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LIMIT CYCLES OF THE GENERALIZED POLYNOMIAL LIÉNARD DIFFERENTIAL SYSTEMS*

Amel Boulfoul†

(*Dept. of Math., 20 August 1955 University, BP26, El Hadaiek 21000, Skikda. Algeria*)

Amar Makhlouf

(*Dept. of Math., LMA Laboratory, Badji Mokhtar University, BP12, El Hadjar 23000, Annaba. Algeria*)

Abstract

Using the averaging theory of first and second order we study the maximum number of limit cycles of generalized Liénard differential systems

$$\begin{cases} \dot{x} = y + \epsilon h_l^1(x) + \epsilon^2 h_l^2(x), \\ \dot{y} = -x - \epsilon(f_n^1(x)y^{2p+1} + g_m^1(x)) + \epsilon^2(f_n^2(x)y^{2p+1} + g_m^2(x)), \end{cases}$$

which bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, where ϵ is a small parameter. The polynomials h_l^1 and h_l^2 have degree l; f_n^1 and f_n^2 have degree n; and g_m^1, g_m^2 have degree m. $p \in \mathbb{N}$ and $[\cdot]$ denotes the integer part function.

Keywords limit cycle; periodic orbit; Liénard differential system; averaging theory

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1 Introduction and Statement of the Main Results

One of the main problems in the theory of differential systems is the study of the existence, number and stability of limit cycles. A limit cycle of a differential system is an isolated periodic orbit in the set of all periodic orbits of the differential system. These last years hundreds of papers have studied the limit cycles of planar polynomial differential systems. The main reason of these studies is the unsolved 16^{th} Hilbert problem, see [7,8,10]. In this paper, we will try to give a partial answer to this problem for the class of generalized Liénard polynomial differential system given by

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†Corresponding author. E-mail: a.boulfoul@univ-skikda.dz

$$\begin{cases} \dot{x} = y + h(x), \\ \dot{y} = -x - f(x)y^{2p+1} - g(x), \end{cases} \tag{1}$$

where $h(x), f(x)$ and $g(x)$ are polynomials in the variable x of degree l, n and m respectively and $p \in \mathbb{N}$. This system was studied when $h(x) = 0$ in [1]. [12] and [13] considered the similar case of differential system (1) for $p = 0$.

Note that when $h(x) = g(x) = 0$ and $p = 0$ system (1) coincides with the classical polynomial Liénard differential system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - f(x)y, \end{cases} \tag{2}$$

where $f(x)$ is a polynomial in the variable x of degree n. A generalization of classical Liénard differential system (2) is the following system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -f(x)y - g(x), \end{cases} \tag{3}$$

where $f(x)$ and $g(x)$ are polynomials in the variable x of degree n and m respectively. We denote by $H(m, n)$ the maximum number of limit cycles of system (3). This number is usually called the Hilbert number for system (3).

- In 1928, Liénard [14] proved that if $m = 1$ and $F(x) = \int_0^x f(s)\mathrm{d}s$ is a continuous odd function, which has a unique root at $x = a$ and is monotone increasing for $x \geq a$, then system (3) has a unique limit cycle.

- In 1973, Rychkov [19] proved that if $m = 1$ and $f(x)$ is an odd polynomial of degree five, then system (3) has at most two limit cycles.

- In 1977, Lins de Melo and Pugh [15] proved that $H(1, 1) = 0$ and $H(1, 2) = 1$.

- In 1988, Coppel [5] proved that $H(2, 1) = 1$.

- In 1997, Dumortier and Li [6] proved that $H(3, 1) = 1$.

- In 2012, Li and Llibre [11] proved that $H(1, 3) = 1$.

A well known method for obtaining results on the limit cycles of polynomial differential systems perturbs the linear center $\dot{x} = y,\ \dot{y} = -x$ inside the class of polynomial differential systems, or inside the class of classical polynomial Liénard differential systems. The limit cycles obtained in this way are sometimes called medium amplitude limit cycles.

In 2010, Llibre, Mereu and Teixeira [16] studied how many limit cycles $\widetilde{H}(m,n)$ can bifurcate from the periodic orbits of the linear center to system (3) using the averaging theory. In fact they compute lower estimations of $\widetilde{H}(m,n)$. More precisely they compute the maximum number of limit cycles $\widetilde{H}_k(m,n)$ which bifurcate from the periodic orbits of the linear center $\dot{x}=y$, $\dot{y}=-x$, using the averaging theory of order k, for $k=1,2,3$. For the limit cycles obtained by bifurcation of the orbits of the center of the Liénard differential systems, see [4,18,22].

