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THE FEFFERMAN INEQUALITY AND DUAL THEOREM FOR QUASI-MARTINGALES

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Abstract: In this paper, the Fefferman inequality and dual theorem for quasi-martingales are studied. By using the correspond results for martingales and the Doob's decomposition, the Fefferman inequality for martingales is extended to the quasi-martingale setting and the dual space of quasi-martingale Hardy space $\widehat{H}_p$ $(1 < p < \infty)$ is described.

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1 Introduction

The history of martingale theory goes back to the early fifties when Doob pointed out the connection between martingales and analytic function. In the course of its development, inequalities of martingale spaces were a concerned research hot spot. People always study the properties operators via the corresponding martingale inequalities, whereby we obtain the relationship between two operators, further, the inclusions of many martingale spaces are established.

Quasi-martingales is an important generalization of martingales. Today, the theory has achieved a satisfactory development and it can perfectly well be applied in complex analysis and in the theory of classical Hardy spaces. In Section 3, we prove the Fefferman inequality for quasi-martingales. Let us briefly describe our main inequality. Let $\widehat{H}_p$ be the quasi-martingale Hardy space and $1 \le p \le 2$, then

$$|E(f_n\overline{\varphi_n})| \le C\|f\|_{\widehat{H}_p}\|\varphi\|_{{}_2\widehat{K}_{p'}}, \quad \varphi \in {}_2\widehat{K}_{p'},$$

where ${}_2\widehat{K}_{p'}$ is a special quasi-martingale space. In Section 4 we describe the dual space of $\widehat{H}_p$. Note that the dual space of martingale Hardy space H_p is $H_{p'}$. However, the case of quasi-martingale is quite different. Let $1 < p < \infty$. We prove the dual space of $\widehat{H}_p$ can be given with the norm

$$\|\phi\| := \|r\|_{H_{p'}} + \|s\|_{\widehat{BD}_{p'}},$$

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where $\widehat{BD}_{p'}$ is a subspace of $l_\infty(L_{p'})$ and $\phi_n = r_n + s_n (n \geq 1)$.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $(\mathcal{F}_n, n \geq 1)$ a non-decreasing sequence of σ-algebras. The expectation operator and the conditional expectation operator are denoted by E and $E_n(\cdot)$. We briefly write L_p instead of the $L_p(\Omega, \mathcal{F}, \mathcal{P})$ space while the norm (or quasinorm) of this space is defined by

$$\|f\|_p := (E|f|^p)^{1/p} (0 < p \leq \infty).$$

A integrable sequence $f = (f_n)_{n\geq 1}$ is said to be a martingale if

(i) it is adapted, i.e., f_n is $\mathcal{F}_n$, measurable for all $n \geq 1$;

(ii) $E_n(f_{n+1}) = f_n$ for all $n \geq 1$.

If additionally, $f = (f_n)_{n\geq 1} \subset L_p$ for some $1 \leq p \leq \infty$, we call f an L_p-martingale. We refer to [1] for more information on martingales.

Now we turn to the definition of quasi-martingales. Let $1 \leq p \leq \infty$. An adapted sequence $f = (f_n)_{n\geq 1}$ in L_1 is called a p quasi-martingale with respect to $(\mathcal{F}_n, n \geq 1)$ if

$$V_p(f) := \sum_{n=1}^{\infty} \|E_{n-1}(df_n)\|_p < \infty. \tag{2.1}$$

If in addition $f = (f_n)_{n\geq 1} \subset L_p$ for some $1 \leq p \leq \infty$, we call f an L_p-quasi-martingale. In this case, we set

$$\|f\|_p := \sup_n \|f_n\|_p + V_p(f).$$

If $\|f\|_p < \infty$, f is called a bounded L_p-quasi-martingale. The quasi-martingale space $\widehat{L}_p$ is defined as the space of all bounded L_p-quasi-martingales, and is equipped with the norm $\|\cdot\|_p$.

