TOTALLY UMBILICAL SUBMANIFOLD ON RIEMANNIAN MANIFOLD WITH AN ORTHOGONAL CONNECTION

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Abstract: In this paper, we investigate the fundamental equations of submanifolds under orthogonal connections and apply the results in totally umbilical submanifolds. By using the method of Cartan to split the torsion tensor into three components, we calculate and attain the fundamental equations. We consider a special orthogonal connection with which the Riemannian curvature has the same properties as the Levi-Civita connection. We use the fundamental equations to argue totally umbilical submanifolds on spaces with constant curvature, which generalizes the results under the Levi-Civita connection.

Keywords: orthogonal connections; fundamental equations in Riemannian manifolds; submanifold; umbilical point

2010 MR Subject Classification: 53C05; 53C17

Document code: A   **Article ID:** 0255-7797(2017)04-0672-13

1 Introduction

Orthogonal connections are affine connections compatible with the metric. Cartan researched general orthogonal connections in the 1920s. An orthogonal connection minus the Levi-Civita equals a tensor which is called torsion. Cartan found that in general the torsion tensor can split into three components: the vectorial torsion, the totally anti-symmetric one and the one of Cartan-type. Taking the scalar curvature of orthogonal connections one attains the Einstein-Cartan-Hilbert functional. Its critical points are Einstein manifolds, in particular the torsion of a critical point is zero.

We review Cartan's classification and Einstein-Cartan theory in Section 2. Under an orthogonal connection, in general, the Bianchi identity is not always hold, so many properties are not as brief as the Levi-Civita connection. We try to find an orthogonal connection which is not the Levi-Civita connection satisfying the Bianchi identity. In this paper, we focus on totally umbilical submanifold in a constant curvature space. We calculate the fundamental equations, and want to use the Causs equation to express the curvature and investigate the

totally umbilical submanifold under an orthogonal connection. To read more results about orthogonal connections, especially properties on subminifolds, please refer to our other work.

2 Preliminaries

We consider an n-dimensional manifold M equipped with some Riemannian metric g. Let ∇ denote the Levi-Civita connection on the tangent bundle. For any affine connection ∇' on the tangent bundle there exists a (2,1)-tensor field A such that

$$\nabla'_X Y = \nabla_X Y + A(X, Y) \tag{2.1}$$

for all vector fields X, Y.

In this article we will require all connections ∇' to be orthogonal, i.e., for all vector fields X, Y, Z, one has

$$X \langle Y, Z \rangle = \langle \nabla'_X Y, Z \rangle + \langle Y, \nabla'_X Z \rangle, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product given by the Riemannian metric g. For any tangent vector X one gets from (2.1) and (2.2) that the endomorphism $A(X, \cdot)$ is skew-adjoint

$$\langle A(X, Y), Z \rangle = -\langle Y, A(X, Z) \rangle. \tag{2.3}$$

Next, we want to express some curvature quantities for ∇' in terms of A and curvature quantities for ∇. To that end we fix some point $p \in M$, and we extend any tangent vectors $X, Y, Z, W \in T_p M$ to vector fields again denoted by X, Y, Z, W being synchronous in p, which means

$$\nabla_V X = \nabla_V Y = \nabla_V Z = \nabla_V W = 0 \quad \text{for any tangent vector } V \in T_p M.$$

Furthermore, we choose a local orthogonal frame of vector fields $E_1, \cdots, E_n$ on a neighbourhood of p, all being synchronous in p, then the Lie bracket $[X, Y] = \nabla_X Y - \nabla_Y X = 0$ vanishes in p, and synchronicity in p implies

$$\nabla'_X \nabla'_Y Z = \nabla_X \nabla_Y Z + (\nabla_X A)(Y, Z) + A(X, A(Y, Z)).$$

Hence, in p the Riemann tensor of ∇' reads as

$$\begin{aligned} \mathrm{Riem}'(X, Y)Z &= \nabla'_X \nabla'_Y Z - \nabla'_Y \nabla'_X Z - \nabla'_{[X,Y]} Z \\ &= \mathrm{Riem}(X, Y)Z + (\nabla_X A)(Y, Z) - (\nabla_Y A)(X, Z) \\ &\quad + A(X, A(Y, Z)) - A(Y, A(X, Z)), \end{aligned} \tag{2.4}$$

where Riem' denotes the Riemann tensor of ∇. We note that $\mathrm{Riem}'(X, Y)Z$ is anti-symmetric in X and Y. And by differentiation of (2.3) we get that $(\nabla_{E_i} A)(E_j, \cdot)$ and $(\nabla_{E_j} A)(E_i, \cdot)$ are skewadjoint, and therefore we have

$$\langle \mathrm{Riem}'(E_i, E_j)E_k, E_l \rangle = -\langle \mathrm{Riem}'(E_i, E_j)E_l, E_k \rangle. \tag{2.5}$$

In general, Riem' does not satisfy the Bianchi identity. The Ricci curvature of ∇' is defined as

$$\mathrm{ric}'(X,Y) = \mathrm{tr}(V \longmapsto \mathrm{Riem}'(V,X)Y),$$

by (2.4) this can be expressed as

$$\mathrm{ric}'(X,Y) = \sum_{i=1}^{n} \langle \mathrm{Riem}'(E_i, X)Y, E_i \rangle$$

$$= \mathrm{ric}(X,Y) + \sum_{i=1}^{n} (\langle (\nabla_{E_i} A)(X,Y), E_i \rangle - \langle (\nabla_X A)(E_i, Y), E_i \rangle) \qquad (2.6)$$

$$+ \sum_{i=1}^{n} (-\langle A(X,Y), A(E_i, E_j) \rangle + \langle A(E_i, Y), A(X, E_i) \rangle),$$

where ric' is the Ricci curvature of ∇'. We have used that $A(E_i, \cdot)$ and $A(X, \cdot)$ are skew-adjoint.

