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THE IMPROVED FOURIER SPLITTING METHOD AND DECAY ESTIMATES OF THE GLOBAL SOLUTIONS OF THE CAUCHY PROBLEMS FOR NONLINEAR SYSTEMS OF FLUID DYNAMICS EQUATIONS*

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Dedicated to Professor Boling Guo on the occasion of his eightieth birthday!

Abstract

Consider the Cauchy problems for an n-dimensional nonlinear system of fluid dynamics equations. The main purpose of this paper is to improve the Fourier splitting method to accomplish the decay estimates with sharp rates of the global weak solutions of the Cauchy problems. We will couple together the elementary uniform energy estimates of the global weak solutions and a well known Gronwall's inequality to improve the Fourier splitting method. This method was initiated by Maria Schonbek in the 1980's to study the optimal long time asymptotic behaviours of the global weak solutions of the nonlinear system of fluid dynamics equations. As applications, the decay estimates with sharp rates of the global weak solutions of the Cauchy problems for n-dimensional incompressible Navier-Stokes equations, for the n-dimensional magnetohydrodynamics equations and for many other very interesting nonlinear evolution equations with dissipations can be established.

Keywords nonlinear systems of fluid dynamics equations; global weak solutions; decay estimates; uniform energy estimates; Fourier transformation; Plancherel's identity; Gronwall's inequality; improved Fourier splitting method

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1 Introduction

1.1 The mathematical model equations

First of all, consider the Cauchy problems for the n-dimensional incompressible Navier-Stokes equations

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$$\frac{\partial \mathbf{u}}{\partial t} - \alpha \triangle \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x}, t), \quad \nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{f} = 0, \tag{1}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \nabla \cdot \mathbf{u}_0 = 0. \tag{2}$$

The real vector valued function $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ represents the velocity of the fluid at position $\mathbf{x}$ and time t. The real scalar function $p = p(\mathbf{x}, t)$ represents the pressure of the fluid at $\mathbf{x}$ and t.

Secondly, consider the Cauchy problems for the n-dimensional magnetohydrodynamics equations

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{1}{\mathrm{RE}} \triangle \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - (\mathbf{A} \cdot \nabla)\mathbf{A} + \nabla P = \mathbf{f}(\mathbf{x}, t), \tag{3}$$

$$\frac{\partial \mathbf{A}}{\partial t} - \frac{1}{\mathrm{RM}} \triangle \mathbf{A} + (\mathbf{u} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{u} = \mathbf{g}(\mathbf{x}, t), \tag{4}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{f} = 0, \quad \nabla \cdot \mathbf{A} = 0, \quad \nabla \cdot \mathbf{g} = 0, \tag{5}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{A}(\mathbf{x}, 0) = \mathbf{A}_0(\mathbf{x}), \quad \nabla \cdot \mathbf{u}_0 = 0, \quad \nabla \cdot \mathbf{A}_0 = 0. \tag{6}$$

In this system, the real vector valued function $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ represents the velocity of the fluid at position $\mathbf{x}$ and time t, the real vector valued function $\mathbf{A} = \mathbf{A}(\mathbf{x}, t)$ represents the magnetic field at position $\mathbf{x}$ and time t. The real scalar function $P(\mathbf{x}, t) = p(\mathbf{x}, t) + \frac{M^2}{2\mathrm{RE}\cdot\mathrm{RM}} |\mathbf{A}(\mathbf{x}, t)|^2$ represents the total pressure, where the real scalar function $p = p(\mathbf{x}, t)$ represents the pressure of the fluid and $\frac{1}{2}|\mathbf{A}(\mathbf{x}, t)|^2$ represents the magnetic pressure. Additionally, $M > 0$ represents the Hartman constant, RE represents the Reynolds constant and RM represents the magnetic Reynolds constant.

Now let us consider the Cauchy problems for the following n-dimensional nonlinear system of fluid dynamics equations

$$\frac{\partial \mathbf{u}}{\partial t} - \alpha \triangle \mathbf{u} + \mathcal{N}(\mathbf{u}, \nabla \mathbf{u}) = \mathbf{f}(\mathbf{x}, t), \tag{7}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}). \tag{8}$$

In this system, $\alpha > 0$ is a positive constant, $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ represents the spatial variable, the dimension $n \geq 3$. Moreover, $\mathbf{u}(\mathbf{x}, t) = (u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), \cdots, u_m(\mathbf{x}, t))$ represents the unknown function, $\mathbf{f}(\mathbf{x}, t) = (f_1(\mathbf{x}, t), f_2(\mathbf{x}, t), \cdots, f_m(\mathbf{x}, t))$ represents the external force, and $\mathcal{N}(\mathbf{u}, \nabla \mathbf{u}) = (N_1(\mathbf{u}, \nabla \mathbf{u}), N_2(\mathbf{u}, \nabla \mathbf{u}), \cdots, N_m(\mathbf{u}, \nabla \mathbf{u}))$ represents the nonlinear function, which is sufficiently smooth, $m \geq n$ is an integer.

The general system (7)-(8) contains the n-dimensional incompressible Navier-Stokes equations (1)-(2) and the n-dimensional magnetohydrodynamics equations (3)-(6) as particular examples. The general system also contains many other interesting nonlinear evolution equations with dissipations as examples.

Many mathematicians have accomplished the existence of the global weak solutions of the Cauchy problems for the n-dimensional incompressible Navier-Stokes equations. They have also established the existence of the global smooth solutions with small initial functions and small external forces. However, the uniform energy estimates of all order derivatives of the global weak solutions with large initial functions or large external forces have been open, see [4,8,9]. For the n-dimensional magnetohydrodynamics equations, there hold very similar results. Very recently, Lei and Lin [1], Lei, Lin and Zhou [2], Peng and Zhou [5] established the existence of large global smooth solution of three-dimensional incompressible Navier-Stokes equations for special cases.

1.2 The main purpose

The decay estimates with sharp rates of the global weak solutions of the Cauchy problems for the n-dimensional incompressible Navier-Stokes equations (1)-(2), the global weak solutions of the Cauchy problems for the n-dimensional magnetohydrodynamics equations (3)-(6) and the global weak solutions of the Cauchy problems for many other interesting nonlinear evolution equations with dissipations are of great interests and importance in applied mathematics. The Fourier splitting method was developed by Maria Schonbek [6,7] to accomplish the optimal long time asymptotic behaviors of the global weak solutions. Today, it has become a very popular tool to study the asymptotic behaviors and thus has been widely used, see [3,10–12] for closely related results. To obtain the optimal decay rate, one must iterate some of the most important steps in the Fourier splitting method for finitely many times, see [6,7]. We will make complete use of the uniform energy estimates (see Lemmas 3, 4 and 8 in Section 2) and the Gronwall's inequality to avoid the iteration process for n-dimensional problems, where $n \geq 3$. Therefore, the main purpose of this paper is to improve the Fourier splitting method so that it may become the most powerful tool to accomplish the decay estimates with sharp rates for the global weak solutions of many nonlinear evolution equations with dissipations.

1.3 The main results — decay estimates with sharp rates

First of all, let us make several reasonable assumptions needed for the main results.

(A1) Suppose that the initial function $\mathbf{u}_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and the external force $\mathbf{f} \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n))$.

(A2) Suppose that there exist real scalar functions $\phi_{kl} \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $\psi_{kl} \in C^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+)$, such that

$$\mathbf{u}_0(\mathbf{x}) = \left(\sum_{l=1}^{n} \frac{\partial \phi_{1l}}{\partial x_l}(\mathbf{x}), \sum_{l=1}^{n} \frac{\partial \phi_{2l}}{\partial x_l}(\mathbf{x}), \cdots, \sum_{l=1}^{n} \frac{\partial \phi_{ml}}{\partial x_l}(\mathbf{x}) \right),$$

$$\mathbf{f}(\mathbf{x},t)=\left(\sum_{l=1}^{n}\frac{\partial\psi_{1l}}{\partial x_l}(\mathbf{x},t),\sum_{l=1}^{n}\frac{\partial\psi_{2l}}{\partial x_l}(\mathbf{x},t),\cdots,\sum_{l=1}^{n}\frac{\partial\psi_{ml}}{\partial x_l}(\mathbf{x},t)\right),$$

and

$$\frac{\partial\phi_{kl}}{\partial x_l}\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n),\quad \frac{\partial\psi_{kl}}{\partial x_l}\in L^1(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+,L^2(\mathbb{R}^n)),$$

for all $k=1,2,\cdots,m$ and $l=1,2,\cdots,n$.