In the following we describe the quasi-martingale Hardy space which is needed in the main results in this paper. For $1 \leq p < \infty$, let $f = (f_n)_{n\geq 1}$ be a p-quasi-martingale,

$$\widehat{H}_p = \{f = (f_n) : \|f\|_{\widehat{H}_p} = \|(\sum_{n=1}^{\infty} |f_n|^2)^{\frac{1}{2}}\|_p + V_p(f) < \infty\}.$$

Note that a basic fact respect to quasi-martingales is that each p-quasi-martingale can be decomposed as a sum of a martingale and a predicable quasi-martingale which we call Doob's decomposition. Doob's decomposition plays an important role in this paper.

Lemma 2.1 (Doob's decomposition) (see [6]) Let $1 \leq p \leq \infty$. Each bounded L_p-quasi-martingale $f = (f_n)_{n\geq 1}$ can be uniquely decomposed as a sum of two sequences $g = (g_n)_{n\geq 1}$ and $h = (h_n)_{n\geq 1}$, where $g = (g_n)_{n\geq 1}$ is a bounded L_p-martingale and $h = (h_n)_{n\geq 1}$ is a predicable p-quasi-martingale with $h_1 = 0$ such that $dh_n = E_{n-1}(dh_n)$.

In the sequel, we use p' to denote the conjugate index of p for $1 \leq p \leq \infty$.

3 The Fefferman Inequality for Quasi-Martingales

Our main result in this section is concerned with the Fefferman inequality for quasi-martingales. We first recall the Fefferman inequality for martingales (see Theorem 2.2.2 of [2]). Let $f \in H_p$, $1 \le p \le 2$, $\varphi \in_2 K_{p'}$. Then

$$|E(f_n\overline{\varphi_n})| \le \sqrt{\frac{2}{p}}\|f\|_{H_p}\|\varphi\|_{{}_2K_{p'}}, \quad \forall n.$$

In this paper, we extend the inequality to the quasi-martingale setting. First we give the definition of ${}_2\widehat{K}_p$.

Definition 3.1 Let $2 \le p \le \infty$, $f = (f_n)_{n\ge1}$ be a L_1-quasi-martingale, f is said to be in ${}_2\widehat{K}_p$, if there exists $\gamma \in L_+^p$ such that

$$E(|f_m - f_{n-1}|^2|\mathcal{F}_n) \le E(\gamma^2|\mathcal{F}_n), \forall m \ge n \ge 1.$$

We define a norm in ${}_2\widehat{K}_p$ by

$$\|f\|_{{}_2\widehat{K}_p} = \inf\{\|\gamma\|_p + V_p(f) : \gamma \text{ runs through all possible ones}\}.$$

Now we are ready to state our main result.

Theorem 3.2 Let $f = (f_n)_{n\ge1} \in \widehat{H}_p$, $1 \le p \le 2$, $\varphi \in {}_2\widehat{K}_{p'}$. Then

$$|E(f_n\overline{\varphi_n})| \le C\|f\|_{\widehat{H}_p}\|\varphi\|_{{}_2\widehat{K}_{p'}}, \quad \forall n,$$

where C is a universal constant.

Proof Let $f_n = g_n + h_n (n \ge 1)$ be the Doob's decomposition of f. Then $g = (g_n)_{n\ge1}$ is a martingale and $\sum\limits_{n=1}^{\infty}\|dh_n\|_p < \infty$. Noting that $(\sum\limits_{n=1}^{\infty}|dh_n|^2)^{\frac{1}{2}} \le \sum\limits_{n=1}^{\infty}|dh_n|$, we have that

$$\|(\sum_{n=1}^{\infty}|dh_n|^2)^{\frac{1}{2}}\|_p \le \sum_{n=1}^{\infty}\|dh_n\|_p < \infty.$$

