Vol. 36 (2016)
No. 3

数 学 杂 志
J. of Math. (PRC)

THE EXTREMAL RANK SOLUTIONS OF THE MATRIX EQUATIONS $AX = B, XC = D$

XIAO Qing-feng

(*Department of Basic, Dongguan Polytechnic, Dongguan 523808, China*)

Abstract: In this paper, we considered the rank range of the solutions of a class of matrix equations. By applying the singular value decomposition of matrix and the properties of Frobenius matrix norm, we obtained the extremal rank and the solution expression of under rank constrained. Some special cases of theses problems are considered, and some results are obtained.

Keywords: optimal control; extremal rank; SVD decomposition; Frobenius matrix norm

2010 MR Subject Classification: 65F05; 65F18

Document code: A **Article ID:** 0255-7797(2016)03-0458-07

1 Introduction

We first introduce some notations to be used. Let $C^{n\times m}$ denote the set of all $n \times m$ complex matrices; $R^{n\times m}$ denote the set of all $n \times m$ real matrices; $OR^{n\times n}$ be the sets of all $n \times n$ orthogonal matrices. The symbols A^T, A^+, A^-, $R(A)$, $N(A)$ and $r(A)$ stand for the transpose, Moore-Penrose generalized inverse, any generalized inverse, range(column space), null space and rank of $A \in R^{n\times m}$, respectively. The symbols E_A and F_A stand for the two projectors $E_A = I - AA^-$ and $F_A = I - A^-A$ induced by A. The matrices I and 0, respectively, denote the identity and zero matrices of sizes implied by context. We use $\langle A, B\rangle = \mathrm{trace}(B^TA)$ to define the inner product of matrices A and B in $R^{n\times m}$. Then $R^{n\times m}$ is a Hilbert inner product space. The norm of a matrix generated by the inner product is the Frobenius norm $\|\cdot\|$, that is $\|A\| = \sqrt{\langle A, A\rangle} = (\mathrm{trace}(A^TA))^{\frac{1}{2}}$.

Researches on extreme ranks of solutions to linear matrix equations was actively ongoing for more than 30 years. For instance, Mitra [1] considered solutions with fixed ranks for the matrix equations $AX = B$ and $AXB = C$; Mitra [2] gave common solutions of minimal rank of the pair of matrix equations $AX = C, XB = D$; Uhlig [3] gave the maximal and minimal ranks of solutions of the equation $AX = B$; Mitra [4] examined common solutions of minimal rank of the pair of matrix equations $A_1X_1B_1 = C_1$ and $A_2X_2B_2 = C_2$. By applying the matrix rank method, recently, Tian [5] obtained the minimal rank of solutions to the matrix equation $A = BX + YC$. In 2003, Tian in [6, 7] investigated the extremal ranks solutions to the complex matrix equation $AXB = C$ and gave some applications. In 2006, Lin and Wang

* **Received date:** 2013-10-08 **Accepted date:** 2013-12-10

Foundation item: Supported by Scientific Research Fund of Dongguan Polytechnic (2012a03).

Biography: Xiao Qingfeng (1977–), male, born at Loudi, Hunan, associate professor, doctor, major in matrix theory and applications.

in [8] studied the extreme ranks of solutions to the system of matrix equations $A_1X = C_1$, $XB_2 = C_2$, $A_3XB_3 = C_3$ over an arbitrary division ring, which was investigated in [9] and [10]. Recently, Xiao et al. considered the extremal ranks, i.e. maximal and minimal ranks to the equation $AX = B$ (see, e.g. [11–15]).

In this paper, we consider the extremal rank solutions of the matrix equations

$$AX = B, XC = D, \tag{1.1}$$

where $A \in R^{p\times m}, B \in R^{p\times n}, C \in R^{n\times q}, D \in R^{m\times q}$ are given matrices.

The paper is organized as follows. At first, we will introduce several lemmas which will be used in the latter sections. In Section 3, applying the matrix rank method, we will discuss the rank of the general solution to the matrix equations $AX = B, XC = D$, where $A \in R^{p\times m}, B \in R^{p\times n}, C \in R^{n\times q}, D \in R^{m\times q}$ are given matrices.

2 Some Lemmas

Lemma 2.1 (see [6]) Let A, B, C, and D be $m \times n$, $m \times k$, $l \times n$, $l \times k$ matrices, respectively. Then

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C(I - A^{-}A)), \tag{2.1}$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} + r\,[A \;\; B] - r(A) + r[E_G(D - CA^{-}B)F_H], \tag{2.2}$$

where $G = CF_A$ and $H = E_AB$.