One obtains the scalar curvature R' of ∇' by taking yet another trace, in p it is given as $R' = \sum_{j=1}^{n} \mathrm{ric}'(E_j, E_j)$. For the following calculation we use that $(\nabla_V A)(X, \cdot)$ is skew-adjoint for any tangent vectors V, X, and we get

$$R' = R + \sum_{i,j=1}^{n} (\langle (\nabla_{E_i} A)(E_j, E_j), E_i \rangle + \langle (\nabla_{E_j} A)(E_i, E_i), E_j \rangle)$$

$$+ \sum_{i,j=1}^{n} (-\langle A(E_j, E_j), A(E_i, E_i) \rangle + \langle A(E_i, E_j), A(E_j, E_i) \rangle)$$

$$= R + 2 \sum_{i,j=1}^{n} \langle (\nabla_{E_i} A)(E_j, E_j), E_i \rangle - \| \sum_{i=1}^{n} A(E_i, E_j) \|^2$$

$$+ \sum_{i,j=1}^{n} \langle A(E_i, E_j), A(E_j, E_i) \rangle),$$

(2.7)

where R denotes the scalar curvature of ∇.

From (2.3) we know that the torsion tensor $A(X, \cdot)$ is skew-adjoint on the tangent space $T_p M$. Any torsion tensor A induces a $(3, 0)$-tensor by setting

$$A_{XYZ} = \langle A(X,Y), Z \rangle \quad \text{for any } X, Y, Z \in T_p M.$$

We define the space of all possible torsion tensors on $T_p M$ by

$$\Upsilon(T_p M) = \{ A \in \otimes^3 T_p^* M \mid A_{XYZ} = -A_{XZY}, \quad \forall X, Y, Z \in T_p M \}.$$

This vector space carries a scalar product

$$\langle A, A' \rangle = \sum_{i,j,k=1}^{n} A_{E_i E_i E_k} A'_{E_i E_j E_k}. \qquad (2.8)$$

For $A \in \Upsilon(T_p M)$ and $Z \in T_p M$ one denotes the trace over the first two entries by

$$C_{12}(A)(Z) = \sum_{i=1}^{n} A_{E_i E_j Z}. \qquad (2.9)$$

Using the definition of inner product of tensors, we denote

$$\| A \|^2 = \langle A, A \rangle, \tag{2.10}$$

$$\langle A, \widehat{A} \rangle = \sum_{i,j,k=1}^{n} A_{E_i E_j E_k} A_{E_j E_i E_k}, \tag{2.11}$$

$$\| c_{12}(A) \|^2 = \sum_{i,j,k=1}^{n} A_{E_i E_i E_k} A_{E_j E_j E_k}, \tag{2.12}$$

here $\widehat{A}$ denotes the (3,0)-tensor obtained from A by interchanging the first two slots, i.e., $\widehat{A}_{XYZ} = A_{YXZ}$ for all tangent vectors X, Y, Z.

Theorem 2.1 For $\dim(M) \geq 3$, one has the following decomposition of $\Upsilon(T_pM)$ into irreducible $O(T_pM)$-subrepresentations

$$\Upsilon(T_pM) = \Upsilon_1(T_pM) \oplus \Upsilon_2(T_pM) \oplus \Upsilon_3(T_pM).$$

This decomposition is orthogonal with respect to $\langle \cdot, \cdot \rangle$, and it is given by

$$\Upsilon_1(T_pM) = \{ A \in \Upsilon(T_pM) \mid \exists V \text{ s.t. } \forall X, Y, Z : A_{XYZ} = \langle X, Y \rangle \langle V, Z \rangle - \langle X, Z \rangle \langle V, Y \rangle \},$$

$$\Upsilon_2(T_pM) = \{ A \in \Upsilon(T_pM) \mid \forall X, Y, Z : A_{XYZ} = -A_{YXZ} \},$$

$$\Upsilon_3(T_pM) = \{ A \in \Upsilon(T_pM) \mid \forall X, Y, Z : A_{XYZ} + A_{YZX} + A_{ZXY} = 0 \text{ and } c_{12}(A)(Z) = 0 \}.$$

For $\dim(M) = 2$ the $O(T_pM)$-representation $\Upsilon(T_pM) = \Upsilon_1(T_pM)$ is irreducible.

Proof Step 1 Proof the decomposition exists.

Suppose any $A \in \Upsilon(T_pM)$, $A = A^{(1)} + A^{(2)} + A^{(3)}$, $A^{(i)} \in \Upsilon_i(T_pM), i = 1, 2, 3$. We denote $A_{E_i E_j E_k}$ by A_{ijk}, and denote $\langle E_i, E_j \rangle$ by δ_{ij}, therefore

$$\sum_{i=1}^{n} A_{iij} = \sum_{i=1}^{n} A_{iij}^{(1)} = \sum_{i=1}^{n} (\delta_{ii} \langle V, E_j \rangle - \delta_{ij} \langle V, E_i \rangle) = (n-1) \langle V, E_j \rangle,$$

we get

$$V = \frac{1}{n-1} \sum_{j=1}^{n} [(\sum_{i=1}^{n} A_{iij}) E_j],$$

so $A^{(1)}$ can be confirmed. Then $A - A^{(1)} = A^{(2)} + A^{(3)}$. Set $A^{(2)} = \frac{1}{6} A_{ijk}^{(2)} W^i \wedge W^j \wedge W^k$, hence

$$(A - A^{(1)})(E_i, E_j, E_k) + (A - A^{(1)})(E_j, E_k, E_i) + (A - A^{(1)})(E_k, E_i, E_j)$$

$$= 3A^{(2)}(E_i, E_j, E_k),$$

$$A_{ijk}^{(2)} = \frac{1}{3} \{ (A - A^{(1)})(E_i, E_j, E_k) + (A - A^{(1)})(E_j, E_k, E_i) + (A - A^{(1)})(E_k, E_i, E_j) \}.$$

Therefore $A^{(2)}$ is confirmed.

