^{'}(t) = \varepsilon f_0(x(t)),\ \ x(t_0) = y_0$ has a solution $x^\varepsilon : [t_0, t_0 + \frac{L}{\varepsilon}] \to R^n$ and $\| y^\varepsilon(t) - x^\varepsilon(t) \| < \xi$ for every $t \in [t_0, t_0 + \frac{L}{\varepsilon}]$.

According to Theorem 3.1 the solution $y^\varepsilon : [t_0, t_0 + \frac{L}{\varepsilon}] \to R^n$, coincides with the solution of the measure functional differential equations with infinite delay

$$y^{'}(s) = \varepsilon f_0(y_s(s), s),\quad y(t) = y_0(\varepsilon).$$

## References

[1] Sanders J A, Verhulst F. Averaging methods in nonlinear dynamical systems[M]. New York: Springer-Verlag, 1985.

[2] Sanders J A, Verhulst F, Murdock J. Averaging methods in nonlinear dynamical systems (2nd ed.)[M]. New York: Spring, 2007.

[3] Verhulst F. Nonlinear differential equations and dynamical systems (2nd ed.)[M]. New York: Spring, 2000.

[4] Slavík A. Measure functional differential equations with infinite delay[J]. Nonl. Anal, 2013, 79: 140–155.

[5] Mesquita J G, Slavík A. Periodic averaging theorems for various types of equations[J]. J. Math. Ann. Appl., 2012, 387: 862–877.

[6] Schwabik Š. Generalized ordinary differential equations[M]. Singapore: World Sci., 1992.

[7] Kurzweil J. Generalized ordinary differential equations and continuous dependence on a parameter[J]. Czechoslovak Math. J., 1957, 82: 418–449.

[8] Slavík A. Averaging dynamic equations on time scales[J]. Appl. Math., 2012, 388: 998–1012.

[9] Federson M, Mesquita J G. Averaging for retarded functional differential equations[J]. Appl. Math., 2011, 388: 77–85.

[10] Schwabik Š. Generalized ordinary differential equations[M]. Singapore: World Sci., 1991.

[11] Schmaedeke W W. Optimal control theory for nonlinear vector differential equations containing measures[J]. J. SIAM Control, 1965, 3: 231–280.

[12] Das P C, Sharma R R. On optimal comtrols for measure delay-differential equations[J]. J. SIAM Control, 1971, 9: 43–61.

[13] Li Baolin, Gou Haide. Bounded variation solutions for retarded functional differential equations[J]. J. Math., 2015, 3, 567–578.

# 无限滞后测度泛函微分方程的平均化

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**摘 要:** 本文研究了无限滞后测度泛函微分方程的平均化. 利用广义常微分方程的平均化方法, 在无限滞后测度泛函微分方程可以转化为广义常微分方程的基础上, 获得了这类方程的周期和非周期平均化定理, 推广了一些相关的结果.

**关键词:** 平均化方法; 测度泛函微分方程; 广义常微分方程; Kurzweil-Stieltjes 积分

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