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BOUNDEDNESS FOR SOME SCHRÖDINGER TYPE OPERATORS ON MORREY SPACES WITH VARIABLE EXPONENT RELATED TO CERTAIN NONNEGATIVE POTENTIALS

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Abstract: In this paper, the boundedness of some Schrödinger type operators and the commutators is considered. Using the boundedness of them on L^p space, we obtain the boundedness of some schrödinger type operators and the commutators on Morrey with variable exponents.

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1 Introduction

In this paper, we consider the schrödinger differential operator

$$\mathcal{L} = -\triangle + V(x) \qquad \text{on } \ \mathbb{R}^n, \ \ n \geq 3,$$

where $V(x)$ is a nonnegative potential belonging to the reverse Hölder class B_q for $q \geq \frac{n}{2}$.

A nonnegative locally L^q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to $B_q (q > 1)$ if there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^q dx\right)^{\frac{1}{q}} \leq C\left(\frac{1}{|B|}\int_B V dx\right)$$

holds for every ball in $\mathbb{R}^n$, see [1].

Shen [1] established L^p estimates for schrödinger type operators with certain potentials. Kurata, Nishigaki and Sugano [2] considered the boundedness of integral operators on generalized Morrey spaces and its application to Schrödinger operators. Recently, paper [3] by Tang and Dong proved the boundedness of some Schrödinger type operators on Morrey spaces related to certain nonnegative potentials.

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It is well known that function spaces with variable exponents were intensively studied during the past 20 years, due to their applications to PDE with non-standard growth conditions and so on, we mention e.g. [4, 5]. A great deal of work was done to extend the theory of potential, singular type operators and their commutators on the classical Lebesgue spaces to the variable exponent case (see [6–8]). Also, many results about potential, singular type operators and their commutators were studied on Morrey Spaces with variable exponent (see [9–12]). Hence, it will be an interesting problem whether we can establish the boundedness of some schrödinger type operators on Morrey spaces with variable exponent related to certain nonnegative potentials. The main purpose of this paper is to answer the above problem.

To meet the requirements in the next sections, here, the basic elements of the theory of the Lebsegue spaces with variable exponent are briefly presented.

Let $p(\cdot):\mathbb{R}^n\to[1,\infty)$ be a measurable function. The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n):=\left\{f \text{ is measurable}: \int_{\mathbb{R}^n}\left|\frac{f(x)}{\lambda}\right|^{p(x)}dx<\infty \text{ for some constant } \lambda>0\right\}.$$

The space $L^{p(\cdot)}_{loc}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n):=\left\{f \text{ is measurable}: f\in L^{p(\cdot)}(K) \text{ for all compact subsets } K\subset\mathbb{R}^n\right\}.$$

$L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with the norm defined by

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}:=\inf\{\lambda>0:\int_{\mathbb{R}^n}\left|\frac{f(x)}{\lambda}\right|^{p(x)}dx\le 1\}.$$

We denote $p_-:=\operatorname{ess}\inf\limits_{x\in\mathbb{R}^n}p(x),\ \ p_+:=\operatorname{ess}\sup\limits_{x\in\mathbb{R}^n}p(x)$.

Let $\mathcal{P}(\mathbb{R}^n)$ be the set of measurable function $p(\cdot)$ on $\mathbb{R}^n$ with value in $[1,\infty)$ such that $1<p_-\le p(\cdot)\le p_+<\infty$.

Given a function $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x):=\sup_{B\ni x}\frac{1}{|B|}\int_B|f(y)|dy,$$

where the supremum is taken over all balls B containing x. $\mathcal{B}(\mathbb{R}^n)$ is the set of $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ satisfying the condition that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

We say a function $p(\cdot):\mathbb{R}^n\longrightarrow\mathbb{R}$ is locally log-Hölder continuous at the origin, if there exists a constant C such that

$$|p(x)-p(0)|\le\frac{C}{\log(e+1/|x|)}$$

for all $x\in\mathbb{R}^n$. If, for some $p(\infty)\in\mathbb{R}$ and $C>0$, there holds

$$|p(x)-p(\infty)|\le\frac{C}{\log(e+|x|)}$$

for all $x \in \mathbb{R}^n$, then we say $p(\cdot)$ is log-Hölder continuous at infinity.