Thus $h = (h_n)_{n\ge1} \in H_p$ and $\|h\|_{H_p} \le 2\sum\limits_{n=1}^{\infty}\|dh_n\|_p$. Therefore, we obtain

$$\|g\|_{H_p} \le \|f\|_{H_p} + \|h\|_{H_p} \le C\|f\|_{H_p}. \tag{3.1}$$

Let $\varphi_n = r_n + s_n (n \ge 1)$ be the Doob's decomposition of φ. We must have that $s = (s_n)_{n\ge1} \in {}_2\widehat{K}_{p'}$. Indeed, by the inequality $|s_m - s_{n-1}| \le \sum\limits_{n=1}^{\infty}|ds_n|, \forall m \ge n$ and the definition of the space ${}_2\widehat{K}_{p'}$, we have that

$$\begin{aligned}\|s\|_{{}_2\widehat{K}_{p'}} &\le \|\sum_{n=1}^{\infty}|ds_n|\|_{p'} + \sum_{n=1}^{\infty}\|ds_n\|_{p'} \\ &\le 2\sum_{n=1}^{\infty}\|ds_n\|_{p'} < \infty.\end{aligned}$$

Thus we have that $s = (s_n)_{n\geq 1} \in_2 \widehat{K}_{p'}$. Therefore, we get that

$$\|r\|_{{}_2K_{p'}} \leq \|\varphi\|_{{}_2\widehat{K}_{p'}} + \|s\|_{{}_2\widehat{K}_{p'}} \leq C\|\varphi\|_{{}_2\widehat{K}_{p'}}. \tag{3.2}$$

By the Fefferman inequality for martingales and Hölder's inequality, we have

$$\begin{aligned} |E(f_n\overline{\varphi_n})| &= |E(g_n+h_n)(\overline{r_n}+\overline{s_n})| \\ &\leq |E(g_n\overline{r_n})| + |E(h_n\overline{r_n})| + |E(h_n\overline{s_n})| + |E(g_n\overline{s_n})| \\ &\leq \sqrt{\tfrac{2}{p}}\|g\|_{H_p}\|r\|_{{}_2K_{p'}} + \|h_n\|_p\|r_n\|_{p'} + \|h_n\|_p\|s_n\|_{p'} + \|g_n\|_p\|s_n\|_{p'} \\ &= \text{I+II+III+IV}. \end{aligned} \tag{3.3}$$

It follows from (3.1) and (3.2) that

$$\text{I} \leq C\|f\|_{\widehat{H}_p}\|\varphi\|_{{}_2\widehat{K}_{p'}}. \tag{3.4}$$

To treat II, we first prove the inequality

$$\|r_n\|_{p'} \leq C\|r\|_{{}_2K_{p'}}. \tag{3.5}$$

For $p' > 2$, by the equality $H_{p'} = {}_2K_{p'}$ (see Theorem 2.2.2 and Theorem 2.2.5 of [2]) and the Burkholder-Gundy inequalities, we get inequality (3.5). For $p' = 2$, since $r = (r_n)_{n\geq 1} \in {}_2\widehat{K}_2$, there exists $\gamma \in L^2_+$ such that

$$E(|r_m - r_{n-1}|^2|\mathcal{F}_n) \leq E(\gamma^2|\mathcal{F}_n), \forall m \geq n \geq 1.$$

Then

$$E(|r_m|^2|\mathcal{F}_0) = E(|r_m - r_{-1}|^2|\mathcal{F}_0) \leq E(|\gamma|^2|\mathcal{F}_0).$$

It is easy to see that for any $n \geq 1$

$$(E(|r_n|^2)^{\frac{1}{2}} \leq (E(|r_{m+1} - r_n|^2)^{\frac{1}{2}} + (E(|r_{m+1}|^2)^{\frac{1}{2}} \leq (E\gamma^2)^{\frac{1}{2}} + (E\gamma^2)^{\frac{1}{2}} \leq 2(E\gamma^2)^{\frac{1}{2}}.$$

Thus we have that $\|r_n\|_2 \leq C\|r\|_{{}_2K_2}$. Using (3.5) and (3.2), we get that

$$\text{II} \leq C\sum_{n=1}^{\infty}\|dh_n\|_p\|r\|_{{}_2K_{p'}} \leq C\|f\|_{\widehat{H}_p}\|\varphi\|_{{}_2\widehat{K}_{p'}}. \tag{3.6}$$