In this work, we provide estimations of $\widetilde{H}(l,m,n)$ for all $l,m,n\geq 1$ computing $\widetilde{H}_k(l,m,n)$ for $k=1,2$. Of course $\widetilde{H}_k(l,m,n)\leq \widetilde{H}(l,m,n)\leq H(l,m,n)$.

Let k be a positive integer. We define $ev(k)$ as the largest even integer $\leq k$, and $od(k)$ as the largest odd integer $\leq k$.

First we consider system (1) with $h(x)=\epsilon h_l^1(x)$, $f(x)=\epsilon f_n^1(x)$ and $g(x)=\epsilon g_m^1(x)$. We obtain the following system

$$\begin{cases}\dot{x}=y+\epsilon h_l^1(x),\\ \dot{y}=-x-\epsilon(f_n^1(x)y^{2p+1}+g_m^1(x)),\end{cases}\tag{4}$$

where the polynomials h_l^1, f_n^1 and g_m^1 have degree l,n and m respectively, $p\in\mathbb{N}$ and ϵ is a small parameter.

Theorem 1 *For $|\epsilon|>0$ sufficiently small, the maximum number of limit cycles of the generalized polynomial Liénard differential systems* (4) *bifurcating from the periodic orbits of linear center $\dot{x}=y$, $\dot{y}=-x$, using the averaging theory of first order is:*

(a) *For $p=0$,*

$$\widetilde{H}_1(l,m,n)=\max\left\{\left[\frac{l-1}{2}\right],\left[\frac{n}{2}\right]\right\}.$$

(b) *For $p\geq 1$, we have three cases:*

(i) *If $1\leq od(l)<2p+1$,*

$$\widetilde{H}_1(l,m,n)=\left[\frac{l+1}{2}\right]+\left[\frac{n}{2}\right].$$

(ii) *If $2p+1\leq od(l)<ev(n)+2p+1$,*

$$\widetilde{H}_1(l,m,n)=\left[\frac{n}{2}\right]+p.$$

(iii) *If $od(l)\geq ev(n)+2p+1$,*

$$\widetilde{H}_1(l,m,n)=\left[\frac{l-1}{2}\right].$$

The proof of Theorem 1 is given in Section 3.

Now we consider system (1) with

$$\begin{aligned} h(x) &= \epsilon h_l^1(x) + \epsilon^2 h_l^2(x), \\ f(x) &= \epsilon f_n^1(x) + \epsilon^2 f_n^2(x), \\ g(x) &= \epsilon g_m^1(x) + \epsilon^2 g_m^2(x). \end{aligned}$$

We obtain the following system

$$\begin{cases} \dot{x} = y + \epsilon h_l^1(x) + \epsilon^2 h_l^2(x), \\ \dot{y} = -x - \epsilon(f_n^1(x) y^{2p+1} + g_m^1(x)) + \epsilon^2(f_n^2(x) y^{2p+1} + g_m^2(x)), \end{cases} \tag{5}$$

where the polynomials h_l^1 and h_l^2 have degree l; f_n^1 and f_n^2 have degree n and g_m^1, g_m^2 have degree m. $p \in \mathbb{N}$ and ϵ is a small parameter.

Theorem 2 *For $|\epsilon| > 0$ sufficiently small, the maximum number of limit cycles of the generalized polynomial Liénard differential systems* (5) *bifurcating from the periodic orbits of linear center $\dot{x} = y$, $\dot{y} = -x$, using the averaging theory of second order is:*

(a) For $p = 0$,

$$\widetilde{H}_2(l,m,n) = \max\left\{\left[\frac{n}{2}\right], \left[\frac{m}{2}\right] + \left[\frac{n-1}{2}\right], \left[\frac{l-1}{2}\right], \left[\frac{m}{2}\right] + \left[\frac{l-2}{2}\right]\right\}.$$

(b) For $p \geq 1$, we have three cases to be considered:

(i) *If $0 \leq \sigma_1 < 2p$,*

$$\widetilde{H}_2(l,m,n) = \max\left\{2\left[\frac{m}{2}\right] + \left[\frac{l}{2}\right] + \left[\frac{n-1}{2}\right], \left[\frac{l+1}{2}\right] + \left[\frac{n}{2}\right]\right\}.$$

(ii) *If $2p \leq \sigma_1 \leq \sigma_2 + 2p$,*

$$\widetilde{H}_2(l,m,n) = \max\left\{\left[\frac{m}{2}\right] + \left[\frac{n-1}{2}\right] + p, \left[\frac{n}{2}\right] + p\right\}.$$

(iii) *If $\sigma_1 > \sigma_2 + 2p$,*

$$\widetilde{H}_2(l,m,n) = \max\left\{\left[\frac{m}{2}\right] + \left[\frac{l-2}{2}\right], \left[\frac{l-1}{2}\right]\right\},$$

where

$$\begin{aligned} \sigma_1 &= \max\left\{ev(l) + ev(m) - 2, od(l) - 1\right\}, \\ \sigma_2 &= \max\left\{od(n) + ev(m) - 1, ev(n)\right\}. \end{aligned}$$

The proof of Theorem 2 is given in Section 4.

