# BIFURCATIONS OF TRAVELING WAVE SOLUTIONS OF INTEGRABLE EVOLUTION EQUATIONS FOR SURFACE WAVES IN DEEP WATER

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**Abstract:** In this paper, we investigate the traveling wave solutions of a small-aspect-ratio wave equation and an integrable evolution equation for surface waves in deep water. By applying the qualitative theory of differential equations, we analyze the phase portraits of the traveling wave systems and obtain the exact explicit representations of solitary wave solutions.

**Keywords:** traveling wave solutions; bifurcations of phase portraits; integrable systems; surface waves equations

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# 1 Introduction

In order to describe the dynamics of monochromatic surface waves in deep water, a asymptotic model for small-aspect-ratio wave was derived in [1] as follows

$$2\sqrt{\frac{k}{g}}\eta_{xxt} = k^2\eta_x - \frac{3}{2}k(\eta\eta_x)_{xx},$$
(1.1)

where g is the gravitation constant and k is wave vector. The equation (1.1) has a kdependent coefficient and it can be considered as belonging to both of the two categories: that of Korteweg-de Vries models (KdV, modified KdV, Benjamin-Bona-Mahony-Peregrine, Camassa-Holm, etc.) describing evolutions of wave profiles and that of NLS-type equations

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(modified NLS [2], Davey-Stewartson [3], etc.) describing modulation of wave profiles and having k-dependent coefficients. Moreover, in order to find a steep rotational Stokes wave, paper [1] also start with equation (1.1) in the frame as

$$2\sqrt{\frac{k}{g}}\eta_{xxt} = k^2\eta_x - \eta_{xxx} - \frac{9}{2}k\eta_x\eta_{xx} - \frac{3}{2}k\eta\eta_{xxx}.$$
 (1.2)

In this paper, since (1.1) and (1.2) are meaningful equations for surface waves in deep water, we will employ the bifurcation method and qualitative theory of dynamical systems [4] to investigate these equations. The phase portraits and the explicit expressions of the bounded traveling wave solutions for the equations will obtained in the paper. To the best of our knowledge, bifurcations of traveling wave solution for above equations have not yet been considered.

It is well known that traveling waves propagation in nonlinear media was the subject of intense investigations in recent years. The study of nonlinear wave equations and their solutions were of great importance in many areas of physics (see [5–9] and the references therein). Traveling wave solution is an important type of solutions for the nonlinear partial differential equations (NLPDEs) which were found to have a variety of traveling wave solutions (see [10, 13, 14, 30, 32]).

In recent years, various powerful methods were developed to construct traveling wave solutions of nonlinear partial differential equations, such as the trigonometric function series method [15], the modified mapping method and the extended mapping method (see [16]), the  $\left(\frac{G'}{G}\right)$  expansion method (see [17, 18]), the homogeneous balance method (see [19, 20]), the tanh and extended methods (see [21]) and so on. Meanwhile, the bifurcation method of phase plane was developed to obtain traveling wave solutions of NLPDEs (see [22–24]). Therefore, it is a good way to understand the behavior of traveling wave solutions of NLPDEs. What is more, breaking three solutions have attracted a great deal of interest (see [25–35]) since Konno et al. (see [37]) first reported the breaking three solutions in a nonlinear oscillation model of an elastic beam with tension.

Motivated by above mentioned works, we consider equation (1.1) and (1.2) by using the bifurcation method and qualitative theory of dynamical systems. The paper is organized as follows. In Section 2, we discuss the dynamical behavior of solutions of small-aspect-ratio wave model (1.2) and give exact parametric expressions of traveling wave solutions for the equations. In Section 3, the dynamical analysis and exact explicit representations of solitary wave solutions of an integrable evolution equation are given. At the last section, we give the conclusions of this paper.

### 2 Dynamical Analysis and Exact Parametric Traveling Wave Solutions

In this section, we investigate the traveling wave solutions of a small-aspect-ratio waves equation (1.1). A breaking three solution and a family of periodic breaking three solutions are found by employing the method of the phase plane. In addition, the relationship between

the loop-soliton solution and the periodic loop solutions is as well investigated. The analysis may be helpful in understanding the significance of dynamical behavior of eq. (1.1).