By $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ we denote the class of all exponents $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ which are log-Hölder continuous at the origin and at infinity with $p(\infty) := \lim\limits_{|x|\to\infty} p(x)$. Obviously, we can show that $p(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \bigcap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ implies $p'(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \bigcap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$. Moreover, we can easily show that $p(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \bigcap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ implies $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, see [13].

Definition 1.1 [9] For any $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, let $k_{p(\cdot)}$ denote the supremum of those $q > 1$ such that $p(\cdot)/q \in \mathcal{B}(\mathbb{R}^n)$. Let $e_{p(\cdot)}$ be the conjugate of $k_{p'(\cdot)}$.

Definition 1.2 [9] Let $p(\cdot) \in L^\infty(\mathbb{R}^n)$ and $1 < p(x) < \infty$. A Lebesgue measurable function $u(x,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ is said to be a Morrey weight function for $L^{p(\cdot)}(\mathbb{R}^n)$ if there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$ and $r > 0$, u fulfills

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} u(x,2^{j+1}r) < Cu(x,r). \tag{1.1}$$

We denote the class of Morrey weight functions by $\mathbb{W}_{p(\cdot)}$.

Next we define the Morrey spaces with variable exponent related to the nonnegative potential V.

For $x \in \mathbb{R}^n$, the function $m_V(x)$ is defined by

$$\frac{1}{m_V(x)} = \sup_{r>0}\left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \le 1 \right\}.$$

Definition 1.3 Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $u(x,r) \in \mathbb{W}_{p(\cdot)}$ and $-\infty < \alpha < \infty$. For $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ and $V \in B_q(q > 1)$, we say the Morrey spaces with variable exponent related to the nonnegative potential V is the collection of all function f satisfying

$$\|f\|_{\mathcal{M}^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)} = \sup_{z\in\mathbb{R}^n, r>0} \frac{(1 + rm_V(z))^\alpha}{u(z,r)} \|\chi_{B(z,r)} f\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty.$$

In particular, when $\alpha = 0$ or $V = 0$, the spaces $\mathcal{M}^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)$ is the Morrey spaces with variable exponent $\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)$ introduced in [9]. It is easy to see that $\mathcal{M}^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n) \subset \mathcal{M}_{p(\cdot),u}(\mathbb{R}^n)$ for $\alpha > 0$ and $\mathcal{M}_{p(\cdot),u}(\mathbb{R}^n) \subset \mathcal{M}^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)$ for $\alpha < 0$. If $p(x)$ is a constant, $u(x,r) = r^\lambda$ and $\lambda \in [0, n/p)$, we have

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \frac{u(x,2^{j+1}r)}{u(x,r)} &= \sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p}(\mathbb{R}^n)}} \frac{(2^{j+1}r)^\lambda}{r^\lambda} \\ &= \sum_{j=0}^{\infty} 2^{(j+1)(\lambda - n/p)} < C. \end{aligned}$$

In this case, the space $\mathcal{M}^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)$ is the Morrey spaces $L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)$ related to the nonnegative potential V, see [3].

Now it is in this position to state our results.

In [1], Shen showed the schrödinger type operators $\nabla(-\triangle+V)^{-1}\nabla, \nabla(-\triangle+V)^{-1/2}$ and $(-\triangle+V)^{-1/2}\nabla$ are the standard Calderón-Zygmund operators provided that $V \in B_n$. In particular, the kernels K of above operators satisfy the following inequality

$$|K(x,y)| \leq \frac{C_k}{(1+|x-y|m_V(x))^k}\frac{1}{|x-y|^n} \tag{1.2}$$

for any $k \in \mathbb{N}$, where C_k denotes a positive constant depend on k. In the rest of this paper, we always assume that T is one of the schrödinger type operators $\nabla(-\triangle+V)^{-1}\nabla, \nabla(-\triangle+V)^{-1/2}$ and $(-\triangle+V)^{-1/2}\nabla$ with $V \in B_n$.