In Section 2, we introduce the averaging theory of first and second orders.

2 Averaging Theory of First and Second Orders

Theorem 3 *We consider the following differential system*

$$\dot{x} = \epsilon F_1(t,x) + \epsilon^2 F_2(t,x) + \epsilon^3 R(t,x,\epsilon), \tag{6}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}$, $R : \mathbb{R} \times D \times (-\epsilon_f, \epsilon_f) \to \mathbb{R}$ are continuous, T-periodic in the first variable, and D is an open subset of $\mathbb{R}$. Assume that the following hypotheses (i), (ii) *hold.*

(i) *$F_1(t,\cdot) \in C^2(D)$, $F_2(t,\cdot) \in C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, R are locally Lipschitz with respect to x, and R is twice differentiable with respect to ϵ.*

We define $F_{k0} : D \to \mathbb{R}$ for $k = 1, 2$ as

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(s,z)\mathrm{d}s,$$
$$F_{20}(z) = \frac{1}{T}\int_0^T [D_z F_1(s,z) y_1(s,z) + F_2(s,z)]\mathrm{d}s,$$

where

$$y_1(s,z) = \int_0^s F_1(t,z)\mathrm{d}t.$$

(ii) *For $V \subset D$ an open and bounded set and for each $\epsilon \in (-\epsilon_f, \epsilon_f)\backslash\{0\}$, there exists an $a \in V$ such that $F_{10}(a) + \epsilon F_{20}(a) = 0$ and $d_B(F_{10} + \epsilon F_{20}, V, a) \neq 0$. The expression $d_B(F_{10} + \epsilon F_{20}, V, a) \neq 0$ means that the Brouwer degree (see* [2]*) of the function $F_{10} + \epsilon F_{20} : V \to \mathbb{R}$ at the fixed point a is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function $F_{10} + \epsilon F_{20}$ at a is not zero.*

Then, for $|\epsilon| > 0$ sufficiently small there exists a T-periodic solution $\varphi(\cdot, \epsilon)$ of the equation (6) *such that $\varphi(0, \epsilon) \to a$ when $\epsilon \to 0$.*

If F_{10} is not identically zero, then the zeros of $F_{10} + \epsilon F_{20}$ are mainly the zeros of F_{10} for ϵ sufficiently small. In this case the previous result provides the averaging theory of first order.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \epsilon F_{20}$ are mainly the zeros of F_{20} for ϵ sufficiently small. In this case the previous result provides the averaging theory of second order.

For a general introduction to averaging theory see [3,20,21].

3 Proof of Theorem 1

In order to apply the first order averaging method we write system (4) in polar coordinates (r, θ) where $x = r\cos\theta$, $y = r\sin\theta$, $r > 0$. If we take $f_n^1(x) = \sum\limits_{i=0}^{n} a_i x^i$,

$g_m^1(x) = \sum\limits_{i=0}^{m} b_i x^i$ and $h_l^1(x) = \sum\limits_{i=0}^{l} c_i x^i$, system (4) can be written in the following way

$$\dot{r} = \epsilon \left(\sum_{i=0}^{l} c_i r^i \cos^{i+1}\theta - \sum_{i=0}^{n} a_i r^{i+2p+1} \cos^i\theta \sin^{2p+2}\theta - \sum_{i=0}^{m} b_i r^i \cos^i\theta \sin\theta \right) + O(\epsilon^2),$$

$$\dot{\theta} = -1 - \frac{\epsilon}{r} \left(\sum_{i=0}^{n} a_i r^{i+2p+1} \cos^{i+1}\theta \sin^{2p+1}\theta + \sum_{i=0}^{m} b_i r^i \sin^i\theta \cos\theta \right.$$
$$\left. - \sum_{i=0}^{l} c_i r^i \cos^i\theta \sin\theta \right) + O(\epsilon^2).$$

If we take θ as a new independent variable, this system becomes

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}\theta} &= -\epsilon \left(\sum_{i=0}^{l} c_i r^i \cos^{i+1}\theta - \sum_{i=0}^{n} a_i r^{i+2p+1} \cos^i\theta \sin^{2(p+1)}\theta - \sum_{i=0}^{m} b_i r^i \cos^i\theta \sin\theta \right) + O(\epsilon^2) \\ &= \epsilon F_1(\theta, r) + O(\epsilon^2).\end{aligned}$$