(A3) Suppose that there holds the condition

$$\int_0^\infty(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{f}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t<\infty.$$

(A4) Suppose that there holds the condition

$$\int_0^\infty(1+t)^{2+n/2}\int_{\mathbb{R}^n}|\mathbf{f}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t<\infty,$$

if

$$\int_{\mathbb{R}^n}\mathbf{u}_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\mathbf{f}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t=0.$$

(A5) Suppose that the nonlinear function satisfies

$$\int_{\mathbb{R}^n}\mathbf{u}(\mathbf{x},t)\cdot\mathcal{N}(\mathbf{u}(\mathbf{x},t),\nabla\mathbf{u}(\mathbf{x},t))\mathrm{d}\mathbf{x}=0,$$

for all $\mathbf{u}\in L^\infty(\mathbb{R}^+,L^2(\mathbb{R}^n))$ with $\nabla\mathbf{u}\in L^2(\mathbb{R}^+,L^2(\mathbb{R}^n))$.

(A6) Suppose that the nonlinear function satisfies the condition

$$|\mathcal{N}(\mathbf{u},\nabla\mathbf{u})|\le C_1|\mathbf{u}||\nabla\mathbf{u}|,$$

for all $\mathbf{u}\in C^1(\mathbb{R}^n)$, where $C_1>0$ is a positive constant, independent of $\mathbf{u}$ and $\nabla\mathbf{u}$.

(A7) Suppose that the nonlinear function satisfies

$$\left|\widehat{\mathcal{N}(\mathbf{u},\nabla\mathbf{u})}(\xi,t)\right|\le C_2|\xi|\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x},$$

for all $\mathbf{u}\in L^\infty(\mathbb{R}^+,L^2(\mathbb{R}^n))$ with $\nabla\mathbf{u}\in L^2(\mathbb{R}^+,L^2(\mathbb{R}^n))$ and for all $(\xi,t)\in\mathbb{R}^n\times\mathbb{R}^+$, where $C_2>0$ is a positive constant, independent of $\mathbf{u}$, $\nabla\mathbf{u}$ and (ξ,t).

(A8) Suppose that there exists a global weak solution to the Cauchy problems for the nonlinear system of fluid dynamics equations (7)-(8):

$$\mathbf{u}\in L^\infty(\mathbb{R}^+,L^2(\mathbb{R}^n)),\quad \nabla\mathbf{u}\in L^2(\mathbb{R}^+,L^2(\mathbb{R}^n)).$$

Theorem 1 (I) *Suppose that assumptions* (A1), (A3), (A5)-(A8) *hold. There holds the decay estimate*

$$(1+t)^{n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_3,$$

for all time $t > 0$, where $C_3 > 0$ is a positive constant, independent of $\mathbf{u}$ *and* $(\mathbf{x}, t)$.

(II) *Suppose that assumptions* (A1)-(A8) *hold. There holds the decay estimate with sharp rate*

$$(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_4,$$

for all time $t > 0$, where $C_4 > 0$ is a positive constant, independent of $\mathbf{u}$ *and* $(\mathbf{x}, t)$.

The global weak solutions of the Cauchy problems for the n-dimensional incompressible Navier-Stokes equations (1)-(2) and for the n-dimensional magnetohydrodynamics equations (3)-(6) satisfy these conditions and results.

2 The Mathematical Analysis and the Proofs of the Main Results

We will couple together the elementary uniform energy estimates, the Fourier transformation, the Plancherel's identity and the Gronwall's inequality to improve the Fourier splitting method to accomplish the decay estimates with sharp rates for the global weak solutions of the Cauchy problems for the nonlinear systems of fluid dynamics equations (7)-(8). The improved Fourier splitting method involves the splitting of the frequency space into two time-dependent subspaces (a small ball with radius proportional to $(1+t)^{-1/2}$ and the exterior of the small ball) and the delicate estimates of the Fourier transformation of the global weak solutions. The key point of the improvement is that for many nonlinear evolution equations with dissipations, we may apply the improved Fourier splitting method to accomplish the decay estimates with sharp rates for the global weak solutions.

2.1 The uniform energy estimates

The main purpose is to use traditional ideas, methods and techniques to establish some uniform energy estimates.

Lemma 1(The Cauchy-Schwartz's inequality) *Let the functions $f \in L^2(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$. There holds the following Cauchy-Schwartz's inequality*

$$\left[\int_{\mathbb{R}^n} f(\mathbf{x})g(\mathbf{x})\mathrm{d}\mathbf{x}\right]^2\le\int_{\mathbb{R}^n}|f(\mathbf{x})|^2\mathrm{d}\mathbf{x}\int_{\mathbb{R}^n}|g(\mathbf{x})|^2\mathrm{d}\mathbf{x}.$$

Lemma 2(The Hölder's inequality) *Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, where $p \ge 1$ and $q \ge 1$ are positive constants, such that $\frac{1}{p}+\frac{1}{q}=1$. There holds the following estimate*

$$\left|\int_{\mathbb{R}^n} f(\mathbf{x})g(\mathbf{x})\mathrm{d}\mathbf{x}\right|\le\left[\int_{\mathbb{R}^n}|f(\mathbf{x})|^p\mathrm{d}\mathbf{x}\right]^{1/p}\left[\int_{\mathbb{R}^n}|g(\mathbf{x})|^q\mathrm{d}\mathbf{x}\right]^{1/q}.$$

Lemma 3 *Suppose that the initial function* $\mathbf{u}_0 \in L^2(\mathbb{R}^n)$ *and the external force* $\mathbf{f} \in L^1(\mathbb{R}^+, L^2(\mathbb{R}^n))$. *There holds the following uniform energy estimate*

$$\left[\int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} + 2\alpha \int_0^t\int_{\mathbb{R}^n} |\nabla \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x}\mathrm{d}\tau\right]^{1/2} \leq \left[\int_{\mathbb{R}^n} |\mathbf{u}_0(\mathbf{x})|^2 \mathrm{d}\mathbf{x}\right]^{1/2} + \int_0^\infty \left[\int_{\mathbb{R}^n} |\mathbf{f}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\right]^{1/2} \mathrm{d}t.$$

In particular, if the initial function $\mathbf{u}_0 \in L^2(\mathbb{R}^n)$ *and the external force* $\mathbf{f} = \mathbf{0}$, *then there holds the following uniform energy estimate*

$$\int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} + 2\alpha \int_0^t\int_{\mathbb{R}^n} |\nabla \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x}\mathrm{d}\tau \leq \int_{\mathbb{R}^n} |\mathbf{u}_0(\mathbf{x})|^2 \mathrm{d}\mathbf{x}.$$

Proof Multiplying system (7) by $2\mathbf{u}$ and integrating the result with respect to $\mathbf{x}$ over $\mathbb{R}^n$ yield the following energy equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} + 2\alpha \int_{\mathbb{R}^n} |\nabla \mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} = 2\int_{\mathbb{R}^n} \mathbf{u}(\mathbf{x},t)\cdot \mathbf{f}(\mathbf{x},t)\mathrm{d}\mathbf{x},$$

where

$$\int_{\mathbb{R}^n} \mathbf{u}(\mathbf{x},t)\cdot \mathcal{N}(\mathbf{u}(\mathbf{x},t), \nabla \mathbf{u}(\mathbf{x},t))\mathrm{d}\mathbf{x} = 0.$$

By using Cauchy-Schwartz's inequality, there hold the following estimates

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \mathbf{u}(\mathbf{x},t)\cdot \mathbf{f}(\mathbf{x},t)\mathrm{d}\mathbf{x}\right| &\leq \left[\int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\right]^{1/2} \left[\int_{\mathbb{R}^n} |\mathbf{f}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\right]^{1/2} \\ &\leq \left[\int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} + 2\alpha \int_0^t\int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x}\mathrm{d}\tau\right]^{1/2} \left[\int_{\mathbb{R}^n} |\mathbf{f}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\right]^{1/2}.\end{aligned}$$

Now the above energy equation becomes the differential inequality

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} + 2\alpha \int_0^t\int_{\mathbb{R}^n} |\nabla \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x}\mathrm{d}\tau\right] \\ &\leq 2\left[\int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} + 2\alpha \int_0^t\int_{\mathbb{R}^n} |\nabla \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x}\mathrm{d}\tau\right]^{1/2} \left[\int_{\mathbb{R}^n} |\mathbf{f}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\right]^{1/2}.\end{aligned}$$

Simplifying the inequality gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} + 2\alpha \int_0^t\int_{\mathbb{R}^n} |\nabla \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x}\mathrm{d}\tau\right]^{1/2} \leq \left[\int_{\mathbb{R}^n} |\mathbf{f}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\right]^{1/2}.$$

Integrating this inequality with respect to time t leads to the desired energy estimate

$$\begin{aligned}&\left\{\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}+2\alpha\int_0^t\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau\right\}^{1/2}\\ \le&\left\{\int_{\mathbb{R}^n}|\mathbf{u}_0(\mathbf{x})|^2\mathrm{d}\mathbf{x}\right\}^{1/2}+\int_0^t\left\{\int_{\mathbb{R}^n}|\mathbf{f}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\mathrm{d}\tau.\end{aligned}$$

If $\mathbf{f}=0$, then the uniform energy estimate

$$\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}+2\alpha\int_0^t\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau\le\int_{\mathbb{R}^n}|\mathbf{u}_0(\mathbf{x})|^2\mathrm{d}\mathbf{x}$$

follows immediately. The proof of Lemma 3 is finished.