We need to ensure that $A^{(3)} = A - A^{(1)} - A^{(2)}$ is a Cartan-type.

For any $X, Y, Z \in T_p M$, since $A^{(1)}_{XYZ} + A^{(1)}_{YZX} + A^{(1)}_{ZXY} = 0$, we have

$$
\begin{aligned}
A^{(3)}_{XYZ} =& A_{XYZ} - A^{(1)}_{XYZ} - A^{(2)}_{XYZ} \\
=& A_{XYZ} - A^{(1)}_{XYZ} \\
& - \frac{1}{18}\{[(A - A^{(1)})_{ijk} + (A - A^{(1)})_{jki} + (A - A^{(1)})_{kij}]W^i \wedge W^j \wedge W^k\}(X, Y, Z) \\
=& A_{XYZ} - A^{(1)}_{XYZ} - \frac{1}{3}(A_{XYZ} + A_{YZX} + A_{ZXY}).
\end{aligned}
$$

In the same way,

$$
\begin{aligned}
A^{(3)}_{YZX} =& A_{YZX} - A^{(1)}_{YZX} - \frac{1}{3}(A_{XYZ} + A_{YZX} + A_{ZXY}), \\
A^{(3)}_{ZXY} =& A_{ZXY} - A^{(1)}_{ZXY} - \frac{1}{3}(A_{XYZ} + A_{YZX} + A_{ZXY}).
\end{aligned}
$$

Add the two sides of the equations, we get $A^{(3)}_{XYZ} + A^{(3)}_{YZX} + A^{(3)}_{ZXY} = 0$, consider

$$
\begin{aligned}
\sum_{i=1}^{n} A^{(3)}_{E_i E_i Z} =& \sum_{i=1}^{n}(A - A^{(1)} - A^{(2)})(E_i, E_i, Z) \\
=& \sum_{i=1}^{n}(A - A^{(1)})(E_i, E_i, Z) \\
=& \sum_{i=1}^{n} A_{E_i E_i Z} - \sum_{i=1}^{n}(\delta_{ii}\langle V, Z\rangle - \langle V, Z\rangle) \\
=& \sum_{i=1}^{n} A_{E_i E_i Z} - (n-1)\langle V, Z\rangle \quad (V \text{ is confirmed above}) \\
=& 0.
\end{aligned}
$$

Hence the decomposition exists.

Step 2 The decomposition is unique.

Let $A = 0 \in \Upsilon(T_p M)$, if $A = A^{(1)} + A^{(2)} + A^{(3)}$, then

$$
0 = \sum_{i=1}^{n} A_{E_i E_i Z} = A^{(1)}_{E_i E_i Z} = (n-1)\langle V, Z\rangle \quad \text{for any} V \in T_p M.
$$

So $V = 0$, i.e., $A^{(1)} = 0$.

$$
0 = A(X, Y, Z) + A(Y, X, Z) + A(Z, X, Y) = 3A^{(2)}(X, Y, Z) \quad \text{for any } X, Y, Z \in T_p M.
$$

So $A^{(2)} = 0$ and $A^{(3)} = 0$.

Step 3 The three space are orthogonal with each other.

$\Upsilon_1 \perp \Upsilon_2$:

$$
\sum_{i,j,k=1}^{n}(\delta_{ij}\langle V, E_k\rangle - \delta_{ik}\langle V, E_j\rangle)A_{ijk} = \sum_{i,j,k=1}^{n}\delta_{ij}\langle V, E_k\rangle A_{ijk} - \sum_{i,j,k=1}^{n}\delta_{ki}\langle V, E_j\rangle A_{kij} = 0;
$$

$\Upsilon_1 \perp \Upsilon_3$:

$$\sum_{i,j,k=1}^{n} (\delta_{ij}\langle V, E_k\rangle - \delta_{ik}\langle V, E_j\rangle)A_{ijk} = \sum_{i,k=1}^{n} A_{iik}\langle V, E_k\rangle - \sum_{j,k=1}^{n} A_{kjk}\langle V, E_j\rangle$$

$$= \sum_{i=1}^{n} A_{E_i E_i V} - \sum_{k=1}^{n} A_{E_k V E_k} = 0.$$

$\Upsilon_2 \perp \Upsilon_3$:

$$\sum_{i,j,k=1}^{n} A_{ijk}^{(2)} A_{ijk}^{(3)} = \frac{1}{3} \sum_{i,j,k=1}^{n} \{A_{ijk}^{(2)} A_{ijk}^{(3)} + \sum_{i,j,k=1}^{n} A_{jki}^{(2)} A_{jki}^{(3)} + \sum_{i,j,k=1}^{n} A_{kij}^{(2)} A_{kij}^{(3)}\}$$

$$= \frac{1}{3} \sum_{i,j,k=1}^{n} A_{ijk}^{(2)}(A_{ijk}^{(3)} + A_{jki}^{(3)} + A_{kij}^{(3)}) = 0.$$

For more about this proof, cf. [12].