By using the notation introduced in Section 2 we have that

$$F_{10}(r) = \frac{-1}{2\pi} \left(\sum_{i=0}^{l} c_i r^i \int_0^{2\pi} \cos^{i+1}\theta \mathrm{d}\theta - \sum_{i=0}^{n} a_i r^{i+2p+1} \int_0^{2\pi} \cos^i\theta \sin^{2(p+1)}\theta \mathrm{d}\theta \right.$$
$$\left. - \sum_{i=0}^{m} b_i r^i \int_0^{2\pi} \cos^i\theta \sin\theta \mathrm{d}\theta \right).$$

In order to calculate the exact expression of F_{10}, we use the following formulas

$$\int_0^{2\pi} \cos^{i+1}(\theta)\mathrm{d}\theta = \begin{cases} 0, & \text{if } i \text{ is even}, \\ \dfrac{\pi}{2^i} \begin{pmatrix} i+1 \\ \dfrac{i+1}{2} \end{pmatrix} = A_{i+1}, & \text{if } i \text{ is odd}, \end{cases}$$

$$\int_0^{2\pi} \cos^i(\theta)\sin(\theta)\mathrm{d}\theta = 0, \quad i = 1, 2, \cdots,$$

$$\int_0^{2\pi} \cos^i(\theta)\sin^{2(p+1)}(\theta)\mathrm{d}\theta$$
$$= \begin{cases} 0, & \text{if } i \text{ is odd}, \\ \dfrac{\pi}{2^{i-1}} \begin{pmatrix} i \\ \dfrac{i}{2} \end{pmatrix} \dfrac{(2p+1)!!}{(2p+2+i)(2p+i)\cdots(i+2)} = B_{i,p+1}, & \text{if } i \text{ is even}, \end{cases}$$

where $(2p+1)!! = 1 \cdot 3 \cdot 5 \cdots (2p+1)$.

Hence

$$F_{10}(r) = \frac{-r}{2\pi}\left(\sum_{\substack{i=1\\ i\,odd}}^{l} c_i r^{i-1}\int_0^{2\pi}\cos^{i+1}\theta \mathrm{d}\theta - \sum_{\substack{i=0\\ i\,even}}^{n} a_i r^{i+2p}\int_0^{2\pi}\cos^{i}\theta\sin^{2(p+1)}\theta \mathrm{d}\theta\right)$$

$$= \frac{-r}{2\pi}\left(\sum_{\substack{i=1\\ i\,odd}}^{l} c_i A_{i+1} r^{i-1} - \sum_{\substack{i=0\\ i\,even}}^{n} a_i B_{i,p+1} r^{i+2p}\right).$$

(a) For $p = 0$, we have that

$$F_{10}(r) = \frac{-r}{2\pi}\big(c_1A_2 + c_3A_4r^2 + c_5A_6r^4 + \cdots + c_{od(l)}A_{od(l)+1}r^{od(l)-1} \\ -a_0B_{0,1} - a_2B_{2,1}r^2 - a_4B_{4,1}r^4\cdots - a_{ev(n)}B_{ev(n),1}r^{ev(n)}\big).$$

Note that in order to find positive roots of F_{10}, we must find the zeros of a polynomial in the variable r^2 of degree equal to

$$\frac{\max\{od(l)-1, ev(n)\}}{2} = \max\left\{\left[\frac{l-1}{2}\right], \left[\frac{n}{2}\right]\right\}.$$

(b) For $p \geq 1$, we distinguish three cases:

(i) If $1 \leq od(l) < 2p+1$,

$$F_{10}(r) = \frac{-r}{2\pi}\big(c_1A_2 + c_3A_4r^2 + c_5A_6r^4 + \cdots + c_{od(l)}A_{od(l)+1}r^{od(l)-1} \\ -a_0B_{0,p+1}r^{2p} - a_2B_{2,p+1}r^{2+2p} - \cdots - a_{ev(n)}B_{ev(n),p+1}r^{ev(n)+2p}\big).$$

The maximum number of positive roots of a polynomial in r^2 in this case is

$$\frac{od(l)-1}{2} + \frac{ev(n)}{2} + 1 = \frac{od(l)+1+ev(n)}{2} = \left[\frac{l+1}{2}\right] + \left[\frac{n}{2}\right].$$

(ii) If $2p+1 \leq od(l) < ev(n)+2p+1$,

$$F_{10}(r) = \frac{-r}{2\pi}\big(c_1A_2 + \cdots - a_0B_{0,p+1}r^{2p} - \cdots + c_{od(l)}A_{od(l)+1}r^{od(l)-1} \\ - \cdots - a_{ev(n)}B_{ev(n),p+1}r^{ev(n)+2p}\big).$$

In this case, The polynomial F_{10} has at most

$$\frac{ev(n)+2p-od(l)+1}{2} + \frac{od(l)-1}{2} = \frac{ev(n)+2p}{2} = \left[\frac{n}{2}\right] + p$$

real positive roots.