2.2 The Fourier transformation of the global weak solutions

Lemma 4 (I) *There holds the following Fourier representation*

$$\begin{aligned}\widehat{\mathbf{u}}(\xi,t)=&\exp(-\alpha|\xi|^2t)\widehat{\mathbf{u}}_0(\xi)+\int_0^t\exp[-\alpha|\xi|^2(t-\tau)]\widehat{\mathbf{f}}(\xi,\tau)\mathrm{d}\tau\\&-\int_0^t\exp[-\alpha|\xi|^2(t-\tau)]\widehat{\mathcal{N}(\mathbf{u},\nabla\mathbf{u})}(\xi,\tau)\mathrm{d}\tau,\end{aligned}$$

for all $(\xi,t)\in\mathbb{R}^n\times\mathbb{R}^+$.

(II) *There holds the following estimate*

$$\begin{aligned}|\widehat{\mathbf{u}}(\xi,t)|\le&|\widehat{\mathbf{u}}_0(\xi)|+\int_0^t|\widehat{\mathbf{f}}(\xi,\tau)|\mathrm{d}\tau\\&+C_5\left[\int_0^t\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau\right]^{1/2}\left[\int_0^t\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau\right]^{1/2},\end{aligned}$$

for all $(\xi,t)\in\mathbb{R}^n\times\mathbb{R}^+$, *where* $C_5>0$ *is a positive constant, independent of* $\widehat{\mathbf{u}}$ *and* (ξ,t).

(III) *There holds the following estimate*

$$|\widehat{\mathbf{u}}(\xi,t)|\le C_6|\xi|,$$

where $C_6>0$ *is a positive constant, independent of* $\widehat{\mathbf{u}}$ *and* (ξ,t), *if*

$$(1+t)^{n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_7,$$

for all time $t>0$ *and for another positive constant* $C_7>0$, *independent of* $\mathbf{u}$ *and* $(\mathbf{x},t)$.

Proof Performing the Fourier transformation to (7) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathbf{u}}(\xi,t)+\alpha|\xi|^2\widehat{\mathbf{u}}(\xi,t)+\widehat{\mathcal{N}(\mathbf{u},\nabla\mathbf{u})}(\xi,t)=\widehat{\mathbf{f}}(\xi,t).$$

Multiplying this equation by the integrating factor $\exp(\alpha|\xi|^2 t)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}[\exp(\alpha|\xi|^2 t)\widehat{\mathbf{u}}(\xi,t)] + \exp(\alpha|\xi|^2 t)\widehat{\mathcal{N}(\mathbf{u},\nabla\mathbf{u})}(\xi,t) = \exp(\alpha|\xi|^2 t)\widehat{\mathbf{f}}(\xi,t).$$

Integrating with respect to time t yields

$$\begin{aligned}&\exp(\alpha|\xi|^2 t)\widehat{\mathbf{u}}(\xi,t) + \int_0^t \exp(\alpha|\xi|^2\tau)\widehat{\mathcal{N}(\mathbf{u},\nabla\mathbf{u})}(\xi,\tau)\mathrm{d}\tau\\ &= \widehat{\mathbf{u}}_0(\xi) + \int_0^t \exp(\alpha|\xi|^2\tau)\widehat{\mathbf{f}}(\xi,\tau)\mathrm{d}\tau.\end{aligned}$$

Finally, we have the representation.

Now let us make estimates about $\widehat{\mathbf{u}}(\xi,t)$. First of all, there hold the following estimates

$$\begin{aligned}\left|\widehat{\mathcal{N}(\mathbf{u},\nabla\mathbf{u})}(\xi,t)\right| &\le C_1\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)||\nabla\mathbf{u}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\\ &\le C_1\left[\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]^{1/2}\left[\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]^{1/2}.\end{aligned}$$

In particular, for the n-dimensional incompressible Navier-Stokes equations (1)-(2), there hold the following estimates

$$\begin{aligned}&|\widehat{(\mathbf{u}\cdot\nabla)\mathbf{u}}| \le |\xi|\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}, \quad |\widehat{\nabla p}(\xi,t)| \le |\xi|\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x},\\ &|\widehat{(\mathbf{u}\cdot\nabla)\mathbf{u}}(\xi,t)| \le \left\{\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\left\{\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2},\\ &|\widehat{\nabla p}(\xi,t)| \le 2\left\{\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\left\{\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2},\end{aligned}$$

for all $(\xi,t)\in\mathbb{R}^n\times\mathbb{R}^+$.

Therefore, there hold the following estimates

$$\begin{aligned}|\widehat{\mathbf{u}}(\xi,t)| &\le |\widehat{\mathbf{u}}_0(\xi)| + \int_0^t|\widehat{\mathbf{f}}(\xi,\tau)|\mathrm{d}\tau\\ &\quad + C_1\int_0^t\left[\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]^{1/2}\left[\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]^{1/2}\mathrm{d}\tau\\ &\le |\widehat{\mathbf{u}}_0(\xi)| + \int_0^t|\widehat{\mathbf{f}}(\xi,\tau)|\mathrm{d}\tau\\ &\quad + C_1\left[\int_0^t\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau\right]^{1/2}\left[\int_0^t\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau\right]^{1/2}.\end{aligned}$$

Moreover, if there exist real scalar functions $\phi_{kl}\in C^1(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ and $\psi_{kl}\in C^1(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^n\times\mathbb{R}^+)$, such that

$$\mathbf{u}_0(\mathbf{x}) = \left(\sum_{l=1}^{n} \frac{\partial \phi_{1l}}{\partial x_l}(\mathbf{x}), \sum_{l=1}^{n} \frac{\partial \phi_{2l}}{\partial x_l}(\mathbf{x}), \cdots, \sum_{l=1}^{n} \frac{\partial \phi_{ml}}{\partial x_l}(\mathbf{x})\right),$$

$$\mathbf{f}(\mathbf{x}, t) = \left(\sum_{l=1}^{n} \frac{\partial \psi_{1l}}{\partial x_l}(\mathbf{x}, t), \sum_{l=1}^{n} \frac{\partial \psi_{2l}}{\partial x_l}(\mathbf{x}, t), \cdots, \sum_{l=1}^{n} \frac{\partial \psi_{ml}}{\partial x_l}(\mathbf{x}, t)\right),$$

for all $k = 1, 2, \cdots, m$ and $l = 1, 2, \cdots, n$, then we have

$$\widehat{\mathbf{u}}_0(\xi) = \mathrm{i}\left(\sum_{l=1}^{n} \xi_l \widehat{\phi}_{1l}(\xi), \sum_{l=1}^{n} \xi_l \widehat{\phi}_{2l}(\xi), \cdots, \sum_{l=1}^{n} \xi_l \widehat{\phi}_{ml}(\xi)\right),$$

$$\widehat{\mathbf{f}}(\xi, t) = \mathrm{i}\left(\sum_{l=1}^{n} \xi_l \widehat{\psi}_{1l}(\xi, t), \sum_{l=1}^{n} \xi_l \widehat{\psi}_{2l}(\xi, t), \cdots, \sum_{l=1}^{n} \xi_l \widehat{\psi}_{ml}(\xi, t)\right).$$

By applying Cauchy-Schwartz's inequality to the Fourier transformations, we get the following estimates

$$\begin{aligned}
|\widehat{\mathbf{u}}_0(\xi)|^2 &= \sum_{k=1}^{m} \left|\sum_{l=1}^{n} \xi_l \widehat{\phi}_{kl}(\xi)\right|^2 \\
&\le \sum_{k=1}^{m} \sum_{l=1}^{n} \xi_l^2 \sum_{l=1}^{n} |\widehat{\phi}_{kl}(\xi)|^2 = |\xi|^2 \sum_{k=1}^{m} \sum_{l=1}^{n} |\widehat{\phi}_{kl}(\xi)|^2, \\
|\widehat{\mathbf{f}}(\xi, t)|^2 &= \sum_{k=1}^{m} \left|\sum_{l=1}^{n} \xi_l \widehat{\psi}_{kl}(\xi, t)\right|^2 \\
&\le \sum_{k=1}^{m} \sum_{l=1}^{n} \xi_l^2 \sum_{l=1}^{n} |\widehat{\psi}_{kl}(\xi, t)|^2 = |\xi|^2 \sum_{k=1}^{m} \sum_{l=1}^{n} |\widehat{\psi}_{kl}(\xi, t)|^2.
\end{aligned}$$