Lemma 2.2 (see [16]) Given $A \in R^{p\times m}, B \in R^{p\times n}, C \in R^{n\times q}, D \in R^{m\times q}$. Let the singular value decompositions of A be,

$$A = U\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}V^T = U_1\Sigma V_1^T, \tag{2.3}$$

where $U = (U_1, U_2) \in OR^{p\times p}$, $U_1 \in R^{p\times k}$, $V = (V_1, V_2) \in OR^{m\times m}$, $V_1 \in R^{m\times k}$, $k = r(A)$, $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \cdots \sigma_k)$, $\sigma_1 \geq \cdots \geq \sigma_k > 0$. Let the singular value decompositions of B be,

$$C = P\begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix}Q^T = P_1\Gamma Q_1^T, \tag{2.4}$$

where $P = (P_1, P_2) \in OR^{n\times n}$, $P_1 \in R^{n\times t}$, $Q = (Q_1, Q_2) \in OR^{q\times q}$, $Q_1 \in R^{q\times t}$, $t = r(C)$, $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \cdots \gamma_t)$, $\gamma_1 \geq \cdots \geq \gamma_t > 0$. Then the matrix equations (1.1) have a solution in $R^{m\times n}$ if and only if

$$BC = AD, \quad AA^{+}B = B, \quad DC^{+}C = C. \tag{2.5}$$

Moreover, its general solution can be expressed as

$$X = DC^{+} + A^{+}B - A^{+}ADC^{+} + (I - A^{+}A)Z(I - CC^{+}), \forall Z \in R^{m\times n}. \tag{2.6}$$

Lemma 2.3 Suppose that matrix equations (1.1) is consistent. Let the singular value decompositions of A and C given by (2.3) and (2.4), respectively. Denote by X the solution of matrix equations (1.1). Then matrix V^TXP can be partitioned into

$$V^TXP = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \tag{2.7}$$

where

$$X_{11} = V_1^TXP_1 = \Sigma^{-1}U_1^TBP_1 \in R^{k\times t}, X_{12} = V_1^TXP_2 = \Sigma^{-1}U_1^TBP_2 \in R^{k\times(n-t)},$$
$$X_{21} = V_2^TXP_1 = V_2^TDQ_1\Gamma^{-1} \in R^{(m-k)\times t}, X_{22} \in R^{(m-k)\times(n-t)}$$

is arbitrary.

Proof By (2.6), Z is arbitrary, we claim from (2.3), (2.4) and (2.7) that X_{22} is arbitrary too. We omit the proof.

3 The Extremal Rank Solutions to (1.1)

Assume the matrix equations (1.1) has a solution $X \in R^{m\times n}$, and the general solution can be written as

$$X = V \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} P^T, \quad \forall X_{22} \in R^{(m-k)\times(n-t)}, \tag{3.1}$$

where

$$X_{11} = \Sigma^{-1}U_1^TBP_1 = V_1^TXP_1, X_{12} = \Sigma^{-1}U_1^TBP_2 = V_1^TXP_2,$$
$$X_{21} = V_2^TDQ_1\Gamma^{-1} = V_2^TXP_1.$$

Let $G_1 = X_{21}F_{X_{11}}$, $H_1 = E_{X_{11}}X_{12}$. Assume the singular value decomposition of G_1 and H_1^+ be, respectively,

$$G_1 = U_{G_1}\begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} V_{G_1}^T = U_{11}\Sigma_1V_{11}^T, \tag{3.2}$$

where $U_{G_1} = (U_{11}, U_{12}) \in OR^{(m-k)\times(m-k)}$, $U_{11} \in R^{(m-k)\times k_1}$, $V_{G_1} = (V_{11}, V_{12}) \in OR^{k\times k}$, $V_{11} \in R^{k\times k_1}$, $k_1 = r(G_1)$, $\Sigma_1 = \mathrm{diag}(\alpha_{11}, \alpha_{21}, \cdots \alpha_{k_11})$, $\alpha_{11} \geq \cdots \geq \alpha_{k_11} > 0$.

$$H_1^+ = P_{H_1^+}\begin{pmatrix} \Gamma_1 & 0 \\ 0 & 0 \end{pmatrix} Q_{H_1^+}^T = P_{11}\Gamma_1Q_{11}^T, \tag{3.3}$$

where $P_{H_1^+} = (P_{11}, P_{12}) \in OR^{(n-t)\times(n-t)}$, $P_{11} \in R^{(n-t)\times t_1}$, $Q_{H_1^+} = (Q_{11}, Q_{12}) \in OR^{k\times k}$, $Q_{11} \in R^{k\times t_1}$, $t_1 = r(H_1^+)$, $\Gamma_1 = \mathrm{diag}(\beta_{11}, \beta_{21}, \cdots \beta_{t_11})$, $\beta_{11} \geq \cdots \geq \beta_{t_11} > 0$.

Now we can establish the existence theorems as follows.