The connections whose torsion tensor is contained in $\Upsilon_1(T_pM) \cong T_pM$ are called vectorial. Those whose torsion tensor is in $\Upsilon_2(T_pM) = \wedge^3 T_p^*M$ are called totally anti-symmetric, and those with torsion tensor in $\Upsilon_3(T_pM)$ are called of Cartan-type.

We note that any Cartan-type torsion tensor $A \in \Upsilon_3(T_pM)$ is trace-free in any pair of entries, i.e., for any Z, one has

$$\sum_{i=1}^{n} A_{E_i E_i Z} = 0, \qquad \sum_{i=1}^{n} A_{E_i Z E_i} = 0, \qquad \sum_{i=1}^{n} A_{Z E_i E_i} = 0.$$

The second equality holds as $A \in \Upsilon(T_pM)$, and the third one follows from the cyclic identity $A_{XYZ} + A_{YZX} + A_{ZXY} = 0$.

Remark 2.2 The invariant quadratic form given in (2.12) has the null space $\Upsilon_2(T_pM) \oplus \Upsilon_3(T_pM)$. More precisely, one has $A \in \Upsilon_2(T_pM) \oplus \Upsilon_3(T_pM)$ if and only if $c_{12}(A)(Z) = 0$ for any $Z \in T_pM$.

Remark 2.3 The decomposition given in Theorem 2.1 is orthogonal with respect to the bilinear form given in (2.11), i.e., for $\alpha, \beta \in \{1, 2, 3\}$, $\alpha \neq \beta$, and $A_\alpha \in \Upsilon_\alpha(T_pM)$, $A_\beta \in \Upsilon_\beta(T_pM)$, one gets $\langle A_\alpha, \hat{A}_\beta \rangle$.

Corollary 2.4 For any orthogonal connection ∇' on some Riemannian manifold of dimension $n \geqslant 3$ there exist a vector field V, a 3-form T and a $(0,3)$-tensor field S with $S_p \in \Upsilon_3(T_pM)$ for any $p \in M$ such that $\nabla'_X Y = \nabla_X Y + A(X, Y)$ takes the form

$$A(X, Y) = \langle X, Y\rangle V - \langle V, Y\rangle X + T(X, Y, \cdot)^\# + S(X, Y, \cdot)^\#,$$

where $T(X, Y, \cdot)^\#$ and $S(X, Y, \cdot)^\#$ are the unique vectors with

$$T(X, Y, z) = \langle T(X, Y, \cdot)^\#, Z\rangle \quad \text{and} \quad S(X, Y, z) = \langle S(X, Y, \cdot)^\#, Z\rangle \quad \text{for all } Z.$$

For any orthogonal connection these V, T, S are unique.

Lemma 2.5 The scalar curvature of an orthogonal connection is given by

$$R' = R + 2(n-1)\mathrm{div}^{\nabla}(V) - (n-1)(n-2)\|V\|^2 - \|T\|^2 + \frac{1}{2}\|S\|^2$$

with V, T, S as in Corollary 2.4, and $\mathrm{div}^{\nabla}(V)$ is the divergence of the vector field V taken with respect to the Levi-Civita connection.

Corollary 2.6 Let M be a closed manifold of dimension $n \geqslant 3$ with Riemannian metric g and orthogonal connection ∇'. Let dvol denote the Riemannian volume measure taken with respect to g. Then the Einstein-Cartan-Hilbert functional is

$$\int_M R' \, d\mathrm{vol} = \int_M R \, d\mathrm{vol} - (n-1)(n-2) \int_M \|V\|^2 \, d\mathrm{vol} - \int_M \|T\|^2 \, d\mathrm{vol} + \frac{1}{2} \int_M \|S\|^2 \, d\mathrm{vol}.$$

3 The Fundamental Equations under Orthogonal Connections and Some Results

Let M to be a submanifold of $\overline{M}$. The signs $\overline{\nabla}'$, $\overline{\nabla}$, $\overline{A}$ and $\overline{R}'$ are orthogonal connection, the Levi-Civita connection, torsion tensor and Riemannian curvature related to $\overline{M}$. The signs ∇', ∇, A and R are orthogonal connection, the Levi-Civita connection, torsion tensor and Riemannian curvature related to M inheriting from $\overline{M}$ and the Riemannian curvature of M.

We have the two orthogonal decomposition

$$\text{Gauss formula: } \overline{\nabla}'_X Y = \nabla'_X Y + B'(X, Y), \quad X, Y \in T_p M,$$
$$\text{Weingarten formula: } \overline{\nabla}'_X \xi = -A_\xi(X) + \nabla'^{\perp}_X \xi, \quad X \in T_p M, \xi \in T_p^{\perp} M.$$

Under the Levi-Civita connection, we denote B' by B.

It is easy to check that ∇' and $\nabla'^{\perp}$ keep compatible with metric, since

$$X\langle Y, Z \rangle = \langle \overline{\nabla}'_X Y, Z \rangle + \langle Y, \overline{\nabla}'_X Z \rangle = \langle \nabla'_X Y, Z \rangle + \langle Y, \nabla'_X Z \rangle,$$
$$X\langle \xi, \eta \rangle = \langle \overline{\nabla}'_X \xi, \eta \rangle + \langle \xi, \overline{\nabla}'_X \eta \rangle = \langle \nabla'^{\perp}_X \xi, \eta \rangle + \langle \xi, \nabla'^{\perp}_X \eta \rangle.$$

And we have the fact that $\langle B'(X, Y), \xi \rangle = \langle A_\xi(X), Y \rangle$.