Recall that there exists a positive constant $C_2 > 0$, independent of $\widehat{\mathbf{u}}(\xi, t)$ and (ξ, t), such that

$$\left|\widehat{\mathcal{N}(\mathbf{u}, \nabla \mathbf{u})}(\xi, t)\right| \le C_2 |\xi| \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x},$$

for all $(\xi, t) \in \mathbb{R}^n \times \mathbb{R}^+$. Now we obtain the following estimate

$$\begin{aligned}
|\widehat{\mathbf{u}}(\xi, t)| &\le |\widehat{\mathbf{u}}_0(\xi)| + \int_0^t |\widehat{\mathbf{f}}(\xi, \tau)| \mathrm{d}\tau + C_2 |\xi| \int_0^t \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x}, \tau)|^2 \mathrm{d}\mathbf{x} \mathrm{d}\tau \\
&\le |\xi| \left[\sum_{k=1}^{m} \sum_{l=1}^{n} |\widehat{\phi}_{kl}(\xi)|^2\right]^{1/2} + |\xi| \int_0^t \left[\sum_{k=1}^{m} \sum_{l=1}^{n} |\widehat{\psi}_{kl}(\xi, \tau)|^2\right]^{1/2} \mathrm{d}\tau \\
&\quad + C_9 |\xi| \int_0^t \frac{1}{(1+\tau)^{n/2}} \mathrm{d}\tau \\
&\le C_{10} |\xi|,
\end{aligned}$$

for all $(\xi, t) \in \mathbb{R}^n \times \mathbb{R}^+$. The proof of Lemma 4 is finished.

2.3 The Fourier splitting method

Now let us review the Fourier splitting method developed in [6, 7] to establish the decay estimates for the global weak solutions of system (7)-(8).

Lemma 5(The Plancherel's identity) *There holds the following Plancherel's identity for all real vector valued functions* $\mathbf{f} \in L^2(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |\mathbf{f}(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\widehat{\mathbf{f}}(\xi)|^2 \mathrm{d}\xi.$$

Lemma 6(The Gronwall's inequality) *Suppose that the nonnegative continuous functions* $f \geq 0$, $g \geq 0$ *and* $h \geq 0$ *satisfy the inequality*

$$g(t) \leq f(t) + \int_0^t g(\tau)h(\tau)\mathrm{d}\tau,$$

for all $t > 0$, *where the derivative* $f' \geq 0$. *Then*

$$g(t) \leq f(t) \exp\left\{\int_0^t h(\tau)\mathrm{d}\tau\right\},$$

for all $t > 0$.

Lemma 7 *Let* $t > 0$ *and define*

$$\Omega(t) = \{\xi \in \mathbb{R}^n : \alpha|\xi|^2(1+t) \leq 2n\}.$$

There holds the following estimate

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\{(1+t)^{2n} \int_{\mathbb{R}^n} |\widehat{\mathbf{u}}(\xi,t)|^2 \mathrm{d}\xi\right\} \\ &\leq 4n(1+t)^{2n-1} \int_{\Omega(t)} |\widehat{\mathbf{u}}(\xi,t)|^2 \mathrm{d}\xi + \frac{1}{2n}(1+t)^{2n+1} \int_{\mathbb{R}^n} |\widehat{\mathbf{f}}(\xi,t)|^2 \mathrm{d}\xi. \end{aligned}$$

Proof Multiplying system (7) by $2\mathbf{u}$ and integrating the result with respect to $\mathbf{x}$ over $\mathbb{R}^n$ yield

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} + 2\alpha \int_{\mathbb{R}^n} |\nabla \mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} = 2\int_{\mathbb{R}^n} \mathbf{u}(\mathbf{x},t)\cdot \mathbf{f}(\mathbf{x},t)\mathrm{d}\mathbf{x},$$

where

$$\int_{\mathbb{R}^n} \mathbf{u}(\mathbf{x},t) \cdot \mathcal{N}(\mathbf{u}(\mathbf{x},t), \nabla \mathbf{u}(\mathbf{x},t))\mathrm{d}\mathbf{x} = 0.$$

Applying the Plancherel's identity to this equation gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^n} |\widehat{\mathbf{u}}(\xi,t)|^2 \mathrm{d}\xi + 2\alpha \int_{\mathbb{R}^n} |\xi|^2 |\widehat{\mathbf{u}}(\xi,t)|^2 \mathrm{d}\xi = 2\int_{\mathbb{R}^n} \widehat{\mathbf{u}}(\xi,t)\cdot \overline{\widehat{\mathbf{f}}}(\xi,t)\mathrm{d}\xi.$$

Multiplying it by $(1+t)^{2n}$ to get the energy equation

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{(1+t)^{2n}\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\right\}+2\alpha(1+t)^{2n}\int_{\mathbb{R}^n}|\xi|^2|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\\ =\;&2n(1+t)^{2n-1}\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi+2(1+t)^{2n}\int_{\mathbb{R}^n}\widehat{\mathbf{u}}(\xi,t)\cdot\overline{\widehat{\mathbf{f}}}(\xi,t)\mathrm{d}\xi.\end{aligned}$$

By applying Cauchy-Schwartz's inequality, we have

$$\begin{aligned}&2(1+t)^{2n}\left|\int_{\mathbb{R}^n}\widehat{\mathbf{u}}(\xi,t)\cdot\overline{\widehat{\mathbf{f}}}(\xi,t)\mathrm{d}\xi\right|\\ \le\;&2n(1+t)^{2n-1}\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi+\frac{1}{2n}(1+t)^{2n+1}\int_{\mathbb{R}^n}|\widehat{\mathbf{f}}(\xi,t)|^2\mathrm{d}\xi.\end{aligned}$$

Now the above energy equation becomes the inequality

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{(1+t)^{2n}\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\right\}+2\alpha(1+t)^{2n}\int_{\mathbb{R}^n}|\xi|^2|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\\ \le\;&4n(1+t)^{2n-1}\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi+\frac{1}{2n}(1+t)^{2n+1}\int_{\mathbb{R}^n}|\widehat{\mathbf{f}}(\xi,t)|^2\mathrm{d}\xi.\end{aligned}$$

Recall that $\Omega(t)=\{\xi\in\mathbb{R}^n:\alpha|\xi|^2(1+t)\le 2n\}$. Then we have the following estimates

$$\begin{aligned}&2\alpha(1+t)^{2n}\int_{\mathbb{R}^n}|\xi|^2|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\\ =\;&2\alpha(1+t)^{2n}\int_{\Omega(t)}|\xi|^2|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi+2\alpha(1+t)^{2n}\int_{\Omega(t)^c}|\xi|^2|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\\ \ge\;&2\alpha(1+t)^{2n}\int_{\Omega(t)^c}|\xi|^2|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\\ \ge\;&4n(1+t)^{2n-1}\int_{\Omega(t)^c}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\\ =\;&4n(1+t)^{2n-1}\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi-4n(1+t)^{2n-1}\int_{\Omega(t)}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi.\end{aligned}$$

Now the energy inequality

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{(1+t)^{2n}\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\right\}+2\alpha(1+t)^{2n}\int_{\mathbb{R}^n}|\xi|^2|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\\ \le\;&4n(1+t)^{2n-1}\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi+\frac{1}{2n}(1+t)^{2n+1}\int_{\mathbb{R}^n}|\widehat{\mathbf{f}}(\xi,t)|^2\mathrm{d}\xi,\end{aligned}$$

becomes the new differential inequality

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{(1+t)^{2n}\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\right\}\\ \le\;&4n(1+t)^{2n-1}\int_{\Omega(t)}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi+\frac{1}{2n}(1+t)^{2n+1}\int_{\mathbb{R}^n}|\widehat{\mathbf{f}}(\xi,t)|^2\mathrm{d}\xi.\end{aligned}$$

The proof of Lemma 7 is finished.

2.4 The improved Fourier splitting method

Lemma 8 (I) *There holds the following decay estimate for the global weak solutions of the Cauchy problems* (7)-(8) *if assumptions* (A1), (A3), (A5)-(A8) *hold:*

$$(1+t)^{n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_{11},$$

for all time $t>0$, where $C_{11}>0$ is a positive constant, independent of $\mathbf{u}$ *and* $(\mathbf{x},t)$.

(II) *There holds the following decay estimate with sharp rate for the global weak solutions of the Cauchy problems for the nonlinear system of fluid dynamics equations* (7)-(8) *if assumptions* (A1)-(A8) *hold:*

$$(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_{12},$$

for all time $t>0$, where $C_{12}>0$ is a positive constant, independent of $\mathbf{u}$ *and* $(\mathbf{x},t)$.