Theorem 1.1 Suppose $V \in B_n$, $-\infty < \alpha < \infty$, $p(x) \in \mathcal{B}(\mathbb{R}^n)$. If $u \in \mathbb{W}_{p(\cdot)}$, then

$$\|Tf\|_{M^{p(\cdot),u}_{\alpha,V}} \leq C\|f\|_{M^{p(\cdot),u}_{\alpha,V}}.$$

Let $b \in \mathrm{BMO}$ (see its definition in [14]), we define the commutator of T by

$$[b,T]f = bTf - T(bf).$$

Theorem 1.2 Suppose $V \in B_n$, $b \in \mathrm{BMO}$, $-\infty < \alpha < \infty$, $p(x) \in \mathcal{B}(\mathbb{R}^n)$. If

$$\sum_{j=1}^{\infty}(j+1)\frac{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}u(z,2^{j+1}r) \leq Cu(z,r), \tag{1.3}$$

then

$$\|[b,T]f\|_{M^{p(\cdot),u}_{\alpha,V}} \leq C\|b\|_{\mathrm{BMO}}\|f\|_{M^{p(\cdot),u}_{\alpha,V}}.$$

Remark 1 We can easily show that u fulfills (1.3) implies $u \in \mathbb{W}_{p(\cdot)}$

Next, we consider the boundedness of fractional integrals related to schrödinger operators.

The $\mathcal{L}$-fractional integral operator is defined by

$$I_\beta f(x) = \mathcal{L}^{-\beta/2}f(x) = \int e^{-t\mathcal{L}}f(x)t^{\beta/2-1}dt = \int_{\mathbb{R}^n} K_\beta(x,y)f(y)dy \quad \text{for} \quad 0<\beta<n.$$

By Lemma 3.3 in [15], one can get the kernel $K_\beta(x,y)$ of I_β satisfy the following inequality

$$\begin{aligned} |K_\beta(x,y)| &\leq \frac{C_k}{(1+|x-y|(m_V(x)+m_V(y)))^k}\frac{1}{|x-y|^{n-\beta}} \\ &\leq \frac{C}{(1+|x-y|m_V(x))^k}\frac{1}{|x-y|^{n-\beta}} \end{aligned} \tag{1.4}$$

for any $k \in \mathbb{N}$ and $0<\beta<n$.

Theorem 1.3 Suppose $V \in B_{n/2}$, $-\infty < \alpha < \infty$, $p(x), q(x) \in \mathcal{B}(\mathbb{R}^n)$ satisfy $p_+ < \frac{n}{\beta}$, $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\beta}{n}$. If exists q_0 satisfying $\frac{n}{n-\beta} < q_0 < \infty$, $\frac{q(\cdot)}{q_0} \in \mathcal{B}(\mathbb{R}^n)$ and $u \in \mathbb{W}_{q(\cdot)}$, then

$$\|I_\beta f\|_{M^{q(\cdot),u}_{\alpha,V}} \leq C\|f\|_{M^{p(\cdot),u}_{\alpha,V}}.$$

Remark 2 $\mathbb{W}_{q(\cdot)} \subset \mathbb{W}_{p(\cdot)}$. Indeed, for $j \geq 0$, by inequality (2.3) in the next section, we have

$$\begin{aligned}\frac{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} &= \frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \frac{\|\chi_{B(z,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \frac{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \\ &\sim 2^{-(j+1)\beta} \frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leq \frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}.\end{aligned}$$

Therefore, using inequality (1.1), we obtain $\mathbb{W}_{q(\cdot)} \subset \mathbb{W}_{p(\cdot)}$.

Let $b \in \mathrm{BMO}$, we define the commutator of I_β by $[b, I_\beta]f = bI_\beta f - I_\beta(bf)$.

Theorem 1.4 Suppose $V \in B_{n/2}$, $b \in \mathrm{BMO}$, $-\infty < \alpha < \infty$, $p(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \bigcap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$. If $p_+ < \frac{n}{\beta}$, $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\beta}{n}$ and

$$\sum_{j=1}^{\infty}(j+1)\frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}} u(z, 2^{j+1}r) \leq Cu(z,r),$$

then

$$\|[b, I_\beta]f\|_{M_{\alpha,V}^{q(\cdot),u}} \leq C\|b\|_{\mathrm{BMO}}\|f\|_{M_{\alpha,V}^{p(\cdot),u}}.$$

For brevity, C always means a positive constant independent of the main parameters and may change from one occurrence to another. $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$, χ_{B_k} be the characteristic function of the set B_k for $k \in \mathbb{Z}$. $|S|$ denotes the Lebesgue measure of S. The exponent $p'(x)$ means the conjugate of $p(x)$, that is, $1/p'(x) + 1/p(x) = 1$.