Now we turn to estimate the two last term separately. By the definitions of the spaces $\widehat{H_p}$ and ${}_2\widehat{K}_{p'}$, we have that

$$\text{III} \leq C\sum_{n=1}^{\infty}\|dh_n\|_p\sum_{n=1}^{\infty}\|ds_n\|_{p'} \leq C\|f\|_{\widehat{H}_p}\|\varphi\|_{{}_2\widehat{K}_{p'}}. \tag{3.7}$$

By the Burkholder-Gundy inequalities and Davis inequalities, we get that $\|g_n\|_p \leq C\|g\|_{H_p}$. Thus we have that

$$\text{IV} \leq C\|g\|_{H_p}\sum_{n=1}^{\infty}\|ds_n\|_{p'} \leq C\|f\|_{\widehat{H}_p}\|\varphi\|_{{}_2\widehat{K}_{p'}}. \tag{3.8}$$

Putting (3.3), (3.4), (3.6), (3.7) and (3.8) together, we obtain that

$$|E(f_n\overline{\varphi_n})| \leq C\|f\|_{\widehat{H}_p}\|\varphi\|_{_2\widehat{K}_{p'}}.$$

4 The Dual Space of Quasi-Martingale Hardy Space $\widehat{H}_p$

In this section we describe the dual space of quasi-martingale Hardy space $\widehat{H}_p$. Note that the dual space of martingale Hardy space H_p is $H_{p'}$. It is natural to ask whether the preceding result can be generalized to the quasi-martingale setting. The answer is unfortunately negative in general. Indeed, the dual space of $\widehat{H}_p$ which is introduced in Theorem 4.2 is difficult from $\widehat{H}_{p'}$. Now we start by introducing a special space which is needed in our main result in this section.

Definition 4.1 Denote by $\widehat{BD}_p$ $(1 \leq p \leq \infty)$ the space of all predicable sequences $f = (f_n)_{n\geq 1}$ (with $f_1 = 0$) for which

$$\|f\|_{\widehat{BD}_p} := \sup_n \|df_n\|_p.$$

Now we are ready to state the following result.

Theorem 4.2 The dual space of $\widehat{H}_p$ ($1 < p < \infty$) can be given with the norm

$$\|\phi\| := \|r\|_{H_{p'}} + \|s\|_{\widehat{BD}_{p'}},$$

where $\phi_n = r_n + s_n (n \geq 1)$.

Proof Let $f = (f_n)_{n\geq 1} \in \widehat{H}_p$ and $f_n = g_n + h_n (n \geq 1)$ be the Doob's decomposition of f. Define a linear functional on $\widehat{H}_p$ by

$$l_\phi(f) = E(\sum_{n=1}^{\infty} dr_n dg_n) + \sum_{n=1}^{\infty} E(ds_n dh_n),$$

where $r = (r_n)_{n\geq 1} \in H_{p'}$, $s = (s_n)_{n\geq 1} \in \widehat{BD}_{p'}$ and $\phi_n = r_n + s_n (n \geq 1)$. By Hölder's inequality, we have that

$$\begin{aligned} l_\phi(f) &\leq E((\sum_{n=1}^{\infty} |dr_n|^2)^{\frac{1}{2}}(\sum_{n=1}^{\infty} |dg_n|^2)^{\frac{1}{2}}) + \sum_{n=1}^{\infty}(\|ds_n\|_{p'}\|dh_n\|_p) \\ &\leq \|(\sum_{n=1}^{\infty} |dr_n|^2)^{\frac{1}{2}}\|_{p'}\|(\sum_{n=1}^{\infty} |dg_n|^2)^{\frac{1}{2}})\|_p + \sup_n \|ds_n\|_{p'} \sum_{n=1}^{\infty} \|dh_n\|_p \\ &= \|r\|_{H_{p'}}\|g\|_{H_p^s} + \|s\|_{\widehat{BD}_{p'}} V_p(f) \\ &\leq \|f\|_{\widehat{\mathcal{H}}_p}(\|r\|_{\mathcal{H}_{p'}} + \|s\|_{\widehat{BD}_{p'}}). \end{aligned}$$

Namely, $l_\phi(f)$ is a bounded linear functional.