It is well known that a traveling wave solution of (1.1) with wave speed c is the solution having the form  $\eta = \phi(\xi)$  with  $\xi = x - ct$ . Substituting the traveling wave solution  $\eta(x, t) = \phi(x - ct)$  for the constant wave speed c into (1.1), we have the following ordinary differential equation

$$-2c\sqrt{\frac{k}{g}}\phi''' = k^2\phi' - \frac{3}{2}k(3\phi'\phi'' + \phi\phi''').$$
(2.1)

Integrating (2.1), we have

$$(4c\sqrt{\frac{k}{g}} - 3k\phi)\phi'' = 3k(\phi')^2 - 2k^2\phi + \bar{g},$$

where  $\overline{g} \in \mathbb{R}$  is an integral constant. Let  $u = 4c\sqrt{k/g} - 3k\phi$ ,  $y = \phi'$ , then we have a plane autonomous system

$$\frac{du}{d\xi} = -3ky, \quad \frac{dy}{d\xi} = \frac{1}{3u} \left( 9ky^2 + 2ku + 3\bar{g} - 8ck\sqrt{\frac{k}{g}} \right).$$
(2.2)

It is easy to see that system (2.2) has the first integral

$$H(u,y) = \frac{9}{2}u^2y^2 + \frac{2}{3}u^3 + \frac{1}{2}Au^2 = h,$$
(2.3)

where  $A = \frac{3\bar{g}}{k} - 8c\sqrt{\frac{k}{g}}$ . All level sets H(u, y) = h  $(h \in \mathbb{R})$  give the invariant curves of (2.2). As well known, system (2.3) has a periodic solution if and only if it has a center. Next, all possible periodic annuli defined by the vector fields of (2.2) when the parameters c, k, g and  $\bar{g}$  vary will be studied. Now, we consider the quadratic Hamiltonian system

$$\frac{\mathrm{d}u}{\mathrm{d}\zeta} = -9kuy, \quad \frac{\mathrm{d}y}{\mathrm{d}\zeta} = 9ky^2 + 2ku + 3\bar{g} - 8ck\sqrt{\frac{k}{g}},\tag{2.4}$$

which is obtained from (2.2) by letting  $d\xi = 3ud\zeta$ . System (2.4) has the same first integral H(u, y) and the same topological phase portraits as system (2.2) except for the straight line u = 0. Clearly, system System (2.4) has two types of singular points of system (2.4), as follows (see Fig. 1). Using qualitative theory of differential equations [29, 30], we can easily verify the following proposition.

**Proposition 2.1** Denote  $h_0 = H(-\frac{1}{2}A, 0) = \frac{1}{24}A^3$ , the points  $P(-\frac{A}{2}, 0)$ ,  $P_1(0, -\frac{\sqrt{-A}}{3})$  and  $P_2(0, \frac{\sqrt{-A}}{3})$ , respectively, then

**Case I** If A < 0, then P is a center,  $P_1$  and  $P_2$  are saddle points, which shown in Fig.1(a). For  $h \in (h_0, 0)$  defined by (2.3), (1.1) has a family of smooth periodic wave solutions (see Fig. 2(a)). For  $h = h_0$  defined by (2.3), (1.1) has a unique periodic cuspon solution shown in Fig. 2(b).

**Case II** If A > 0, then P is saddle point (Fig. 1(b)); for  $h = h_0$  defined by (2.3), (1.1) has a unique breaking three solution shown in Fig. 2(c); for  $h \in (0, h_0)$ , there exists a

family of uncountably infinite many periodic breaking three solutions of (1.1) shown in Fig. 2(d). Moreover, the periodic loop solutions converge to the breaking three solutions as h

Now, we will give the exact parametric representations of smooth traveling wave solutions, periodic cuspons, breaking three solution and periodic breaking three solutions of the small-aspect-ratio waves equation (1.1).

(a) Smooth periodic wave solutions.