2 Proofs of Theorems

In order to prove our result, we need some conclusions as follows.

Lemma 2.1 [16] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then the following conditions are equivalent:

(1) $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$;

(2) $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$;

(3) $p(\cdot)/q \in \mathcal{B}(\mathbb{R}^n)$ for some $1 < q < p_-$;

(4) $(p(\cdot)/q)' \in \mathcal{B}(\mathbb{R}^n)$ for some $1 < q < p_-$.

Lemma 2.1 ensures that $k_{p(\cdot)}$ is well-defined and satisfies $1 < k_{p(\cdot)} \leq p_-$. Moreover, $p_+ \geq e_{p(\cdot)}$.

Lemma 2.2 [17] If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and all $g \in L^{p'(\cdot)}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq r_p\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where $r_p := 1 + 1/p_- - 1/p_+$.

Lemma 2.3 [6] If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then there exists $C > 0$ such that for all balls B in $\mathbb{R}^n$,

$$C^{-1}|B| \leq \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C|B|.$$

Lemma 2.4 [9] Let $p(x) \in \mathcal{B}(\mathbb{R}^n)$ and $1 < p_- \leq p_+ < \infty$. There exist $C_1, C_2 > 0$ such that for any $B \in \mathbb{B}$,

$$C_1|B|^{\frac{1}{p_B}} \leq \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C_2|B|^{\frac{1}{p_B}},$$

where $\frac{1}{\bar{p}_B}=\frac{1}{|B|}\displaystyle\int_B \frac{1}{p(x)}dx$.

Lemma 2.5 [9] Let $p(x)\in\mathcal{B}(\mathbb{R}^n)$. For any $1<q<k_{p(\cdot)}$ and $1<s<k_{p'(\cdot)}$, there exist constants $C_1, C_2>0$ such that for any $x_0\in\mathbb{R}^n$ and $r>0$, we have

$$C_2 2^{jn(1-\frac{1}{s})}\le \frac{\|\chi_{B(x_0,2^jr)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x_0,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\le C_1 2^{\frac{jn}{q}},\quad \forall j\in\mathbb{N}.$$

The next lemma can be get by inequality (1.4) and Corollary 2.12 in [6].

Lemma 2.6 [6] Let $\beta>0, p(x), q(x)\in\mathcal{P}(\mathbb{R}^n)$ satisfy $p_+<\frac{\beta}{n}$ and $\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{\beta}{n}$. If exists q_0 satisfying $\frac{n}{n-\beta}<q_0<\infty$ and $\frac{q(\cdot)}{q_0}\in\mathcal{B}(\mathbb{R}^n)$, then $\|I_\beta f\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ for some $C>0$.

Theorem 1 in [8] and inequality (1.4) are rewrited as the following lemma.

Lemma 2.7 Suppose that $p(\cdot)\in\mathcal{P}_0^{\log}(\mathbb{R}^n)\bigcap\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ satisfies $p_+<\frac{\beta}{n}$ and $\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{\beta}{n}$. then we have

$$\|[b,I_\beta]\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C\|b\|_{\mathrm{BMO}}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

for $f\in L^{p(\cdot)}(\mathbb{R}^n)$ and $b\in\mathrm{BMO}(\mathbb{R}^n)$.

Using Corollary 2.5 and Corollary 2.10 in [6] and the inequality (1.2), we can get the following result.

Lemma 2.8 Let T be a Calderón-Zygmund operator and let b be a BMO function. If $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, then there exists a constant C independent of the function $f\in L^{p(\cdot)}(\mathbb{R}^n)$ such that

$$\|Tf\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\quad\text{and}\quad \|[b,T]f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|b\|_{\mathrm{BMO}}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2.9 [18] Let k be a positive integer. Then we have that for all $b\in\mathrm{BMO}(\mathbb{R}^n)$ and all $i,j\in\mathbb{Z}$ with $i>j$,

$$C^{-1}\|b\|_{\mathrm{BMO}}^k\le\sup_B\frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b-b_B)^k\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|b\|_{\mathrm{BMO}}^k;$$
$$\|(b-b_{B_i})^k\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C(j-i)^k\|b\|_{\mathrm{BMO}}^k\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2.10 [1, 3] Suppose $V\in B_q$ with $q\ge n/2$. Then there exist positive constants C and k_0 such that