Theorem 3.1 (Guass Equation)

$$\overline{R}'(X, Y, Z, W) = R'(X, Y, Z, W) + \langle B'(X, Z), B'(Y, W) \rangle - \langle B'(X, W), B'(Y, Z) \rangle$$

for any $X, Y, Z, W \in T_p M$.

Proof

$$\overline{R}'(X, Y, Z, W)$$
$$= \langle \overline{R}'(X, Y)Z, W \rangle$$
$$= \langle \overline{\nabla}'_X \overline{\nabla}'_Y Z - \overline{\nabla}'_Y \overline{\nabla}'_X Z - \overline{\nabla}'_{[X,Y]} Z, W \rangle$$
$$= \langle \overline{\nabla}'_X (\nabla'_Y Z + B'(Y, Z)) - \overline{\nabla}'_Y (\nabla'_X Z + B'(X, Z)) - \nabla'_{[X,Y]} Z - B'([X, Y], Z), W \rangle$$
$$= \langle \nabla'_X \nabla'_Y Z - A_{B'(Y,Z)}(X) - \nabla'_Y \nabla'_X Z + A_{B'(X,Z)}(Y) - \nabla'_{[X,Y]} Z, W \rangle$$
$$= R'(X, Y, Z, W) - \langle A_{B'(Y,Z)}(X), W \rangle + \langle A_{B'(X,Z)}(Y), W \rangle$$
$$= R'(X, Y, Z, W) + \langle B'(X, Z), B'(Y, W) \rangle - \langle B'(X, W), B'(Y, Z) \rangle.$$

Theorem 3.2 (Codazzi Equation)

$$(\overline{R}'(X, Y)Z)^\perp = (\widetilde{\nabla}'_X B)(Y, Z) - (\widetilde{\nabla}'_Y B)(X, Z) + B'(A(X, Y), Z) - B'(A(Y, X), Z)$$

for any $X, Y, Z \in T_p M$, which $(\widetilde{\nabla}'_X B)(Y, Z) = \nabla'^\perp_X B'(Y, Z) - B'(\nabla'_X Y, Z) - B'(Y, \nabla'_X Z)$.

Proof

$$(\overline{R}'(X, Y)Z)^\perp = \{\overline{\nabla}'_X \overline{\nabla}'_Y Z - \overline{\nabla}'_Y \overline{\nabla}'_X Z - \overline{\nabla}'_{[X,Y]} Z\}^\perp$$
$$= \{\overline{\nabla}'_X (\nabla'_Y Z + B'(Y, Z))\}^\perp - \{\overline{\nabla}'_Y (\nabla'_X Z + B'(X, Z))\}^\perp - \{\overline{\nabla}'_{[X,Y]} Z\}^\perp$$
$$= B'(X, \nabla'_Y Z) + \nabla'^\perp_X B'(Y, Z) - B'(Y, \nabla'_X Z) - \nabla'^\perp_Y B'(X, Z) - B'([X, Y], Z),$$

while

$$B'([X, Y], Z) = B'(\nabla_X Y - \nabla_Y X, Z)$$
$$= B'(\nabla'_X Y - A(X, Y) - \nabla'_Y X + A(Y, X), Z)$$
$$= B'(\nabla'_X Y, Z) - B'(\nabla'_Y X, Z) - B'(A(X, Y), Z) + B'(A(Y, X), Z).$$

So the equation is found.

Theorem 3.3 (Ricci Equation)

$$(\overline{R}'(X, Y)\xi)^\perp = R'^\perp(X, Y)\xi + B'(Y, A_\xi(X)) - B'(X, A_\xi(Y))$$

for any $X, Y \in T_p M$, $\xi \in T_p^\perp M$, which $R'^\perp(X, Y)\xi = \nabla'^\perp_X \nabla'^\perp_Y \xi - \nabla'^\perp_Y \nabla'^\perp_X \xi - \nabla'^\perp_{[X,Y]} \xi$.

Proof

$$\overline{R}'(X, Y)\xi = \overline{\nabla}'_X \overline{\nabla}'_Y \xi - \overline{\nabla}'_Y \overline{\nabla}'_X \xi - \overline{\nabla}'_{[X,Y]} \xi$$
$$= \overline{\nabla}'_X(-A_\xi(Y) + \nabla'^\perp_Y \xi) - \overline{\nabla}'_Y(-A_\xi(X) + \nabla'^\perp_X \xi) - \overline{\nabla}'_{[X,Y]} \xi,$$
$$(\overline{R}'(X, Y)\xi)^\perp = -B'(X, A_\xi(Y)) + \nabla'^\perp_X \nabla'^\perp_Y \xi + B'(Y, A_\xi(X)) - \nabla'^\perp_Y \nabla'^\perp_X \xi - \nabla'^\perp_{[X,Y]} \xi$$
$$= R'^\perp(X, Y)\xi) + B'(Y, A_\xi(X)) - B'(X, A_\xi(Y)).$$

Proposition 3.4 If $\overline{A}(X, Y) \in T_p M$ for any $X, Y \in T_p M$, then $B'(X, Y) = B'(Y, X) = B(X, Y)$.

Proof

$$\overline{A}(X,Y) = \overline{\nabla}'_X Y - \overline{\nabla}_X Y = \nabla'_X Y - B'(X,Y) - \nabla_X Y - B(X,Y) \in T_p M,$$

then $B'(X,Y) - B(X,Y) = 0$, $B'(X,Y) = B(X,Y)$.