(iii) If $od(l) > ev(n)+2p+1$,

$$F_{10}(r) = \frac{-r}{2\pi}\big(c_1A_2 + \cdots - a_{ev(n)}B_{ev(n),p+1}r^{ev(n)+2p} + \cdots + c_{od(l)}A_{od(l)+1}r^{od(l)-1}\big).$$

This polynomial has at most $[\frac{l-1}{2}]$ positive roots. Hence Theorem 1 is proved.

4 Proof of Theorem 2

For proving Theorem 2 we shall use the second order averaging theory. We consider the differential system (5)

$$\begin{cases} \dot{x} = y + \epsilon h_l^1(x) + \epsilon^2 h_l^2(x), \\ \dot{y} = -x - \epsilon\left(f_n^1(x)y^{2p+1} + g_m^1(x)\right) - \epsilon^2\left(f_n^2(x)y^{2p+1} + g_m^2(x)\right), \end{cases}$$

where

$$h_l^2(x) = \sum_{i=0}^{l} \hat{c}_i x^i, \quad f_n^2(x) = \sum_{i=0}^{n} \hat{a}_i x^i \quad \text{and} \quad g_m^2(x) = \sum_{i=0}^{m} \hat{b}_i x^i.$$

Then system (5) in polar coordinates (r, θ), $r > 0$ becomes

$$\begin{aligned} \dot{r} = \epsilon & \left(\frac{r\cos\theta h_l^1(r\cos\theta) - f_n^1(r\cos\theta)(r\sin\theta)^{2p+2} - r\sin\theta g_m^1(r\cos\theta)}{r}\right) \\ & + \epsilon^2\left(\frac{r\cos\theta h_l^2(r\cos\theta) - f_n^2(r\cos\theta)(r\sin\theta)^{2p+2} - r\sin\theta g_m^2(r\cos\theta)}{r}\right) + O(\epsilon^3), \\ \dot{\theta} = -1 - \epsilon & \left(\frac{r\cos\theta f_n^1(r\cos\theta)y^{2p+1} + r\cos\theta g_m^1(r\cos\theta) + y h_l^1(r\cos\theta)}{r^2}\right) \\ & - \epsilon^2\left(\frac{r\cos\theta f_n^2(r\cos\theta)(r\sin\theta)^{2p+1} + r\cos\theta g_m^2(r\cos\theta) + r\sin\theta h_l^2(r\cos\theta)}{r^2}\right) + O(\epsilon^3). \end{aligned} \tag{7}$$

Now taking θ as a new independent variable, system (5) becomes

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}\theta} &= -\epsilon\Big(\frac{r\cos\theta h_l^1(r\cos\theta) - f_n^1(r\cos\theta)(r\sin\theta)^{2p+2} - r\sin\theta g_m^1(r\cos\theta)}{r}\Big) \\ & \quad -\epsilon^2\Big(\frac{r\cos\theta h_l^2(r\cos\theta) - f_n^2(r\cos\theta)(r\sin\theta)^{2p+2} - r\sin\theta g_m^2(r\cos\theta)}{r} \\ & \quad -\frac{1}{r^3}\big(r\cos\theta h_l^1(r\cos\theta) - f_n^1(r\cos\theta)(r\sin\theta)^{2p+2} - r\sin\theta g_m^1(r\cos\theta)\big) \\ & \quad \times\big(r\cos\theta f_n^1(r\cos\theta)(r\sin\theta)^{2p+1} + r\cos\theta g_m^1(r\cos\theta) + r\sin\theta h_l^1(r\cos\theta)\big)\Big) + O(\epsilon^3) \\ &= \epsilon F_1(\theta, r) + \epsilon^2 F_2(\theta, r) + O(\epsilon^3). \end{aligned}$$

Now we determine the corresponding function

$$F_{20} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\mathrm{d}}{\mathrm{d}r}F_1(\theta, r)\cdot\int_0^{\theta}F_1(\phi, r)\mathrm{d}\phi + F_2(\theta, r)\right)\mathrm{d}\theta.$$