Conversely, assume that l is an arbitrary bounded linear functional on $\widehat{H}_p$. It is easy to see that l is also a bounded linear functional on H_p. Since ${H_p}^* = H_{p'}$, there exists a sequence $r=(r_n)_{n\geq 1}\in H_{p'}$ such that

$$l(r)=E(\sum_{n=1}^{\infty}dr_n dg_n)\quad (g=(g_n)_{n\geq 1}\in H_p)$$

and

$$\|r\|_{H_{p'}}\leq C\|l\|. \tag{4.1}$$

On the other hand, let Q_p be the subspace of $l_1(L_p)$ of all sequences $db=(db_n)_{n\geq 1}$ such that $b=(b_n)_{n\geq 1}$ is a predictable quasi-martingale in $\widehat{H}_p$ with $b_1=0$. Then we have that

$$\|db\|_{l_1(L_p)}\leq \|b\|_{\widehat{H}_p}\leq 2\|db\|_{l_1(L_p)}$$

for any $db=(db_n)_{n\geq 1}\in Q_p$. Define a functional on Q_p by

$$l_2(db)=l(b),\quad db=(db_n)_{n\geq 1}\in Q_p.$$

Then l_2 is a continuous linear functional on Q_p and $\|l_2\|\leq 2\|l\|$. By the Hahn-Banach theorem, l_2 extends to a functional on $l_1(L_p)$. Since $(l_1(L_p))^*=l_\infty(L_{p'})$, the representation theorem allows us to find a sequence $s'=(s'_n)_{n\geq 1}\in l_\infty(L_{p'})$ such that

$$l_2(s)=\sum_{n=1}^{\infty}E(s'_n h_n)\quad (h=(h_n)_{n\geq 1}\in l_1(L_p)) \tag{4.2}$$

and $\|s'\|_{l_\infty(L_{p'})}=\sup_n\|s'_n\|_{p'}\leq C\|l_2\|$. Set $s_1=0$ and $s_n=\sum\limits_{k=1}^{n}E_{k-1}(s'_k)(n\geq 2)$. For any $db=(db_n)_{n\geq 1}\in Q_p$, noting that $db=(db_n)_{n\geq 1}$ is predicable, it follows from (4.2) that

$$\begin{aligned} l_2(db) &= \sum_{n=1}^{\infty}E(E_{n-1}(s'_n db_n))\\ &= \sum_{n=1}^{\infty}E(db_n E_{n-1}(s'_n))\\ &= \sum_{n=1}^{\infty}E(ds_n db_n). \end{aligned}$$

It remains to show that $s=(s_n)_{n\geq 1}\in\widehat{BD}_{p'}$. This is true since $s=(s_n)_{n\geq 1}$ is predicable with $s_1=0$ and

$$\|s\|_{\widehat{BD}_{p'}}=\sup_n\|ds_n\|_{p'}\leq\sup_n\|s'_n\|_{p'}\leq C\|l_2\|\leq C\|l\|. \tag{4.3}$$

Putting (4.1) and (4.3) together, we have that

$$\|r\|_{\mathcal{H}_{p'}}+\|s\|_{\widehat{BD}_{p'}}\leq C\|l\|.$$

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拟鞅的Fefferman不等式和对偶定理

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摘要: 本文讨论了拟鞅的Fefferman不等式和Hardy空间的对偶空间. 利用鞅的相关结果和Doob分解的方法, 把鞅的Fefferman不等式推广到拟鞅情形, 并描述了拟鞅的Hardy空间$\widehat{\mathcal{H}}_p$在$1 < p < \infty$ 时的对偶空间.

关键词: 拟鞅; Hardy空间; 对偶空间; 范数

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