First, corresponding to Fig.1(a), when A < 0, a family of smooth periodic wave solutions of (1.1) exist, which correspond to a family of periodic orbits defined by H(u, y) = h, where  $h \in (h_0, 0)$ . The numerator of (2.3) can be decomposed into

$$y^{2} = \frac{4}{27u^{2}} \left( -u^{3} - \frac{3A}{4}u^{2} + \frac{3}{2}h \right) = -\frac{4}{27u^{2}} \left[ (u - \alpha)(u - \beta)(u - \gamma) \right],$$

where  $\alpha > \beta > 0 > \gamma$  are function of  $c, k, g, \overline{g}$ , which can be rigorously determined by the formula for cubic algebraic equations. Then for  $\beta < u < \alpha$ , and by  $y = -\frac{1}{3k} \frac{\mathrm{d}u}{\mathrm{d}\xi}$ , we have

$$\xi = \frac{\sqrt{3}}{2k} \int_{\beta}^{u} \frac{z}{\sqrt{(\alpha - z)(z - \beta)(u - \gamma)}} dz.$$

Then we obtain the following exact parametric representations of smooth periodic wave solutions of (1.1) as follows (see [36])

$$\begin{cases} u(\tau) = \frac{(\alpha - \gamma)\beta - \gamma(\alpha - \beta)(\mu \operatorname{sn}(\tau, \lambda))^2}{(\alpha - \gamma) - (\alpha - \beta)(\mu \operatorname{sn}(\tau, \lambda))^2}, \\ \xi(\tau) = \frac{1}{k}\sqrt{\frac{3}{\alpha - \gamma}} \left[\gamma \tau + (\beta - \gamma)\Pi\left(\operatorname{arcsin}(\mu \operatorname{sn}(\tau, \lambda), \lambda^2, \lambda)\right)\right], \end{cases}$$

where  $\lambda^2 = (\alpha - \beta)/(\alpha - \gamma)$ ,  $\operatorname{sn}(\tau, \lambda)$  is Jacobian elliptic functions with the modulus  $\lambda$ ,  $\Pi(\cdots)$  is the elliptic integral of the third kind and  $\mu$  is a appropriate parameters.

(b) Periodic cuspons

Corresponding to Fig. 1(a), when A < 0 and h = 0, a periodic cuspon of (1.1) exists, which corresponds to the heteroclinic orbits defined by H(u, y) = h = 0. We have the traveling wave solution of (1.1)

$$\xi = -\frac{\sqrt{3}}{2k} \int_{u}^{-\frac{3}{4}A} \frac{1}{\sqrt{-z - \frac{3}{4}A}} dz = \frac{\sqrt{3}}{k} \sqrt{\left(-u - \frac{4}{3}A\right)}.$$

Let  $T = \frac{3}{2}\sqrt{\frac{-A}{k^2}}$ , we obtain the following periodic cuspon (see Fig. 2(b))

$$u = -\frac{k^2}{3}(x - ct)^2 - \frac{3}{4}A, \ (2n - 1)T \le x - ct \le (2n + 1)T.$$

(c) Breaking three solutions

approaches  $h_0$ .



Fig.2: wave profiles. (a) smooth periodic wave solutions, (b) periodic cuspons, (c) breaking three solutions, (d) periodic breaking three solutions.

Corresponding to Fig. 1(a), when A > 0 and  $h = h_0$ , the equilibrium point P(-A/2, 0) is a saddle point. By using the first equation of system (2.2) to perform the integration along the three orbits for the initial value  $u(0) = -\frac{A}{4}$  and  $u(0) = \frac{A}{4}$ , respectively, we have

$$\xi = -\frac{\sqrt{3}}{2k} \int_{-\frac{A}{4}}^{u} \frac{z}{\sqrt{\left(\frac{A}{4} - z\right)(z + \frac{A}{2})^2}} dz = \frac{\sqrt{3}}{2k} \int_{-\frac{A}{4}}^{u} \frac{z}{(z + \frac{A}{2})\sqrt{\left(\frac{A}{4} - z\right)}} dz$$