For this we put $F_{10} \equiv 0$ which is equivalent to

$$\begin{cases} c_i = 0 & \text{for } i \text{ is odd}, \\ a_i = 0 & \text{for } i \text{ is even}. \end{cases}$$

First, we have that

$$\frac{\mathrm{d}}{\mathrm{d}r}F_1(\theta,r) = -\sum_{\substack{i=2\\ i\,even}}^{l} ic_ir^{i-1}\cos^{i+1}\theta + \sum_{\substack{i=1\\ i\,odd}}^{n}(i+2p+1)a_ir^{i+2p}\cos^i\theta\sin^{2(p+1)}\theta$$
$$+\sum_{i=1}^{m} ib_ir^{i-1}\cos^i\theta\sin\theta,$$

and

$$\int_0^\theta F_1(\phi,r)\mathrm{d}\phi = -\sum_{\substack{i=0\\ i\,even}}^{l} c_ir^i\int_0^\theta\cos^{i+1}\phi\mathrm{d}\phi + \sum_{\substack{i=1\\ i\,odd}}^{n} a_ir^{i+2p+1}\int_0^\theta\cos^i\phi\sin^{2(p+1)}\phi\mathrm{d}\phi$$
$$+\sum_{i=0}^{m} b_ir^i\int_0^\theta\cos^i\phi\sin\phi\mathrm{d}\phi$$
$$= -\sum_{\substack{i=0\\ i\,even}}^{l}\mathcal{A}_{i+1}(\theta)c_ir^i + \sum_{\substack{i=1\\ i\,odd}}^{n}\mathcal{B}_{i,p+1}(\theta)a_ir^{i+2p+1} + \sum_{i=0}^{m} b_ir^i\left(\frac{1-\cos^{i+1}\theta}{i+1}\right),$$

where

$$\mathcal{A}_j(\theta) = \int_0^\theta\cos^j\phi\mathrm{d}\phi$$
$$= \sum_{\substack{k=1\\ k=odd}}^{j-2}\frac{(j-k)!}{j!}\frac{(j-k)^2\cdot(j-(k-2))^2\cdots(j-1)^2}{(j-k)^2}\sin\theta\cos^{j-k}\theta$$
$$+\frac{(j-1)^2\cdot(j-3)^2\cdots2^2}{j!}\sin\theta,$$

and

$$\mathcal{B}_{i,p+1}(\theta) = \int_0^\theta\cos^i\phi\sin^{2(p+1)}\phi\mathrm{d}\phi$$
$$= -\frac{\cos^{i+1}\theta}{i+2(p+1)}\left(\sin^{2p+1}\theta + \sum_{k=1}^{p}\frac{(2p+1)(2p-1)\cdots(2p-2k+3)}{(2p+i)(2p+i-2)\cdots(2p+i-2k)}\sin^{2p-2k+1}\theta\right)$$
$$+\frac{(2p+1)!!}{(2p+2+i)(2p+i)\cdots(i+2)}\int_0^\theta\cos^i\phi\mathrm{d}\phi.$$

For more details see [9].

So

$$\int_0^{2\pi}\frac{\mathrm{d}}{\mathrm{d}r}F_1(\theta,r)y_1(\theta,r)\mathrm{d}\theta$$
$$= \sum_{\substack{i=0\\ i\,even}}^{m}\sum_{\substack{k=2\\ k\,even}}^{l}\frac{k}{i+1}b_ic_kr^{i+k-1}\int_0^{2\pi}\cos^{i+k+2}\theta\mathrm{d}\theta$$

$$
\begin{aligned}
&-\sum_{\substack{i=1\\ i\,odd}}^{n}\sum_{\substack{k=0\\ k\,even}}^{m}\frac{i+2p+1}{k+1}a_ib_kr^{i+k+2p}\int_0^{2\pi}\cos^{i+k+1}\theta\sin^{2(p+1)}\theta\mathrm{d}\theta\\
&+\sum_{\substack{i=1\\ i\,odd}}^{n}\sum_{\substack{k=0\\ k\,even}}^{m}ka_ib_kr^{i+k+2p}\int_0^{2\pi}\cos^{k}\theta\sin\theta\mathcal{B}_{i,p+1}(\theta)\mathrm{d}\theta\\
&-\sum_{\substack{i=2\\ i\,even}}^{m}\sum_{\substack{k=0\\ k\,even}}^{l}ib_ic_kr^{i+k-1}\int_0^{2\pi}\cos^{i}\theta\sin\theta\mathcal{A}_{k+1}(\theta)\mathrm{d}\theta.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
\int_0^{2\pi}F_2(r,\theta)\mathrm{d}\theta=&-\sum_{\substack{i=1\\ i\,odd}}^{1}\hat{c}_iA_{i+1}r^i+\sum_{\substack{i=0\\ i\,even}}^{n}\hat{a}_ir^{i+2p+1}\int_0^{2\pi}\cos^{i}\theta\sin^{2(p+1)}\theta\mathrm{d}\theta\\
&-2\sum_{\substack{i=1\\ i\,odd}}^{n}\sum_{\substack{k=0\\ k\,even}}^{m}a_ib_kr^{i+k+2p}\int_0^{2\pi}\cos^{i+k+1}\theta\sin^{2(p+1)}\theta\mathrm{d}\theta\\
&+\sum_{\substack{i=0\\ i\,even}}^{m}\sum_{\substack{k=0\\ k\,even}}^{l}b_ic_kr^{i+k-1}\int_0^{2\pi}\cos^{i+k}\theta\cos2\theta\mathrm{d}\theta.
\end{aligned}
$$