Proof (I) Recall that there exists a positive constant $C_5>0$, independent of $\widehat{\mathbf{u}}(\xi,t)$ and (ξ,t), such that

$$\begin{aligned}|\widehat{\mathbf{u}}(\xi,t)|\le{}&|\widehat{\mathbf{u}}_0(\xi)|+\int_0^t|\widehat{\mathbf{f}}(\xi,\tau)|\mathrm{d}\tau\\&+C_5\left[\int_0^t\!\!\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau\right]^{1/2}\left[\int_0^t\!\!\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau\right]^{1/2},\end{aligned}$$

for all $(\xi,t)\in\mathbb{R}^n\times\mathbb{R}^+$.

Therefore, by using Lemma 7, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{(1+t)^{2n}\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\right\}\\
&\le 4n(1+t)^{2n-1}\int_{\Omega(t)}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi+\frac{1}{2n}(1+t)^{2n+1}\int_{\mathbb{R}^n}|\widehat{\mathbf{f}}(\xi,t)|^2\mathrm{d}\xi\\
&\le\frac{1}{2n}(1+t)^{2n+1}\int_{\mathbb{R}^n}|\widehat{\mathbf{f}}(\xi,t)|^2\mathrm{d}\xi+4n(1+t)^{2n-1}\int_{\Omega(t)}\left\{|\widehat{\mathbf{u}}_0(\xi)|+\int_0^t|\widehat{\mathbf{f}}(\xi,\tau)|\mathrm{d}\tau\right.\\
&\quad\left.+C_5\left[\int_0^t\!\!\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau\right]^{1/2}\left[\int_0^t\!\!\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau\right]^{1/2}\right\}^2\mathrm{d}\xi\\
&\le\frac{1}{2n}(1+t)^{2n+1}\int_{\mathbb{R}^n}|\widehat{\mathbf{f}}(\xi,t)|^2\mathrm{d}\xi\\
&\quad+C_{13}(1+t)^{3n/2-1}\left\{\int_{\mathbb{R}^n}|\mathbf{u}_0(\mathbf{x})|\mathrm{d}\mathbf{x}+\int_0^\infty\!\!\int_{\mathbb{R}^n}|\mathbf{f}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&\quad+C_{14}(1+t)^{3n/2-1}\left\{\int_0^t\!\!\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau\right\}\left\{\int_0^t\!\!\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau\right\}.\end{aligned}$$

Integrating this inequality with respect to time t yields

$$\begin{aligned}&(1+t)^{2n}\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\\ \leq&\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}_0(\xi)|^2\mathrm{d}\xi+\frac{1}{2n}(1+t)^{3n/2}\int_0^t(1+\tau)^{1+n/2}\int_{\mathbb{R}^n}|\widehat{\mathbf{f}}(\xi,\tau)|^2\mathrm{d}\xi\mathrm{d}\tau\\ &+C_{15}(1+t)^{3n/2}\left\{\int_{\mathbb{R}^n}|\mathbf{u}_0(\mathbf{x})|\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{f}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\ &+C_{16}(1+t)^{3n/2}\left\{\int_0^t\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau\right\}\left\{\int_0^t\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau\right\}.\end{aligned}$$

That is

$$\begin{aligned}&(1+t)^{n/2}\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\\ \leq&\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}_0(\xi)|^2\mathrm{d}\xi+\frac{1}{2n}\int_0^\infty(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\widehat{\mathbf{f}}(\xi,t)|^2\mathrm{d}\xi\mathrm{d}t\\ &+C_{17}\left\{\int_{\mathbb{R}^n}|\mathbf{u}_0(\mathbf{x})|\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{f}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\ &+C_{18}\left\{\int_0^t\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau\right\}\left\{\int_0^\infty\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\},\end{aligned}$$

where $C_{17}>0$ and $C_{18}>0$ are positive constants, independent of $\mathbf{u}$, $\widehat{\mathbf{u}}$, $(\mathbf{x},t)$ and (ξ,t). Note that

$$\int_0^t\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau=\frac{1}{(2\pi)^n}\int_0^t\frac{1}{(1+\tau)^{n/2}}\left[(1+\tau)^{n/2}\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}(\xi,\tau)|^2\mathrm{d}\xi\right]\mathrm{d}\tau.$$

Recall that there holds the following uniform energy estimate

$$\begin{aligned}&\left\{\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}+2\alpha\int_0^t\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau\right\}^{1/2}\\ \leq&\left\{\int_{\mathbb{R}^n}|\mathbf{u}_0(\mathbf{x})|^2\mathrm{d}\mathbf{x}\right\}^{1/2}+\int_0^\infty\left\{\int_{\mathbb{R}^n}|\mathbf{f}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\mathrm{d}t.\end{aligned}$$

By using Gronwall's inequality, it follows that

$$\begin{aligned}&(1+t)^{n/2}\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\\ \leq&\left\{\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}_0(\xi)|^2\mathrm{d}\xi+\frac{1}{2n}\int_0^\infty(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\widehat{\mathbf{f}}(\xi,t)|^2\mathrm{d}\xi\mathrm{d}t\right.\\ &\left.+\,C_{19}\left[\int_{\mathbb{R}^n}|\mathbf{u}_0(\mathbf{x})|\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{f}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right]^2\right\}\\ &\cdot\exp\left\{C_{20}\left[\int_0^\infty\frac{1}{(1+t)^{n/2}}\mathrm{d}t\right]\left[\int_0^\infty\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right]\right\},\end{aligned}$$

where $C_{19} > 0$ and $C_{20} > 0$ are positive constants, independent of $\widehat{\mathbf{u}}(\xi,t)$ and (ξ,t).

(II) Recall that

$$|\widehat{\mathbf{u}}(\xi,t)| \leq C_6|\xi|,$$

for all $(\xi,t) \in \mathbb{R}^n \times \mathbb{R}^+$. Now we have the following estimates

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{(1+t)^{2n}\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\right\}\\
&\leq 4n(1+t)^{2n-1}\int_{\Omega(t)}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi + \frac{1}{2n}(1+t)^{2n+1}\int_{\mathbb{R}^n}|\widehat{\mathbf{f}}(\xi,t)|^2\mathrm{d}\xi\\
&\leq 4n(1+t)^{2n-1}\int_{\Omega(t)}C_{21}|\xi|^2\mathrm{d}\xi + \frac{1}{2n}(1+t)^{2n+1}\int_{\mathbb{R}^n}|\widehat{\mathbf{f}}(\xi,t)|^2\mathrm{d}\xi.
\end{aligned}$$

Integrating the inequality in time t yields the estimate

$$\begin{aligned}
&(1+t)^{2n}\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\\
&\leq \int_{\mathbb{R}^n}|\widehat{\mathbf{u}}_0(\xi)|^2\mathrm{d}\xi + \frac{1}{2n}(1+t)^{3n/2-1}\int_0^t(1+\tau)^{2+n/2}\int_{\mathbb{R}^n}|\widehat{\mathbf{f}}(\xi,\tau)|^2\mathrm{d}\xi\mathrm{d}\tau + C_{22}(1+t)^{3n/2-1}.
\end{aligned}$$

That is

$$\begin{aligned}
&(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\\
&\leq \int_{\mathbb{R}^n}|\widehat{\mathbf{u}}_0(\xi)|^2\mathrm{d}\xi + C_{23} + C_{24}\int_0^{\infty}(1+t)^{2+n/2}\int_{\mathbb{R}^n}|\widehat{\mathbf{f}}(\xi,t)|^2\mathrm{d}\xi\mathrm{d}t.
\end{aligned}$$

The proof of Lemma 8 is finished.

Therefore, the proofs of the main results stated in Theorem 1 are finished.

Remark Both the global weak solutions of the Cauchy problems for the n-dimensional incompressible Navier-Stokes equations (1)-(2) and the global weak solutions of the Cauchy problems for the n-dimensional magnetohydrodynamics equations (3)-(6) enjoy the decay estimates.

3 Conclusions and Remarks

3.1 Summary

The main purpose of this paper is to improve the Fourier splitting method to simplify the mathematical analysis to accomplish the decay estimates with sharp rates.

We considered the Cauchy problems for the following n-dimensional nonlinear system of fluid dynamics equations

$$\frac{\partial \mathbf{u}}{\partial t} - \alpha\triangle\mathbf{u} + \mathcal{N}(\mathbf{u},\nabla\mathbf{u}) = \mathbf{f}(\mathbf{x},t), \quad \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}).$$

The general system contains the n-dimensional incompressible Navier-Stokes equations (1)-(2) and the n-dimensional magnetohydrodynamics equations (3)-(6) as particular examples. There holds the following decay estimate with sharp rate

$$(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_{25},$$

for all time $t>0$, where $C_{25}>0$ is a positive constant, independent of $\mathbf{u}$ and $(\mathbf{x},t)$. The uniform energy estimate played a key role in the mathematical analysis.