(1) $m_V(x)\sim m_V(y)$ if $|x-y|\le\frac{C}{m_V(x)}$;

(2) $m_V(y)\le C(1+|x-y|m_V(x))^{k_0}m_V(x)$;

(3) $m_V(y)\ge\frac{Cm_V(x)}{(1+|x-y|m_V(x))^{k_0/(k_0+1)}}$.

We will give the proofs of Theorems 1.3 and 1.4 below. The arguments for Theorems 1.1 and 1.2 are similar, we omit the details here.

Proof of Theorem 1.3 Without loss of generality, we may assume that $\alpha<0$. Let $f\in\mathcal{M}_{p(\cdot),u}$. For any $z\in\mathbb{R}^n$ and $r>0$, we write $f(x)=f_0(x)+\sum\limits_{j=1}^{\infty}f_j(x)$, where $f_0=f\chi_{B(z,2r)}$, $f_j=f\chi_{B(z,2^{j+1}r)\setminus B(z,2^jr)}$ for $j\ge1$. Hence we have

$$\|(I_\beta f)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le\|(I_\beta f_0)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}+\sum_{j=1}^{\infty}\|(I_\beta f_j)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

By Lemma 2.6, we obtain

$$\frac{(1+rm_V(z))^\alpha}{u(z,r)}\|(I_\beta f_0)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\frac{(1+rm_V(z))^\alpha}{u(z,r)}\|f\chi_{B(z,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Because inequality (1.1) and Lemma 2.5 imply that $u(x,r) \ge Cu(x,2r)$. Therefore, we obtain

$$\begin{aligned}\frac{(1+rm_V(z))^\alpha}{u(z,r)}\|(I_\beta f_0)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\le C\frac{(1+rm_V(z))^\alpha}{u(z,2r)}\|f\chi_{B(z,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C\frac{(1+2rm_V(z))^\alpha}{u(z,2r)}\|f\chi_{B(z,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)}.\end{aligned}$$

Furthermore, for any $j \ge 1$, $x \in B(z,r)$ and $y \in B(z,2^{j+1}r)\backslash B(z,2^j r)$, we note that $|x-y| \ge |y-z|-|x-z| > C2^j r$. Thus we get

$$|(I_\beta f_j)(x)| \le C(2^j r)^{\beta-n}\int_{B(z,2^{j+1}r)}\frac{1}{(1+2^j rm_V(x))^k}|f(y)|dy.$$

Using Lemma 2.10, we derive the estimate

$$\begin{aligned}1+2^j rm_V(x) &\ge 1+2^j r\frac{Cm_V(z)}{(1+|x-z|m_V(z))^{k_0/k_0+1}}\\ &\ge C\frac{1+2^j rm_V(z)}{(1+rm_V(z))^{k_0/(k_0+1)}}\\ &\ge C(1+2^j rm_V(z))^{1/(k_0+1)}.\end{aligned} \tag{2.1}$$

Thus we get that

$$|(I_\beta f_j)(x)| \le C(2^j r)^{\beta-n}\int_{B(z,2^{j+1}r)}\frac{1}{(1+2^j rm_V(z))^{k/(k_0+1)}}|f(y)|dy.$$

Lemma 2.2 ensures that

$$\int_{B(z,2^{j+1}r)}|f(y)|dy \le C\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}$$

for some constant $C>0$.

Subsequently, taking the norm $\|\cdot\|_{L^{q(\cdot)}(\mathbb{R}^n)}$, we have

$$\begin{aligned}\|(I_\beta f_j)\chi_{B(z,2r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le &C\frac{(2^j r)^{\beta-n}}{(1+2^j rm_V(z))^{k/(k_0+1)}}\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\times\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{2.2}$$

Applying Lemma 2.3 with $B = B(z,2^{j+1}r)$, we obtain

$$\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C\frac{(2^{j+1}r)^n}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