Then F_{20} is the polynomial

$$
\begin{aligned}
F_{20}(r)=r\Bigg(&\sum_{\substack{i=1\\ i\,odd}}^{n}\sum_{\substack{k=0\\ k\,even}}^{m}\lambda_{i,k}\,a_ib_k\,r^{i+k+2p-1}+\sum_{\substack{i=0\\ i\,even}}^{m}\sum_{\substack{k=2\\ k\,even}}^{l}\mu_{i,k}\,b_ic_k\,r^{i+k-2}\\
&+\sum_{\substack{i=1\\ i\,odd}}^{l}A_{i+1}\,\hat{c}_i\,r^{i-1}+\sum_{\substack{i=0\\ i\,even}}^{n}B_{i,p+1}\,\hat{a}_i\,r^{i+2p}\Bigg),
\end{aligned}
\tag{8}
$$

where

$$
\begin{cases}
\lambda_{i,k}=\dfrac{-(2k+i+2p+3)}{k+1}B_{i+k+1,p+1}+kE_{i+k,p+1},\quad \text{with}\\
E_{i+k,p+1}=\displaystyle\int_0^{2\pi}\cos^{k}\theta\sin\theta\mathcal{B}_{i,p+1}(\theta)\mathrm{d}\theta,
\end{cases}
$$

and

$$
\begin{cases}
\mu_{i,k}=\dfrac{k}{i+1}A_{i+k+2}-iD_{i,k+1}+C_{i+k},\quad \text{with}\\
D_{i,k+1}=\displaystyle\int_0^{2\pi}\cos^{i}\theta\sin\theta\mathcal{A}_{k+1}(\theta)\mathrm{d}\theta,\quad C_{i+k}=\int_0^{2\pi}\cos^{i+k}\theta\cos2\theta\mathrm{d}\theta.
\end{cases}
$$

(a) For $p=0$, equation (8) becomes

$$F_{20}(r)=r\Big[\lambda_{1,0}a_1b_0+(\lambda_{1,2}a_1b_2+\lambda_{3,0}a_3b_0)r^2+\cdots+\sum_{\substack{i+k=\\od(n)+ev(m)}}\lambda_{i,k}\,a_ib_k\,r^{od(n)+ev(m)-1}$$
$$+\mu_{0,2}b_0c_2+(\mu_{2,2}b_2c_2+\mu_{0,4}b_0c_4)r^2+\cdots+\sum_{\substack{i+k=\\ev(m)+ev(l)}}\mu_{i,k}b_ic_kr^{ev(m)+ev(l)-2}$$
$$+A_2\hat{c}_1+A_4\hat{c}_3r^2+\cdots+A_{od(l)+1}\hat{c}_{od(l)}r^{od(l)-1}$$
$$+B_{0,p+1}\hat{a}_0+B_{2,p+1}\hat{a}_2r^2+\cdots+B_{ev(n),p+1}\hat{a}_{ev(n)}r^{ev(n)}\Big].$$

This polynomial has at most

$$\max\left\{\frac{od(n)+ev(m)-1}{2},\frac{ev(m)+ev(l)-2}{2},\frac{od(l)-1}{2},\frac{ev(n)}{2}\right\}$$
$$=\max\left\{\left[\frac{m}{2}\right]+\left[\frac{n-1}{2}\right],\left[\frac{m}{2}\right]+\left[\frac{l-2}{2}\right],\left[\frac{l-1}{2}\right],\left[\frac{n}{2}\right]\right\}$$

real positive roots.