3.2 Summary about the n-dimensional incompressible Navier-Stokes equations

Consider the Cauchy problems for the n-dimensional incompressible Navier-Stokes equations

$$\begin{aligned}&\frac{\partial\mathbf{u}}{\partial t}-\alpha\triangle\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\mathbf{f}(\mathbf{x},t),\quad\nabla\cdot\mathbf{u}=0,\quad\nabla\cdot\mathbf{f}=0,\\&\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}),\quad\nabla\cdot\mathbf{u}_0=0.\end{aligned}$$

Suppose that the initial function $\mathbf{u}_0\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and the external force $\mathbf{f}\in L^1(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+,L^2(\mathbb{R}^n))$. There exists a global weak solution $\mathbf{u}\in L^\infty(\mathbb{R}^+,L^2(\mathbb{R}^n))$, such that $\nabla\mathbf{u}\in L^2(\mathbb{R}^+,L^2(\mathbb{R}^n))$.

There holds the following uniform energy estimate

$$\begin{aligned}&\left\{\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}+2\alpha\int_0^t\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\mathrm{d}\tau\right\}^{1/2}\\&\le\left\{\int_{\mathbb{R}^n}|\mathbf{u}_0(\mathbf{x})|^2\mathrm{d}\mathbf{x}\right\}^{1/2}+\int_0^t\left\{\int_{\mathbb{R}^n}|\mathbf{f}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right\}^{1/2}\mathrm{d}\tau.\end{aligned}$$

Due to the divergence free conditions $\nabla\cdot\mathbf{u}_0=0$ and $\nabla\cdot\mathbf{f}=0$, it is necessarily true that $\int_{\mathbb{R}^n}\mathbf{u}_0(\mathbf{x})\mathrm{d}\mathbf{x}=\mathbf{0}$ and $\int_{\mathbb{R}^n}\mathbf{f}(\mathbf{x},t)\mathrm{d}\mathbf{x}=\mathbf{0}$, for all $t>0$.

Suppose that there exist real scalar functions $\phi_{kl}\in C^1(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ and $\psi_{kl}\in C^1(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^n\times\mathbb{R}^+)$, such that

$$\begin{aligned}\mathbf{u}_0(\mathbf{x})&=\left(\sum_{l=1}^n\frac{\partial\phi_{1l}}{\partial x_l}(\mathbf{x}),\sum_{l=1}^n\frac{\partial\phi_{2l}}{\partial x_l}(\mathbf{x}),\cdots,\sum_{l=1}^n\frac{\partial\phi_{nl}}{\partial x_l}(\mathbf{x})\right),\\\mathbf{f}(\mathbf{x},t)&=\left(\sum_{l=1}^n\frac{\partial\psi_{1l}}{\partial x_l}(\mathbf{x},t),\sum_{l=1}^n\frac{\partial\psi_{2l}}{\partial x_l}(\mathbf{x},t),\cdots,\sum_{l=1}^n\frac{\partial\psi_{nl}}{\partial x_l}(\mathbf{x},t)\right),\end{aligned}$$

and that

$$\frac{\partial\phi_{kl}}{\partial x_l}\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n),\quad\frac{\partial\psi_{kl}}{\partial x_l}\in L^1(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+,L^2(\mathbb{R}^n)),$$

for all $k=1,2,\cdots,n$ and $l=1,2,\cdots,n$.

There holds the following decay estimate for the global weak solutions of the Cauchy problems for the n-dimensional incompressible Navier-Stokes equations

$$(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_{26},$$

for all time $t>0$, where $C_{26}>0$ is a positive constant, independent of $\mathbf{u}$ and $(\mathbf{x},t)$.

Let us review some important open problems about the global smooth solutions and their influences. Suppose that the initial function $\mathbf{u}_0\in L^1(\mathbb{R}^n)\cap H^{2m+1}(\mathbb{R}^n)$ and the external force $\mathbf{f}\in L^1(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+,L^2(\mathbb{R}^n))\cap L^2(\mathbb{R}^+,H^{2m}(\mathbb{R}^n))$. The following uniform energy estimates have been open

$$\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_{27},$$
$$\int_{\mathbb{R}^n}|\triangle\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_{28},$$
$$\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_{29},$$
$$\int_{\mathbb{R}^n}|\nabla\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_{30},$$

for all positive integers $m\ge 1$ and for all time $t>0$, where $C_{27}>0$, $C_{28}>0$, $C_{29}>0$ and $C_{30}>0$ are positive constants, independent of $\mathbf{u}$ and $(\mathbf{x},t)$. Therefore, the existence of the global smooth solution $\mathbf{u}\in L^\infty(\mathbb{R}^+,H^{2m+1}(\mathbb{R}^n))$ such that $\nabla\mathbf{u}\in L^2(\mathbb{R}^+,H^{2m+1}(\mathbb{R}^n))$ has been open.

Suppose that the initial function $\mathbf{u}_0\in L^1(\mathbb{R}^n)\cap H^{2m+1}(\mathbb{R}^n)$ and the external force $\mathbf{f}\in L^1(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+,L^2(\mathbb{R}^n))\cap L^2(\mathbb{R}^+,H^{2m}(\mathbb{R}^n))$. Suppose that there exists a global smooth solution to the Cauchy problems for the n-dimensional incompressible Navier-Stokes equations (1)-(2): $\mathbf{u}\in L^\infty(\mathbb{R}^+,H^{2m+1}(\mathbb{R}^n))$, such that $\nabla\mathbf{u}\in L^2(\mathbb{R}^+,H^{2m+1}(\mathbb{R}^n))$, where $m\ge 1$ is a positive integer. There hold the following decay estimates with sharp rates

$$(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_{31},$$
$$(1+t)^{2+n/2}\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_{32},$$
$$(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_{33},$$
$$(1+t)^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le C_{34},$$

and

$$
\begin{aligned}
&(1+t)^{1/2+n/2}\|\mathbf{u}(\cdot,t)\|_{L^\infty} \le C_{35},\\
&(1+t)^{1+n/2}\|\nabla\mathbf{u}(\cdot,t)\|_{L^\infty} \le C_{36},\\
&(1+t)^{m+1/2+n/2}\|\triangle^m\mathbf{u}(\cdot,t)\|_{L^\infty} \le C_{37},\\
&(1+t)^{m+1+n/2}\|\nabla\triangle^m\mathbf{u}(\cdot,t)\|_{L^\infty} \le C_{38},
\end{aligned}
$$

for all positive integers $m \ge 1$ and for all time $t > 0$, where $C_{31} > 0$, $C_{32} > 0$, $C_{33} > 0$, $C_{34} > 0$, $C_{35} > 0$, $C_{36} > 0$, $C_{37} > 0$, $C_{38} > 0$ are positive constants, independent of $\mathbf{u}$ and $(\mathbf{x}, t)$.

Now let us consider a very interesting question for the global smooth solutions of the Cauchy problems for the n-dimensional incompressible Navier-Stokes equations (1)-(2). As $t \to \infty$, how do the following exact limits

$$
\begin{aligned}
&\lim_{t\to\infty}\left\{(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\},\\
&\lim_{t\to\infty}\left\{(1+t)^{2+n/2}\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\},\\
&\lim_{t\to\infty}\left\{(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}|\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\},\\
&\lim_{t\to\infty}\left\{(1+t)^{2m+2+n/2}\int_{\mathbb{R}^n}|\nabla\triangle^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\},
\end{aligned}
$$

depend on the nonlinear function, the initial function, the nonhomogeneous function, the dimension n and the order of the derivative (that is, the integer m)? Do the global smooth solutions carry initial information (e.g. mass, energy and momentum of physical objects) to the very end? In another word, can the global smooth solutions "remember the very beginning" at "the very end"? We will couple together existing ideas, methods, results and new ideas to generate a very different method to solve these complicated mathematical problems and accomplish very general results.

Again consider the Cauchy problems for the n-dimensional incompressible Navier-Stokes equations

$$
\begin{aligned}
&\frac{\partial\mathbf{u}}{\partial t} - \alpha\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x},t), \quad \nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{f} = 0,\\
&\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \nabla\cdot\mathbf{u}_0 = 0.
\end{aligned}
$$

The following estimate has been open

$$
\left|\widehat{\mathcal{N}(\mathbf{u},\nabla\mathbf{u})}(\xi,t)\right| \le |\xi||\widehat{\mathbf{u}}(\xi,t)|\kappa_1(|\widehat{\mathbf{u}}(\xi,t)|) + |\xi|^{2-\varepsilon}|\widehat{\mathbf{u}}(\xi,t)|\kappa_2(|\widehat{\mathbf{u}}(\xi,t)|),
$$

for all $(\xi, t) \in \mathbb{R}^n \times \mathbb{R}^+$, where $\kappa_1 = \kappa_1(t)$ and $\kappa_2 = \kappa_2(t)$ are positive, continuous, increasing functions defined on $(0, \infty)$, $0 < \varepsilon \ll 1$ is a constant.