Then we obtain the following parametric representations of the traveling wave solutions of (1.1) (see Fig. 2(c))

$$\begin{cases} u(\tau) = \frac{3A}{4} \left(\frac{1+\tau}{1-\tau}\right)^2 - \frac{A}{4},\\ \xi(\tau) = -\frac{3\sqrt{A}}{2k} \left(\frac{1+\tau}{1-\tau}\right) - \frac{\sqrt{A}}{2k} \ln|\tau| + g_0, \end{cases}$$

where

$$g_0 = \frac{3}{2k} \left( \sqrt{2A} - \sqrt{\frac{A}{3}} \ln \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \right).$$

(d) Periodic breaking three solutions

Corresponding to Fig. 2(d), when A > 0, the graph defined by H(u, y) = h,  $h \in (0, h_0)$  consists of two open-end curves, passing through the points  $(\beta, 0)$  and  $(\alpha, 0)$ , respectively, where  $-A/2 < \beta < 0 < \alpha$ .

By the algebra curve the numerator of (2.3), we have the similar representations of smooth periodic wave solutions as in (a) by doing similar procedure, we also have

$$\pm \xi = \frac{\sqrt{3}}{2k} \int_{u}^{\alpha} \frac{z}{\sqrt{(\alpha-z)(z-\beta)(z-\gamma)}} dz, \qquad (2.5)$$

where  $\alpha > 0 > \beta > \gamma$  are function of  $c, k, g, \overline{g}$ , and we obtain the following exact parametric representations of smooth periodic wave solutions of (1.1) (see Fig. 2(d)) as follows

$$\begin{cases} u(\tau) = \frac{(\alpha - \gamma)\beta - \gamma(\alpha - \beta)(\mu \operatorname{sn}(\tau, \lambda))^2}{(\alpha - \gamma) - (\alpha - \beta)(\mu \operatorname{sn}(\tau, \lambda))^2}, \\ \xi(\tau) = \frac{1}{k}\sqrt{\frac{3}{\alpha - \gamma}} \left[\gamma\tau + (\beta - \gamma)\Pi\left(\operatorname{arcsin}(\mu \operatorname{sn}(\tau, \lambda), \lambda^2, \lambda)\right)\right], \end{cases}$$

where  $\lambda^2 = (\alpha - \beta)/(\alpha - \gamma)$ ,  $\operatorname{sn}(\tau, \lambda)$  is Jacobian elliptic functions with the modulus  $\lambda$ ,  $\Pi(\cdots)$  is the elliptic integral of the third kind and  $\mu$  is a appropriate parameter.

#### **3** Dynamical Analysis and Exact Traveling Wave Solutions of (1.2)

In this section, we investigate the periodic traveling wave solutions of (1.2) which has a great relationship with the steep rotational Stokes wave equation. Moreover, the results have some different from (1.1). First, by substituting  $\eta(x,t) = \phi(\xi)$  with  $\xi = x - ct$  for the constant wave speed c into (1.2), we have the following ordinary differential equation

$$-2c\sqrt{\frac{k}{g}}\phi^{\prime\prime\prime} = k^2\phi^{\prime} - \phi^{\prime\prime\prime} - \frac{9}{2}k\phi^{\prime}\phi^{\prime\prime} - \frac{3}{2}k\phi\phi^{\prime\prime\prime}.$$
(3.1)

Integrating (3.1) with respect to  $\xi$ , and let  $u = 1 - 2c\sqrt{\frac{k}{g}} + \frac{3}{2}k\phi$ ,  $y = \phi'$ , then we have a plane autonomous system

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = \frac{3}{2}ky, \quad \frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{-9ky^2 + 4ku + 8ck\sqrt{\frac{k}{g}} + 6\overline{g} - 4k}{6u},\tag{3.2}$$

where  $\overline{g} \in R$  is an integral constant. It is easy to see that system (3.2) has the first integral

$$H(u,y) = \frac{9}{2}u^2y^2 - \frac{4}{3}u^3 - Qu^2 = h,$$
(3.3)

where  $Q = 4c\sqrt{\frac{k}{g}} + 3\frac{\overline{g}}{k} - 2$ , all level sets H(u, y) = h  $(h \in \mathbb{R})$  give the invariant curves of (3.2). As well known, system (3.3) has a periodic solution if and only if it has a center. Now, we first consider the quadratic Hamiltonian system