(b) For $p\geq 1$, we denote by

$$\sigma_1=\max\{ev(l)+ev(m)-2,od(l)-1\}\quad\text{and}\quad\sigma_2=\max\{od(n)+ev(m)-1,ev(n)\}.$$

Note that

$$\begin{cases}\sigma_1=od(l)-1, & \text{if and oly if } m=1,\\ \sigma_2=ev(n), & \text{if and only if } m=1,\end{cases}$$

so, we have three cases:

(i) If $0\leq\sigma_1<2p$, in this case equation (8) can be written as

$$F_{20}(r)=r\Bigg[(\mu_{0,2}b_0c_2+A_2\hat{c}_1)+(\mu_{2,2}b_2c_2+\mu_{0,4}b_0c_4+A_4\hat{c}_3)r^2+\cdots$$
$$+\Bigg(\sum_{\substack{i+k-2=\sigma_1=j-1\\ i\ even\\ k\ even\\ j\ odd}}\mu_{i,k}b_ic_k+A_{j+1}\hat{c}_j\Bigg)r^{\sigma_1}+\cdots+(\lambda_{1,0}a_1b_0+B_{0,p+1}\hat{a}_0)r^{2p}+\cdots$$
$$+\Bigg(\sum_{\substack{i+k-1=\sigma_2=j\\ i\ odd\\ k\ even\\ j\ even}}\lambda_{i,k}a_ib_k+B_{j,p+1}\hat{a}_j\Bigg)r^{\sigma_2+2p}\Bigg].$$

This polynomial has at most

$$\max\left\{\frac{ev(l)+ev(m)+od(n)+ev(m)-1}{2},+\frac{od(l)+1+ev(n)}{2}\right\}$$
$$=\max\left\{2\left[\frac{m}{2}\right]+\left[\frac{l}{2}\right]+\left[\frac{n-1}{2}\right],\left[\frac{l+1}{2}\right]+\left[\frac{n}{2}\right]\right\}$$

real potive roots.

(ii) If $2p \le \sigma_1 \le \sigma_2 + 2p$,

$$F_{20}(r) = r\Bigg[(\mu_{0,2}b_0c_2 + A_2\hat{c}_1) + (\mu_{2,2}b_2c_2 + \mu_{0,4}b_0c_4 + A_4\hat{c}_3)r^2 + \cdots + (\lambda_{1,0}a_1b_0 + B_{0,p+1}\hat{a}_0)r^{2p} + \cdots + \Bigg(\sum_{\substack{i+k-2=\sigma_1=j-1\\ i\ even\\ k\ even\\ j\ odd}} \mu_{i,k}b_ic_k + A_{j+1}\hat{c}_j\Bigg)r^{\sigma_1} + \cdots + \Bigg(\sum_{\substack{i+k-1=\sigma_2=j\\ i\ odd\\ k\ even\\ j\ even}} \lambda_{i,k}a_ib_k + B_{j,p+1}\hat{a}_j\Bigg)r^{\sigma_2+2p}\Bigg].$$

The maximum number of positive real roots that can has F_{20} is at most

$$\max\left\{\left[\frac{ev(m)+od(n)+2p-1}{2}\right],\left[\frac{ev(n)+2p}{2}\right]\right\}$$
$$=\max\left\{\left[\frac{m}{2}\right]+\left[\frac{n-1}{2}\right]+p,\left[\frac{n}{2}\right]+p\right\}.$$

(iii) If $\sigma_1 \ge \sigma_2 + 2p$, the polynomial F_{20} is

$$F_{20}(r) = r\Bigg[(\mu_{0,2}b_0c_2 + A_2\hat{c}_1) + (\mu_{2,2}b_2c_2 + \mu_{0,4}b_0c_4 + A_4\hat{c}_3)r^2 + \cdots + (\lambda_{1,0}a_1b_0 + B_{0,p+1}\hat{a}_0)r^{2p} + \cdots + \Bigg(\sum_{\substack{i+k-1=\sigma_2=j\\ i\ odd\\ k\ even\\ j\ even}} \lambda_{i,k}a_ib_k + B_{j,p+1}\hat{a}_j\Bigg)r^{\sigma_2+2p} + \cdots + \Bigg(\sum_{\substack{i+k-2=\sigma_1=j-1\\ i\ even\\ k\ even\\ j\ odd}} \mu_{i,k}b_ic_k + A_{j+1}\hat{c}_j\Bigg)r^{\sigma_1}\Bigg],$$

which has at most

$$\max\left\{\left[\frac{ev(l)+ev(m)-2}{2}\right],\left[\frac{od(l)-1}{2}\right]\right\}=\max\left\{\left[\frac{m}{2}\right]+\left[\frac{l-2}{2}\right],\left[\frac{l-1}{2}\right]\right\}$$

positive roots. Hence Theorem 2 follows.

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