Definition 3.5 We define the mean curvature vector by $H' = \frac{1}{n}\text{tr}B'$. If for any $X \in TM$, $\nabla'^{\perp}_X H' = 0$, we call M is a submanifold with parallel mean curvature vector.

It is easy to check that if M is a submanifold of $\overline{M}$ with parallel mean curvature vector, we have $\| H' \|$ is a constant. Since for any $X \in TM$, $X\langle H', H' \rangle = 2\langle \nabla'^{\perp}_X H', H' \rangle = 0$.

Definition 3.6 M is a submanifold of $\overline{M}$, $x \in M$, $\xi \in T^{\perp}M$, then

(1) If $A_{\xi}(x) : T_x M \to T_x M$ satisfies $A_{\xi}(x) = \lambda_{\xi}(x) \cdot Id$, which $\lambda_{\xi}(x)$ is a constant related to point x, and Id is identity mapping.Then we call x is a umbilical point related to normal vector ξ.

(2) If for all $x \in M$, x is a umbilical point related to ξ. Then we call M is umbilical related to normal vector ξ.

(3) If M is umbilical related to any normal vector $\xi \in T^{\perp}M$. Then we call M is a totally umbilical submanifold.

Proposition 3.7 Let M^n to be a submanifold of $\overline{M}^m$, then M is a totally umbilical submanifold if and only if $B'(X,Y) = g(X,Y)H'$, $\forall X, Y \in T_p M$.

Proof If M is a totally umbilical submanifold, then

$$B'(X,Y) = \sum_{\alpha}\langle B'(X,Y), E_{\alpha}\rangle E_{\alpha} = \sum_{\alpha}\langle A_{\alpha}(X), Y\rangle E_{\alpha} = g(X,Y)\sum_{\alpha}\lambda_{\alpha}E_{\alpha},$$

$$H' = \frac{1}{n}\sum_i B'(E_i, E_i) = \frac{1}{n}\sum_i [\sum_{\alpha}\langle B'(E_i, E_i), E_{\alpha}\rangle E_{\alpha}]$$

$$= \frac{1}{n}\sum_i (\sum_{\alpha}\langle E_i, E_i\rangle \lambda_{\alpha}E_{\alpha}) = \sum_{\alpha}\lambda_{\alpha}E_{\alpha},$$

$$B'(X,Y) = g(X,Y)H'.$$

If $B'(X,Y) = g(X,Y)H', \forall X, Y \in T_p M$, then $\forall \xi \in T^{\perp}pM$,

$$\langle B'(X,Y), \xi \rangle = g(X,Y)\langle H', \xi \rangle = \langle\langle H', \xi\rangle X, Y\rangle,$$

while $\langle B'(X,Y), \xi \rangle = \langle A_{\xi}(X), Y\rangle$, hence $A_{\xi}(X) = \langle H', \xi\rangle X$.

Proposition 3.8 Let M to be a submanifold of $\overline{M}$, then M is a totally geodesic if and only if M is totally umbilical and $H' \equiv 0$.

Proof If $B' \equiv 0$, then $\forall X, Y \in T_p M$, $\forall \xi \in T^{\perp}_p M$,

$$0 = \langle B'(X,Y), \xi \rangle = \langle A_{\xi}(X), Y\rangle, \ A_{\xi}(X) = 0,$$

so M is totally umbilical. By Proposition 3.7, $B' = gH'$, then $H' \equiv 0$.

If M is totally umbilical and $H' \equiv 0$, then $B' = gH' = 0$, so $B' \equiv 0$, M is a totally geodesic submanifold.

Under the Levi-Civita connections, the Riemannian curvature R has the following properties

(i) $R(X, Y, Z, W) = -R(Y, X, Z, W)$;

(ii) $R(X, Y, Z, W) = -R(X, Y, W, Z) = R(Y, X, W, Z)$;

(iii) $R(X, Y, Z, W) = R(Z, W, X, Y)$;

(iv) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$. $\qquad$ (3.1)

But under orthogonal connections, (iii), (iv) do not always hold.

We usually denote $G(X, Y, Z, W) \triangleq \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle$. It is easy to check that G has the same properties (3.1) as R.

In the rest of this section, we argue Lemma 3.9, Theorem 3.10, Corallary 3.11, and Theorem 3.12 in 3-dimensional Reimannian manifold equipped with an orthogonal connection which torsion T is a totally anti-symmetric satisfying $T = fW^1 \wedge W^2 \wedge W^3 (W^1, W^2, W^3$ is the dual bases of $E_1, E_2, E_3)$, f is a constant.

Lemma 3.9 (M, g) is under the conditions above, then the first Bianchi identity is founded, that is to say

$$R'(X, Y)Z + R'(Y, Z)X + R'(Z, X)Y = 0, \quad \forall X, Y, Z \in T_p M.$$

Proof   At any point $p \in M$, we choose parallel unit vector fields E_1, E_2, E_3 as the bases in the neighborhood of p.