$$\frac{\mathrm{d}x}{\mathrm{d}\zeta} = 9kuy, \quad \frac{\mathrm{d}y}{\mathrm{d}\zeta} = -9ky^2 + 4ku + 2kQ, \tag{3.4}$$

which is obtained from (3.2) by letting  $d\xi = 6ud\zeta$ . System (3.4) has the same first integral H(u, y) and the same topological phase portraits as system (3.2) except for the straight line u = 0. Clearly, system (3.4) also has two types of singular points, as follows (see Fig. 1). Using qualitative theory of differential equations, we can easily verify the following statement.

Proposition 3.1 Denote

$$h_0 = H(-\frac{1}{2}Q, 0) = -\frac{Q^3}{12},$$

and the points  $P(-\frac{1}{2}Q,0)$ ,  $P_1(0,-\sqrt{2Q/9})$  and  $P_2(0,\sqrt{2Q/9})$ , respectively. Then

**Case I** If Q > 0, then P is a center;  $P_1$  and  $P_2$  are saddle points (see Fig. 3(a)). For  $h \in (h_0, 0)$  defined by (3.3), (1.2) has a family of smooth periodic wave solutions(Fig. 4(a)). For h = 0 defined by (3.4), (1.2) has a unique periodic cuspon shown in Fig. 4(b).

**Case II** If Q < 0, then P is saddle points (see Fig. 3(b); For  $h = h_0$  defined by (3.4), (1.2) has a unique breaking three solution which is shown in Fig. 4(c). For  $h \in (0, h_0)$ , there exists a family of uncountably infinite many periodic loop solutions of (1.2) shown in Fig. 3(d). Moreover, the periodic breaking three solutions converge to the breaking three solutions as h approaches  $h_0$ .



Fig.4: (color online) wave profiles. (a) smooth periodic wave solutions, (b) periodic cuspons, (c) breaking three solutions, (d) periodic breaking three solutions.

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In the following, we will give the exact representations of the smooth periodic traveling wave solutions, periodic cuspons, the breaking three solution and periodic breaking three solutions of equation (1.2).

(a) Smooth periodic wave solutions.

Corresponding to Fig. 3(a), when Q > 0, a family of smooth periodic wave solutions of (1.1) exist, which correspond to a family of periodic orbits defined by  $H(u, y) = h \in (h_0, 0)$ , we have

$$y^{2} = \frac{8}{27u^{2}} \left( u^{3} + \frac{3}{4}Qu^{2} + \frac{3}{4}h \right).$$
(3.5)

By using the first equation of system (3.2), we have

$$\xi = \frac{\sqrt{6}}{2k} \int_{\gamma}^{a} \frac{z}{\sqrt{(\alpha - z)(\beta - z)(z - \gamma)}} dz,$$

where  $\gamma < \beta < 0 < \alpha$  are function of  $c, k, g, \overline{g}$ , which can be rigorously determined by the formula for cubic algebraic equations. Then we obtain the following exact parametric representations of smooth periodic wave solutions of of eq. (1.2)

$$\begin{cases} u(\tau) = \gamma + (\beta - \gamma)(\mu \operatorname{sn}(\tau, \lambda))^2, \\ \xi(\tau) = \frac{\sqrt{6}}{k\sqrt{\alpha - \gamma}} [\alpha \tau - (\alpha - \gamma)E(\operatorname{arcsin}(\mu \operatorname{sn}(\tau, \lambda)))], \end{cases}$$

where  $\lambda^2 = (\beta - \gamma)/(\alpha - \gamma)$ ,  $\operatorname{sn}(\tau, \lambda)$  is Jacobian elliptic functions with the modulus  $\lambda$ ,  $E(\cdot)$  is the elliptic integral of the second kind and  $\mu$  is a appropriate parameters.