If this estimate is true, then there exists a unique global smooth solution $\mathbf{u} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$ to (1)-(2) and there hold the following decay estimates with sharp rates

$$(1+t)^{2m+1+n/2} \int_{\mathbb{R}^n} |\triangle^m \mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \le C_{39},$$

for all positive integers $m \ge 1$ and for all time $t > 0$, where $C_{39} > 0$ is a positive constant, independent of $\mathbf{u}$ and $(\mathbf{x}, t)$.

If there exists a global smooth solution $\mathbf{u} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$ to the Cauchy problems for the n-dimensional incompressible Navier-Stokes equations (1)-(2), then there holds the following solution representation

$$\begin{aligned}
\mathbf{u}(\mathbf{x},t) = & \frac{1}{(4\pi\alpha t)^{n/2}} \int_{\mathbb{R}^n} \exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha t}\right] \mathbf{u}_0(\mathbf{y})\mathrm{d}\mathbf{y} \\
& + \int_0^t \left\{ \frac{1}{[4\pi\alpha(t-\tau)]^{n/2}} \int_{\mathbb{R}^n} \exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha(t-\tau)}\right] \mathbf{f}(\mathbf{y},\tau)\mathrm{d}\mathbf{y} \right\} \mathrm{d}\tau \\
& - \int_0^t \left\{ \frac{1}{[4\pi\alpha(t-\tau)]^{n/2}} \int_{\mathbb{R}^n} \exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha(t-\tau)}\right] [(\mathbf{u}(\mathbf{y},\tau)\cdot\nabla)\mathbf{u}(\mathbf{y},\tau)]\mathrm{d}\mathbf{y} \right\} \mathrm{d}\tau \\
& - \int_0^t \left\{ \frac{1}{[4\pi\alpha(t-\tau)]^{n/2}} \int_{\mathbb{R}^n} \exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha(t-\tau)}\right] \nabla p(\mathbf{y},\tau)\mathrm{d}\mathbf{y} \right\} \mathrm{d}\tau.
\end{aligned}$$

3.3 Summary about the n-dimensional magnetohydrodynamics equations

Consider the Cauchy problems for the n-dimensional magnetohydrodynamics equations

$$\begin{aligned}
& \frac{\partial \mathbf{u}}{\partial t} - \frac{1}{\mathrm{RE}}\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{A}\cdot\nabla)\mathbf{A} + \nabla P = \mathbf{f}(\mathbf{x},t), \\
& \frac{\partial \mathbf{A}}{\partial t} - \frac{1}{\mathrm{RM}}\triangle\mathbf{A} + (\mathbf{u}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{u} = \mathbf{g}(\mathbf{x},t), \\
& \nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{f} = 0, \quad \nabla\cdot\mathbf{A} = 0, \quad \nabla\cdot\mathbf{g} = 0, \\
& \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{A}(\mathbf{x},0) = \mathbf{A}_0(\mathbf{x}), \quad \nabla\cdot\mathbf{u}_0 = 0, \quad \nabla\cdot\mathbf{A}_0 = 0.
\end{aligned}$$

Suppose that the initial functions

$$\mathbf{u}_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \quad \mathbf{A}_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

Suppose that the external forces

$$\begin{aligned}
& \mathbf{f} \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)), \\
& \mathbf{g} \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)).
\end{aligned}$$

There exists a global weak solution

$$\mathbf{u} \in L^{\infty}(\mathbb{R}^+, L^2(\mathbb{R}^n)), \quad \mathbf{A} \in L^{\infty}(\mathbb{R}^+, L^2(\mathbb{R}^n)),$$

such that

$$\nabla \mathbf{u} \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^n)), \quad \nabla \mathbf{f} \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^n)).$$

There holds the following uniform energy estimate

$$\begin{aligned}&\left\{\int_{\mathbb{R}^n}[|\mathbf{u}(\mathbf{x},t)|^2+|\mathbf{A}(\mathbf{x},t)|^2]\mathrm{d}\mathbf{x}+\int_0^t\int_{\mathbb{R}^n}\left[\frac{2}{\mathrm{RE}}|\nabla\mathbf{u}(\mathbf{x},\tau)|^2+\frac{2}{\mathrm{RM}}|\nabla\mathbf{A}(\mathbf{x},\tau)|^2\right]\mathrm{d}\mathbf{x}\mathrm{d}\tau\right\}^{1/2}\\ \leq&\left\{\int_{\mathbb{R}^n}[|\mathbf{u}_0(\mathbf{x})|^2+|\mathbf{A}_0(\mathbf{x})|^2]\mathrm{d}\mathbf{x}\right\}^{1/2}+\int_0^t\left\{\int_{\mathbb{R}^n}[|\mathbf{f}(\mathbf{x},\tau)|^2+|\mathbf{g}(\mathbf{x},\tau)|^2]\mathrm{d}\mathbf{x}\right\}^{1/2}\mathrm{d}\tau.\end{aligned}$$

Suppose that there exist real scalar functions

$$\begin{aligned}&\phi_{kl} \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \quad \psi_{kl} \in C^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+),\\ &\kappa_{kl} \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \quad \omega_{kl} \in C^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+),\end{aligned}$$

such that

$$\begin{aligned}\mathbf{u}_0(\mathbf{x}) &= \left(\sum_{l=1}^n \frac{\partial \phi_{1l}}{\partial x_l}(\mathbf{x}), \sum_{l=1}^n \frac{\partial \phi_{2l}}{\partial x_l}(\mathbf{x}), \cdots, \sum_{l=1}^n \frac{\partial \phi_{nl}}{\partial x_l}(\mathbf{x})\right),\\ \mathbf{f}(\mathbf{x},t) &= \left(\sum_{l=1}^n \frac{\partial \psi_{1l}}{\partial x_l}(\mathbf{x},t), \sum_{l=1}^n \frac{\partial \psi_{2l}}{\partial x_l}(\mathbf{x},t), \cdots, \sum_{l=1}^n \frac{\partial \psi_{nl}}{\partial x_l}(\mathbf{x},t)\right),\\ \mathbf{A}_0(\mathbf{x}) &= \left(\sum_{l=1}^n \frac{\partial \kappa_{1l}}{\partial x_l}(\mathbf{x}), \sum_{l=1}^n \frac{\partial \kappa_{2l}}{\partial x_l}(\mathbf{x}), \cdots, \sum_{l=1}^n \frac{\partial \kappa_{nl}}{\partial x_l}(\mathbf{x})\right),\\ \mathbf{g}(\mathbf{x},t) &= \left(\sum_{l=1}^n \frac{\partial \omega_{1l}}{\partial x_l}(\mathbf{x},t), \sum_{l=1}^n \frac{\partial \omega_{2l}}{\partial x_l}(\mathbf{x},t), \cdots, \sum_{l=1}^n \frac{\partial \omega_{nl}}{\partial x_l}(\mathbf{x},t)\right),\end{aligned}$$

and that

$$\begin{aligned}&\frac{\partial \phi_{kl}}{\partial x_l} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \quad \frac{\partial \psi_{kl}}{\partial x_l} \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)),\\ &\frac{\partial \kappa_{kl}}{\partial x_l} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \quad \frac{\partial \omega_{kl}}{\partial x_l} \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)),\end{aligned}$$

for all $k = 1, 2, \cdots, n$ and $l = 1, 2, \cdots, n$.

There holds the following decay estimate with sharp rate

$$(1+t)^{1+n/2} \int_{\mathbb{R}^n} [|\mathbf{u}(\mathbf{x},t)|^2 + |\mathbf{A}(\mathbf{x},t)|^2] \mathrm{d}\mathbf{x} \leq C_{40},$$

for all time $t > 0$, where $C_{40} > 0$ is a positive constant, independent of $(\mathbf{u}, \mathbf{A})$ and $(\mathbf{x}, t)$.