Using the above inequality on (2.2), we obtain

$$\begin{aligned}
&\|\chi_{B(z,2r)}(I_\alpha f_j)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
\leq& C\frac{(2^j r)^{\beta-n}}{(1+2^j r m_V(z))^{k/(k_0+1)}}\frac{\|\chi_{B(z,r)}(x)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}(2^{j+1}r)^n}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\\
\leq& C\frac{(2^j r)^{\beta}}{(1+2^j r m_V(z))^{k/(k_0+1)}}\frac{\|\chi_{B(z,r)}(x)\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

In view of the fact that for any ball B, we have

$$\frac{1}{|B|}\int_B \frac{1}{p(x)}dx-\frac{1}{|B|}\int_B\frac{1}{q(x)}dx=\frac{1}{\bar{p}_B}-\frac{1}{\bar{q}_B}=\frac{\beta}{n}.$$

Lemma 2.4 implies that

$$C_2|B|^{\frac{\beta}{n}}\leq\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\leq C_1|B|^{\frac{\beta}{n}} \tag{2.3}$$

for some constants $C_1>C_2>0$ independent of B.

Hence using (2.3) with $B=B(z,2^{j+1}r)$, we have

$$C_2\frac{(2^{j+1}r)^\beta}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\leq\frac{1}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}.$$

Therefore

$$\begin{aligned}
&\|\chi_{B(z,r)}(I_\beta f_j)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
\leq& C\frac{1}{(1+2^j r m_V(z))^{k/(k_0+1)}}\frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
=& C\frac{1}{(1+2^j r m_V(z))^{k/(k_0+1)}}\frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\frac{u(z,2^{j+1}r)}{u(z,2^{j+1}r)}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Thus we arrive at the inequality

$$\begin{aligned}
\|\chi_{B(z,r)}(I_\beta f_j)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\leq& C\frac{(1+2^j r m_V(z))^{-\alpha}}{(1+2^j r m_V(z))^{k/(k_0+1)}}\frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\\
&u(z,2^{j+1}r)\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)}.
\end{aligned}$$

Taking $k=(-[\alpha]+1)(k_0+1)$, we obtain

$$\sum_{j=1}^\infty\|\chi_{B(z,r)}(I_\beta f_j)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\leq C\sum_{j=1}^\infty\frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}u(z,2^{j+1}r)\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)}.$$

As $u\in\mathbb{W}_{q(\cdot)}$ and $\alpha<0$, we have

$$\begin{aligned}
\frac{(1+rm_V(z))^\alpha}{u(z,r)}\sum_{j=1}^\infty\|\chi_{B(z,r)}(I_\beta f_j)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\leq& C(1+rm_V(z))^\alpha\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)}\\
\leq& C\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, we have $\|I_\beta f\|_{M^{q(\cdot),u}_{\alpha,V}} \le C\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)}$, and hence the proof of Theorem 1.3 is completed.

Proof of Theorem 1.4 Without loss of generality, we may assume that $\alpha < 0$. Let $f \in \mathcal{M}_{p(\cdot),u}$. For any $z \in \mathbb{R}^n$ and $r > 0$, write $f(x) = f_0(x) + \sum\limits_{j=1}^{\infty} f_j(x)$, where $f_0 = f\chi_{B(z,2r)}$, $f_j = f\chi_{B(z,2^{j+1}r)\setminus B(z,2^j r)}$ for $j \ge 1$. Hence we have

$$\begin{aligned}\frac{(1+rm_V(z))^\alpha}{u(z,r)}\|([b,I_\beta]f)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le&\frac{(1+rm_V(z))^\alpha}{u(z,r)}\|([b,I_\beta]f_0)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &+\frac{(1+rm_V(z))^\alpha}{u(z,r)}\sum_{j=1}^{\infty}\|([b,I_\beta]f_j)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ =&D_1+D_2.\end{aligned}$$

First, we estimate D_1.

Lemma 2.7 shows that $\|[b,I_\beta]f_0\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{\mathrm{BMO}}\|f_0\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. Thus, we find that

$$\begin{aligned}D_1 &\le C\|b\|_{\mathrm{BMO}}\frac{(1+rm_V(z))^\alpha}{u(z,r)}\|\chi_{B(z,2r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C\|b\|_{\mathrm{BMO}}\frac{(1+rm_V(z))^\alpha}{u(z,2r)}\|\chi_{B(z,2r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C\|b\|_{\mathrm{BMO}}\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)},\end{aligned}$$

because inequality (1.1) and Lemma 2.5 imply that $u(z,2r) \le Cu(z,r)$.