Since the curvature tensor $R'(X, Y)Z$ at point p is not related to the extensions of $X|_p, Y|_p, Z|_p$, we let the extensions to be

$$X = \sum_{i=1}^{3} X^i E_i, \quad Y = \sum_{i=1}^{3} Y^i E_i, \quad Z = \sum_{i=1}^{3} Z^i E_i,$$

which X^i, Y^i, Z^i are constants, $i = 1, 2, 3$, then

$$R'(X, Y)Z + R'(Y, Z)X + R'(Z, X)Y$$
$$= \sum_{i,j,k=1}^{3} X^i Y^j Z^k [R'(E_i, E_j)E_k + R'(E_j, E_k)E_i + R'(E_k, E_i)E_j].$$

If i, j, k are at least two identical,

$$R'(E_i, E_j)E_k + R'(E_j, E_k)E_i + R'(E_k, E_i)E_j = 0.$$

We consider i, j, k are different from each other, then

$$\nabla_{E_i} T(E_j, E_k) = 0 \quad \text{and} \quad T(E_i, T(E_j, E_k)) = 0.$$

Without of loss generality, we let $i = 1, j = 2, k = 3$,

$$R'(E_1, E_2)E_3$$
$$= R(E_1, E_2)E_3 + \nabla_{E_1} T(E_2, E_3) - \nabla_{E_2} T(E_1, E_3) + T(E_1, T(E_2, E_3)) - T(E_2, T(E_1, E_3))$$
$$= R(E_1, E_2)E_3,$$

so $R'(E_1, E_2)E_3 + R'(E_2, E_3)E_1 + R'(E_3, E_1)E_2 = 0$. That is to say, in the case of i, j, k are different from each other,

$$R'(E_i, E_j)E_k + R'(E_j, E_k)E_i + R'(E_k, E_i)E_j = 0.$$

Therefore

$$R'(X, Y)Z + R'(Y, Z)X + R'(Z, X)Y = 0, \quad \forall X, Y, Z \in T_p M.$$

So property (iv) of (3.1) is founded. Since $\dim M = 3$, we have $\nabla T = 0$, then

$$\langle R'(X, Y)Z, W \rangle = \langle R'(Z, W)X, Y \rangle,$$

corresponding to the (iii) of (3.1).

After all, (3.1) holds for an orthogonal in 3-dimension under the conditions above.

Theorem 3.10 If $\dim M = 3$, under an orthogonal connection above, then the curvature tensor of M at point p is determined by the all (sections') sectional curvatures.

Proof Becuase (3.1) holds for R', we prove the theorem as following. In order to proof the theorem, we only need to prove that if there is another $(0, 4)$-tensor $\widetilde{R}'(X, Y, Z, W)$ satisfying (3.1), and for any linearly independent vectors $X, Y \in T_p M$, it always hold that $\widetilde{R}'(X, Y, X, Y) = R'(X, Y, X, Y)$, then for any $X, Y, Z, W \in T_p M$, we have $\widetilde{R}'(X, Y, Z, W) = R'(X, Y, Z, W)$. So let $S(X, Y, Z, W) = \widetilde{R}'(X, Y, Z, W) - R'(X, Y, Z, W)$, the argument above is equivalent to that if for any $X, Y \in T_p M$, $S(X, Y, X, Y) = 0$, then $S \equiv 0$. Obviously, S is a $(0,4)$-tensor satisfying (3.1). Expanding $S(X + Z, Y, X + Z, Y) = 0$, we have

$$S(X, Y, Z, Y) = 0, \quad \forall X, Y, Z \in T_p M.$$

Then expanding $S(X, Y + W, Z, Y + W) = 0$, we have

$$S(X, Y, Z, W) + S(X, W, Z, Y) = 0, \quad \forall X, Y, Z, W \in T_p M.$$

Via (iv) $S(X, Y, Z, W) + S(X, Z, W, Y) + S(X, W, Y, Z) = 0$, we obtain

$$2S(X, Y, Z, W) = S(X, Z, Y, W).$$

Likewise,

$$2S(X, Z, Y, W) = S(X, Y, Z, W).$$

Hence, for any $X, Y, Z, W \in T_p M$, $S(X, Y, Z, W) = 0$.

Corollary 3.11 Let (M, g) a Riemannian manifold, $\dim M = 3$, under an orthogonal connection above, then M is a isotropic manifold if and only if fixing any $p \in M$,

$$K(X, Y) = \frac{R'(X, Y, Z, W)}{G(X, Y, Z, W)} \quad \text{is a constant}, \quad \forall X, Y, Z, W \in T_p M.$$

Theorem 3.12 $\overline{M}(\overline{c})$ is constant curvature Riemannian manifold which $\dim\overline{M} = 3$ and curvature is $\overline{c}$, equipped with an orthogonal connection above, denoted by $\overline{\nabla}'$(the torsion is $\overline{T}$). Let M be a submanifold of $\overline{M}(\overline{c})$ which is connected and is totally umbilical, then

(1)M is a submanifold with a parallel mean curvature vector, and $R^\perp(X, Y)\xi \equiv 0$ under the Levi-Civita connection.

(2)M is a submanifold of constant curvature, which curvature is $K_M = \overline{c} + \|H'\|^2$.

Proof At first, for convenience, we proof the case of $\overline{M}$ under the Levi-Civita connection and n-dimension

Since $\overline{M}$ is a constant curvature manifold, then

$$\overline{R}(X, Y)Z = \overline{c}(\langle Y, Z\rangle X - \langle X, Z\rangle Y), \quad \Rightarrow \quad (\overline{R}(X, Y)Z)^\perp = 0.$$