Let us review some open problems and their influences about system (3)-(6). Suppose that the initial functions

$$\mathbf{u}_0 \in L^1(\mathbb{R}^n) \cap H^{2m+1}(\mathbb{R}^n), \quad \mathbf{A}_0 \in L^1(\mathbb{R}^n) \cap H^{2m+1}(\mathbb{R}^n).$$

Suppose that the external forces

$$\begin{aligned}\mathbf{f} &\in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^n)),\\ \mathbf{g} &\in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^n)).\end{aligned}$$

The following uniform energy estimates have been open

$$\begin{aligned}&\int_{\mathbb{R}^n} [|\nabla \mathbf{u}(\mathbf{x}, t)|^2 + |\nabla \mathbf{A}(\mathbf{x}, t)|^2] \mathrm{d}\mathbf{x} \leq C_{41},\\ &\int_{\mathbb{R}^n} [|\triangle \mathbf{u}(\mathbf{x}, t)|^2 + |\triangle \mathbf{A}(\mathbf{x}, t)|^2] \mathrm{d}\mathbf{x} \leq C_{42},\\ &\int_{\mathbb{R}^n} [|\triangle^m \mathbf{u}(\mathbf{x}, t)|^2 + |\triangle^m \mathbf{A}(\mathbf{x}, t)|^2] \mathrm{d}\mathbf{x} \leq C_{43},\\ &\int_{\mathbb{R}^n} [|\nabla\triangle^m \mathbf{u}(\mathbf{x}, t)|^2 + |\nabla\triangle^m \mathbf{A}(\mathbf{x}, t)|^2] \mathrm{d}\mathbf{x} \leq C_{44},\end{aligned}$$

for all positive integers $m \geq 1$ and for all time $t > 0$, where $C_{41} > 0$, $C_{42} > 0$, $C_{43} > 0$, $C_{44} > 0$ are positive constants, independent of $(\mathbf{u}, \mathbf{A})$ and $(\mathbf{x}, t)$.

The existence of the global smooth solution of the Cauchy problems for the n-dimensional magnetohydrodynamics equations (3)-(6):

$$\mathbf{u} \in L^\infty(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n)), \quad \mathbf{A} \in L^\infty(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n)),$$

such that

$$\nabla \mathbf{u} \in L^2(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n)), \quad \nabla \mathbf{A} \in L^2(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n)),$$

has been open.

Suppose that there exists a global smooth solution to the Cauchy problems for the n-dimensional magnetohydrodynamics equations:

$$\mathbf{u} \in L^\infty(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n)), \quad \mathbf{A} \in L^\infty(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n)),$$

such that

$$\nabla \mathbf{u} \in L^2(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n)), \quad \nabla \mathbf{A} \in L^2(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n)),$$

where $m \geq 1$ is a positive integer.

For the global smooth solution of the n-dimensional magnetohydrodynamics equations (3)-(6), there hold the following decay estimates with sharp rates

$$
\begin{aligned}
&(1+t)^{1+n/2}\int_{\mathbb{R}^n}[|\mathbf{u}(\mathbf{x},t)|^2+|\mathbf{A}(\mathbf{x},t)|^2]\mathrm{d}\mathbf{x}\le C_{45},\\
&(1+t)^{2+n/2}\int_{\mathbb{R}^n}[|\nabla\mathbf{u}(\mathbf{x},t)|^2+|\nabla\mathbf{A}(\mathbf{x},t)|^2]\mathrm{d}\mathbf{x}\le C_{46},\\
&(1+t)^{2m+1+n/2}\int_{\mathbb{R}^n}[|\triangle^m\mathbf{u}(\mathbf{x},t)|^2+|\triangle^m\mathbf{A}(\mathbf{x},t)|^2]\mathrm{d}\mathbf{x}\le C_{47},\\
&(1+t)^{2m+2+n/2}\int_{\mathbb{R}^n}[|\nabla\triangle^m\mathbf{u}(\mathbf{x},t)|^2+|\nabla\triangle^m\mathbf{A}(\mathbf{x},t)|^2]\mathrm{d}\mathbf{x}\le C_{48},
\end{aligned}
$$

and

$$
\begin{aligned}
&(1+t)^{1/2+n/2}[\|\mathbf{u}(\cdot,t)\|_{L^\infty}+\|\mathbf{A}(\cdot,t)\|_{L^\infty}]\le C_{49},\\
&(1+t)^{1+n/2}[\|\nabla\mathbf{u}(\cdot,t)\|_{L^\infty}+\|\nabla\mathbf{A}(\cdot,t)\|_{L^\infty}]\le C_{50},\\
&(1+t)^{m+1/2+n/2}[\|\triangle^m\mathbf{u}(\cdot,t)\|_{L^\infty}+\|\triangle^m\mathbf{A}(\cdot,t)\|_{L^\infty}]\le C_{51},\\
&(1+t)^{m+1+n/2}[\|\nabla\triangle^m\mathbf{u}(\cdot,t)\|_{L^\infty}+\|\nabla\triangle^m\mathbf{A}(\cdot,t)\|_{L^\infty}]\le C_{52},
\end{aligned}
$$

for all positive integers $m\ge 1$ and for all time $t>0$, where $C_{45}>0$, $C_{46}>0$, $C_{47}>0$, $C_{48}>0$, $C_{49}>0$, $C_{50}>0$, $C_{51}>0$, $C_{52}>0$ are positive constants, independent of $(\mathbf{u},\mathbf{A})$ and $(\mathbf{x},t)$. The proofs follow from the Fourier splitting method.

Let $\alpha_1=\frac{1}{\mathrm{RE}}$ and $\alpha_2=\frac{1}{\mathrm{RM}}$. If there exists a global smooth solution $\mathbf{u}\in C^\infty(\mathbb{R}^n\times\mathbb{R}^+)$, $\mathbf{A}\in C^\infty(\mathbb{R}^n\times\mathbb{R}^+)$ to the Cauchy problems for the n-dimensional magnetohydrodynamics equations (3)-(6), then there hold the following solution representations

$$
\begin{aligned}
\mathbf{u}(\mathbf{x},t)=&\frac{1}{(4\pi\alpha_1 t)^{n/2}}\int_{\mathbb{R}^n}\exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha_1 t}\right]\mathbf{u}_0(\mathbf{y})\mathrm{d}\mathbf{y}\\
&+\int_0^t\left\{\frac{1}{[4\pi\alpha_1(t-\tau)]^{n/2}}\int_{\mathbb{R}^n}\exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha_1(t-\tau)}\right]\mathbf{f}(\mathbf{y},\tau)\mathrm{d}\mathbf{y}\right\}\mathrm{d}\tau\\
&-\int_0^t\left\{\frac{1}{[4\pi\alpha_1(t-\tau)]^{n/2}}\int_{\mathbb{R}^n}\exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha_1(t-\tau)}\right][(\mathbf{u}(\mathbf{y},\tau)\cdot\nabla)\mathbf{u}(\mathbf{y},\tau)]\mathrm{d}\mathbf{y}\right\}\mathrm{d}\tau\\
&-\int_0^t\left\{\frac{1}{[4\pi\alpha_1(t-\tau)]^{n/2}}\int_{\mathbb{R}^n}\exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha_1(t-\tau)}\right][(\mathbf{A}(\mathbf{y},\tau)\cdot\nabla)\mathbf{A}(\mathbf{y},\tau)]\mathrm{d}\mathbf{y}\right\}\mathrm{d}\tau\\
&-\int_0^t\left\{\frac{1}{[4\pi\alpha_1(t-\tau)]^{n/2}}\int_{\mathbb{R}^n}\exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha_1(t-\tau)}\right]\nabla P(\mathbf{y},\tau)\mathrm{d}\mathbf{y}\right\}\mathrm{d}\tau,
\end{aligned}
$$

$$\begin{aligned}\mathbf{A}(\mathbf{x},t) &= \frac{1}{(4\pi\alpha_2 t)^{n/2}}\int_{\mathbb{R}^n}\exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha_2 t}\right]\mathbf{A}_0(\mathbf{y})\mathrm{d}\mathbf{y}\\ &\quad+\int_0^t\left\{\frac{1}{[4\pi\alpha_2(t-\tau)]^{n/2}}\int_{\mathbb{R}^n}\exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha_2(t-\tau)}\right]\mathbf{g}(\mathbf{y},\tau)\mathrm{d}\mathbf{y}\right\}\mathrm{d}\tau\\ &\quad-\int_0^t\left\{\frac{1}{[4\pi\alpha_2(t-\tau)]^{n/2}}\int_{\mathbb{R}^n}\exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha_2(t-\tau)}\right][(\mathbf{u}(\mathbf{y},\tau)\cdot\nabla)\mathbf{A}(\mathbf{y},\tau)]\mathrm{d}\mathbf{y}\right\}\mathrm{d}\tau\\ &\quad-\int_0^t\left\{\frac{1}{[4\pi\alpha_2(t-\tau)]^{n/2}}\int_{\mathbb{R}^n}\exp\left[-\frac{|\mathbf{x}-\mathbf{y}|^2}{4\alpha_2(t-\tau)}\right][(\mathbf{A}(\mathbf{y},\tau)\cdot\nabla)\mathbf{u}(\mathbf{y},\tau)]\mathrm{d}\mathbf{y}\right\}\mathrm{d}\tau.\end{aligned}$$

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