Theorem 3.1 Given $A \in R^{p\times m}, B \in R^{p\times n}, C \in R^{n\times q}, D \in R^{m\times q}$. the singular value decompositions of the matrices A, C and G_1, H_1^+ are given by (2.3), (2.4) and (3.2), (3.3), respectively. Then equations (1.1) has a solution X if and only if

$$BC = AD, \quad AA^+B = B, \quad DC^+C = C. \tag{3.4}$$

In this case, let Ω be the set of all solutions of equations (1.1), then the extreme ranks of X are as follows:

(1) The minimal rank of X is

$$\min_{X\in\Omega} r(X) = r(B) + r(D) - r(BC). \tag{3.5}$$

The general expression of A satisfying (3.5) is

$$X = X_0 + V_2U_{11}U_{11}^T\tilde{Y}P_{11}P_{11}^TP_2^T, \tag{3.6}$$

where $X_0 = DC^+ + A^+B - A^+ADC^+ + (I - AA^+)DC^+(A^+BCC^+)^+A^+B(I - CC^+)$, and $\tilde{Y} \in R^{(m-k)\times(n-t)}$ is arbitrary matrix.

(2) The maximal rank of X is

$$\max_{X\in\Omega} r(X) = \min(m + r(B) - r(A), n + r(D) - r(C)). \tag{3.7}$$

The general expression of X satisfying (3.7) is

$$X = X_0 + V_2YP_2^T, \tag{3.8}$$

where

$$X_0 = DC^+ + A^+B - A^+ADC^+ + (I - AA^+)DC^+(A^+BCC^+)^+A^+B(I - CC^+),$$

and the arbitrary matrix $Y \in R^{(m-k)\times(n-t)}$ satisfies

$$r(E_{G_1}YF_{H_1}) = r(BC) + \min(m - r(A) - r(D), n - r(B) - r(C)).$$

Proof Suppose the matrix equation (1.1) has a solution X, then it follows from Lemma 2.2 that (3.4) hold. In this case, let Ω be the set of all solutions of equations (1.1). By (3.1),

$$r(X) = r\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}. \tag{3.9}$$

By Lemma 2.1, we have

$$r(X) = r\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix} + r\begin{bmatrix} X_{11} & X_{12} \end{bmatrix} - r(X_{11}) + r[E_{G_1}(X_{22} - X_{21}X_{11}^+X_{12})F_{H_1}], \tag{3.10}$$

where $G_1 = X_{21}F_{X_{11}}$, $H_1 = E_{X_{11}}X_{12}$.

$$\begin{aligned} r\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix} &= r\begin{bmatrix} V_1^TXP_1 \\ V_2^TXP_1 \end{bmatrix} = r\begin{bmatrix} V_1^TDQ_1\Gamma^{-1} \\ V_2^TDQ_1\Gamma^{-1} \end{bmatrix} = r(V^TDQ_1\Gamma^{-1}) \\ &= r(DQ_1) = r(DQ_1Q_1^T) = r(DC^+C) = r(D), \\ r\,[X_{11} \quad X_{12}] &= r\left[\Sigma^{-1}U_1^TBP_1 \quad \Sigma^{-1}U_1^TBP_2\right] = r(\Sigma^{-1}U_1^TB(P_1,P_2)) \\ &= r(\Sigma^{-1}U_1^TB) = r(U_1^TB) = r(U_1U_1^TB) = r(AA^+B) = r(B), \\ r(X_{11}) &= r(\Sigma^{-1}U_1^TBP_1) = r(U_1^TBP_1\Gamma) = r(U_1U_1^TBP_1\Gamma Q_1^T) = r(AA^+BC) = r(BC). \end{aligned}$$

(1) By (3.10),

$$\begin{aligned} \min_{X\in\Omega} r(X) &= r(B)+r(D)-r(BC)+\min_{X_{22}} r[E_{G_1}(X_{22}-X_{21}X_{11}^+X_{12})F_{H_1}] \\ &= r(B)+r(D)-r(BC). \end{aligned}$$

Then (3.4) hold. By Lemma 2.2, The general expression of X satisfying (3.5) can be expressed as

$$X = DC^+ + A^+B - A^+ADC^+ + V_2X_{21}X_{11}^+X_{12}P_2^T + V_2YP_2^T, \tag{3.11}$$

where $Y \in R^{(m-k)\times(n-t)}$ satisfies $E_{G_1}YF_{H_1} = 0$.