Via the Codazzi equation $(\overline{R}(X, Y)Z)^\perp = (\widetilde{\nabla}_X B)(Y, Z) - (\widetilde{\nabla}_Y B)(X, Z)$, we have

$$(\widetilde{\nabla}_X B)(Y, Z) = (\widetilde{\nabla}_Y B)(X, Z).$$

Using Proposition 3.7, $B(X, Y) = \langle X, Y\rangle H$, we have

$$\begin{aligned}
(\widetilde{\nabla}_X B)(Y, Z) =& \nabla_X^\perp B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z) \\
=& \nabla_X^\perp(\langle Y, Z\rangle H) - \langle\nabla_X Y, Z\rangle H - \langle Y, \nabla_X Z\rangle H \\
=& (X\langle Y, Z\rangle - \langle\nabla_X Y, Z\rangle - \langle Y, \nabla_X Z\rangle)H - \langle Y, Z\rangle\nabla_X^\perp H \\
=& \langle Y, Z\rangle\nabla_X^\perp H, \\
(\widetilde{\nabla}_Y B)(X, Z) =& \langle Y, Z\rangle\nabla_X^\perp H.
\end{aligned}$$

So

$$\langle Y, Z\rangle \nabla_X^\perp H = \langle X, Z\rangle\nabla_Y^\perp H \quad \text{for any} \ X, Y, Z \in T_p M.$$

We pick $Y = Z \neq 0$, $X \perp Y$, then $\nabla_X^\perp H = 0$ for any $X, Y \in T_p M$, $\xi \in T_p^\perp M$, i.e., H is parallel related to $T_p^\perp M$.

Next, proof of $R^\perp \equiv 0$.

Since $\overline{M}$ is constant curvature manifold, then $\overline{R}(X, Y)\xi = \overline{c}(\langle Y, \xi\rangle X - \langle X, \xi\rangle Y)$. We have $(\overline{R}(X, Y)\xi)^\perp = 0$, while $B = B(X, \lambda_\xi Y) = \lambda_\xi B(X, Y) = B(Y, A_\xi(X))$.

Via the Ricci equation $(\overline{R}(X, Y)\xi)^\perp = R^\perp(X, Y)\xi + B(Y, A_\xi(X)) - B(X, A_\xi(Y))$, we have $R^\perp(X, Y)\xi = 0$.

(2) Via the Gauss equation, $\forall X, Y, Z, W \in T_p M$, we have

$$\begin{aligned}
R(X, Y, Z, W) =& \overline{R}(X, Y, Z, W) + \langle B(X, Z), B(Y, W)\rangle - \langle B(X, W), B(Y, Z)\rangle \\
=& \overline{c}(\langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle) + (\langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle)\|H\|^2 \\
=& (\overline{c} + \|H\|^2)\langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle,
\end{aligned}$$

so M is a constant curvature manifold with sectional curvature $\overline{c} + \|H\|^2$, while $\overline{T}$ is a totally anti-symmetric tensor, T is also a totally anti-symmetric tensor. Because of $\dim\overline{M} = 3$,

dim$M \leqslant 2$, $T \equiv 0$. So the Codazzi equation is the same as the case of the Levi-Civita connection, i.e.,

$$(\overline{R}'(X,Y)Z)^{\perp} = (\widetilde{\nabla}'_X B)(Y,Z) - (\widetilde{\nabla}'_Y B)(X,Z).$$

Combined with Corollary 3.11, we can get the result.

References

[1] Cartan. Sur les varits connexion affne et la thorie de la rlativit gnralise (premire partie)[J]. Ann. Norm. Sup., 1923, 40: 325–412.

[2] Cartan. Sur les varits connexion affne et la thorie de la rlativit gnralise (premire partie suite)[J]. Ann. Norm. Sup., 1924, 41: 1–25.

[3] Cartan. Sur les varits connexion affne et la thorie de la rlativit gnralise (deuxime partie)[J]. Ann. Norm. Sup., 1925, 42: 17–88.

[4] Agricola I. The Srn lectures on non-integrable geometries with torsion[J]. Arch. Math. (Brno), 2006, 42: 5–84.

[5] Tricerri F, Vanhecke L. Homogeneous structures on Riemannian manifolds[M]. London Math. Soc. Lecture Note Ser., Vol. 83, Cambridge: Cambridge University Press, 1983.

[6] Pfaffle F, Stephan C A. On gravity, torsion and the spectral action principle[J]. J. Funct. Anal., 2012, 262: 1529–1565.

[7] Kobayashi S, Nomizu K. Foundations of differential geometry[M]. New York: Interscience Publishers, 1969.

[8] Bai Z G, Shen Y B. An introduction to Riemann geometry (revised ed.)[M]. Beijing: Higher Edu. Press, 2004.

[9] Isaac Chavel. Riemannian geometry, a modern introduction (2nd ed.)[M]. Cambridge: Cambridge Cambridge University Press, 2006.

[10] Berger M. Quelques formules de variation pour une structure riemannienne[J]. Ann. Norm. Sup., 1970, 3(4): 285–294.

[11] Tricerri F, Vanhecke L. Homogeneous structures on Riemannian manifolds[M]. London Math. Soc., Lecture Note Ser., Vol.83 Cambridge: Cambridge University Press, 1983.

[12] Nie Zhi. On the Guass equation and Chern connections of Finsler submanifolds[J]. J. Math., 2004, 24(5): 537–542.

黎曼流形在正交联络下的全脐点子流形

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摘要: 本文研究了正交联络下子流形基本方程以及在全脐点子流形中的应用. 利用Cartan的方法将挠率张量分解成三个部分, 计算得到正交联络下的三个基本方程, 并考虑一个特殊的正交联络, 证明了其黎曼曲率会有类似于Levi-Civita 联络下的性质. 利用基本方程得到常曲率空间中的全脐点子流形的性质, 推广了Levi-Civita联络下的相应结果.

关键词: 正交联络; 黎曼流形的基本方程; 子流形; 脐点

MR(2010)主题分类号: 53C05; 53C17　　　中图分类号: O186.12