(b) Periodic cuspons

Corresponding to Fig. 3(b), when Q > 0 and h = 0, a periodic cuspon of (1.2) exists, which corresponds to the heteroclinic orbits defined by H(u, y) = h = 0. We have the following traveling wave solution of (1.2)

$$\xi = \frac{\sqrt{6}}{2k} \int_{-\frac{3}{4}Q}^{u} \frac{1}{\sqrt{z + \frac{3}{4}Q}} dz = \frac{\sqrt{6}}{k} \sqrt{u + \frac{3}{4}Q}.$$
(3.6)

Thus we have the periodic cusp wave solutions of equation (1.2) (see Fig. 4(b))

$$u(x,t) = \frac{k^2}{6}(x - ct - 2nT)^2 - \frac{3}{4}Q,$$
  
(2n-1)T \le x - ct \le (2n+1)T, n \in \mathbb{N}, T = \sqrt{\frac{9Q}{2k^2}}.

(c) Breaking three solutions

Corresponding to Fig. 3(b), when Q < 0 and  $h = h_0$ , the equilibrium point P(-Q/2, 0) is a saddle point. By using the first equation of system (3.2) to perform the integration along the three orbits for the initial value u(0) = Q/4 and u(0) = -Q/4, respectively, we have

$$\xi = \frac{\sqrt{6}}{2k} \int_{-\frac{1}{4}Q}^{u} \frac{z}{\sqrt{\left(z - \frac{Q}{4}\right)\left(z + \frac{1}{2}Q\right)^2}} dz.$$

Then we obtain the following parametric representations of the traveling wave solutions of (1.2) (see Fig. 4(c))

$$\begin{cases} u(\tau) = \frac{Q}{4} - \frac{3Q}{4} \left(\frac{1+t}{1-t}\right)^2, \\ \xi(\tau) = -\frac{3\sqrt{-2Q}}{2k} \left(\frac{1+t}{1-t} + \frac{1}{3}\log|t|\right) + g_0, \end{cases}$$

where  $g_0 = \frac{\sqrt{-3Q}}{2k} + \frac{-2Q}{2k} \log \frac{\sqrt{2}-1}{\sqrt{2}+1}$ .

(d) Periodic breaking three solutions

Corresponding to Fig. 3(b), when Q < 0 and  $h \in (0, h_0)$ , the graph defined by  $H(u, y) = h \in (0, h_0)$  consists of two open-end curves, passing through the points  $(\gamma, 0)$  and  $(\beta, 0)$ , respectively, where  $\gamma < 0 < \beta < -\frac{Q}{2}$ . By calculating, we obtain following exact parametric representations of the periodic breaking three solutions of (1.2) [see Fig. 4(d)]

$$\begin{cases} u(\tau) = \gamma + (\beta - \gamma)(\mu \operatorname{sn}(\tau, \lambda))^2, \\ \xi(\tau) = \frac{\sqrt{6}}{k\sqrt{\alpha - \gamma}} [\alpha \tau - (\alpha - \gamma)E(\operatorname{arcsin}(\mu \operatorname{sn}(\tau, \lambda)))], \end{cases}$$

where  $\lambda^2 = (\beta - \gamma)/(\alpha - \gamma)$ ,  $\operatorname{sn}(\tau, \lambda)$  is Jacobian elliptic functions with the modulus  $\lambda$  and  $\mu$  is a appropriate parameter.

# 4 Conclusions

In this paper, by using the qualitative theory of differential equations, a small-aspectratio wave equation (1.1) and an integrable evolution equation (1.2) for surface waves in deep water are studied. The phase portraits of the traveling wave systems are analyzed (see Fig. 1 and Fig. 3) and exact explicit representations of solitary wave solutions such as smooth periodic wave solutions, periodic cuspons, breaking three solution and periodic breaking three solutions (see Fig. 2 and Fig. 4) are give in Section 2 and Section 3, respectively. By comparing the results of these two equations, the phase portraits and exact explicit representations of solitary wave solutions are obtained under some different parameter conditions.

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# 深水表面波可积发展方程的行波解与分支

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**摘要:** 本文研究了small-aspect-ratio波方程和深水表面波可积发展方程的行波解问题.利用微分方程 定性理论的方法,分析了行波系统的相图分支,获得了孤立波解的精确表达式. 关键词: 行波解;相图分支;可积系统;表面波方程

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