Next, we estimate D_2.

For any $j \ge 1$, $x \in B(z,r)$ and $y \in B(z,2^{j1}r)\setminus B(z,2^j r)$, we note that $|x-y| \ge |y-z| - |x-z| > C2^j r$. Using inequality (2.1) and Lemma 2.2, we obtain

$$\begin{aligned}|([b,I_\beta]f_j)(x)| \le&\frac{1}{(1+2^j rm_V(z))^{k/(k_0+1)}}\int_{B(z,2^{j+1}r)\setminus B(z,2^j r)}\frac{|(b(x)-b(y))f(y)|}{|x-y|^{n-\beta}}dy\\ \le&\frac{(2^j r)^{\beta-n}}{(1+2^j rm_V(z))^{k/(k_0+1)}}\int_{B(z,2^{j+1}r)}|(b(x)-b(y))f(y)|dy\\ \le&\frac{(2^j r)^{\beta-n}}{(1+2^j rm_V(z))^{k/(k_0+1)}}\\ &\times\left(|b(x)-b_{B(z,r)}|\int_{B(z,2^{j+1}r)}|f(y)|dy+\int_{B(z,2^{j+1}r)}|(b_{B(z,r)}-b(y))f(y)|dy\right)\\ \le&\frac{(2^j r)^{\beta-n}}{(1+2^j rm_V(z))^{k/(k_0+1)}}\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\times\left(|b(x)-b_{B(z,r)}|\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}+\|(b_{B(z,r)}-b)\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\right).\end{aligned}$$

Subsequently, taking the norm $\|\cdot\|_{L^{q(\cdot)}(\mathbb{R}^n)}$ and using Lemma 2.9, we have

$$\begin{aligned}&\|([b,I_\beta]f_j)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ \le&C(j+1)\frac{(2^j r)^{\beta-n}}{(1+2^j rm_V(z))^{k/(k_0+1)}}\\ &\times\|b\|_{\mathrm{BMO}}\|f\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

The arguments here are quite similar to the proof of Theorem 1.3, so we have

$$\begin{aligned}\|([b,I_\beta]f_j)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq & C\|b\|_{\mathrm{BMO}}(j+1)\frac{1}{(1+2^jrm_V(z))^{k/(k_0+1)}}\frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\\ & \frac{u(z,2^{j+1}r)}{u(z,2^{j+1}r)}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ \leq & C\|b\|_{\mathrm{BMO}}(j+1)\frac{(1+2^jrm_V(z))^{-\alpha}}{(1+2^jrm_V(z))^{k/(k_0+1)}}\frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\\ & u(z,2^{j+1}r)\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)}.\end{aligned}$$

Taking $k=(-[\alpha]+1)(k_0+1)$, we obtain

$$\begin{aligned}&\sum_{j=1}^{\infty}\|([b,I_\beta]f_j)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ \leq & C\|b\|_{\mathrm{BMO}}\sum_{j=1}^{\infty}(j+1)\frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}u(z,2^{j+1}r)\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)}.\end{aligned}$$

As u fulfills (1.3) and $\alpha<0$, we obtain

$$\begin{aligned}D_2=&\frac{(1+rm_V(z))^\alpha}{u(z,r)}\sum_{j=1}^{\infty}\|([b,I_\beta]f_j)\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ \leq & C(1+rm_V(z))^\alpha\|b\|_{\mathrm{BMO}}\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)}\leq C\|b\|_{\mathrm{BMO}}\|f\|_{M^{p(\cdot),u}_{\alpha,V}(\mathbb{R}^n)}.\end{aligned}$$

Consequently we have proved Theorem 1.4.

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一类 Schrödinger 型算子在关于非负位势的变指数 Morrey 空间上的有界性

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摘要: 本文考虑了一类 Schrödinger 型算子和其交换子的有界性问题. 利用其在L^p空有界性间上的, 获得了一类 Schrödinger 型算子和其交换子在变指数 Morrey 空间上的有界性.

关键词: Morrey 空间; 交换子; Schrödinger 型算子; 变指数

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