By (3.1),

$$X_{11} = \Sigma^{-1}U_1^TBP_1 \in R^{k\times t}, X_{12} = \Sigma^{-1}U_1^TBP_2 \in R^{k\times(n-t)}, X_{21} = V_2^TDQ_1\Gamma^{-1} \in R^{(m-k)\times t}.$$

By (2.3), (2.4), $A^+ = V_1\Sigma^{-1}U_1^T$, $CC^+ = P_1P_1^T$. Then

$$V_1X_{11}P_1^T = A^+BCC^+, P_1X_{11}^+V_1^T = (A^+BCC^+)^+, X_{11}^+ = P_1^T(A^+BCC^+)^+V_1.$$

Thus we obtain

$$\begin{aligned} V_2X_{21}X_{11}^+X_{12}P_2^T &= V_2V_2^TDQ_1\Gamma^{-1}P_1^T(A^+BCC^+)^+V_1\Sigma^{-1}U_1^TBP_2P_2^T \\ &= (I-AA^+)DC^+(A^+BCC^+)^+A^+B(I-CC^+). \end{aligned} \tag{3.12}$$

By (3.2), (3.3),

$$G_1G_1^+ = U_{11}U_{11}^T, \quad E_{G_1} = I - U_{11}U_{11}^T = U_{12}U_{12}^T,$$
$$H_1^+H_1 = P_{11}P_{11}^T, \quad F_{H_1} = I - P_{11}P_{11}^T = P_{12}P_{12}^T.$$

Thus $E_{G_1}YF_{H_1} = 0$, i,e. $U_{12}U_{12}^TYP_{12}P_{12}^T = 0$, we have

$$Y = U_{11}U_{11}^T\tilde{Y}P_{11}P_{11}^T, \tag{3.13}$$

where $\tilde{Y} \in R^{(m-k)\times(n-t)}$ is arbitrary.

Taking (3.12), (3.13) into (3.11) yields (3.6).

(2) By (3.10),

$$\begin{aligned}\max_{X\in\Omega} r(X) &= r(B) + r(D) - r(BC) + \max_{X_{22}} r[E_{G_1}(X_{22} - X_{21}X_{11}^{+}X_{12})F_{H_1}] \\ &= r(B) + r(D) - r(BC) + \min(r(E_{G_1}), r(F_{H_1})).\end{aligned}$$

Since E_{G_1} and F_{H_1} are idempotent matrices, we have

$$\begin{aligned} r(E_{G_1}) &= \mathrm{trace}(E_{G_1}) = m - k - r(G_1G_1^{+}) = m - k - r(G_1) \\ &= m - k - r(X_{21}(I - X_{11}^{+}X_{11})) = m - k - r\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix} + r(X_{11}) \\ &= m - k - r(D) + r(BC) = m + r(BC) - r(A) - r(D), \\ r(F_{H_1}) &= \mathrm{trace}(F_{H_1}) = n - t - r(H_1^{+}H_1) = n - t - r(H_1) \\ &= n - t - r((I - X_{11}X_{11}^{+})X_{12}) = n - t - r\,[X_{11} \quad X_{12}] + r(X_{11}) \\ &= n - t - r(B) + r(BC) = n + r(BC) - r(B) - r(C). \end{aligned}$$

Then the maximal rank of the matrix equations (1.1) is

$$\begin{aligned}\max_{X\in\Omega} r(X) &= \min(m + r(BC) - r(A) - r(D), n + r(BC) - r(B) - r(C)) \\ &\quad + r(B) + r(D) - r(BC) = \min(m + r(B) - r(A), n + r(D) - r(C)).\end{aligned}$$

By Lemma 2.2, The general expression of X satisfying (3.7) can be expressed as

$$X = X_0 + V_2YP_2^T,$$

where $X_0 = DC^{+} + A^{+}B - A^{+}ADC^{+} + (I - AA^{+})DC^{+}(A^{+}BCC^{+})^{+}A^{+}B(I - CC^{+})$, and the arbitrary matrix $Y \in R^{(m-k)\times(n-t)}$ satisfies

$$r(E_{G_1}YF_{H_1}) = r(BC) + \min(m - r(A) - r(D), n - r(B) - r(C)).$$

The proof is completed.

The result in (3.5) implies Theorem 3 in (see [2]) as a corollary.

Corollary Assume $r(B) \le r(D)$, and matrix equations (1.1) is consistent. Then the matrix equations (1.1) have solution with rank of $r(D)$ if and only if $r(BC) = r(B)$.

References

[1] Mitra S. Fixed rank solutions of linear matrix equations[J]. Sankhya Ser. A., 1972, 35: 387–392.

[2] Mitra S. The matrix equation $AX = C, XB = D$[J]. Linear Algebra Appl., 1984, 59: 171–181.

[3] Uhlig F. On the matrix equation $AX = B$ with applications to the generators of controllability matrix[J]. Linear Algebra Appl., 1987, 85: 203–209.

[4] Mitra S. A pair of simultaneous linear matrix equations $A_1X_1B_1 = C_1, A_2X_2B_2 = C_2$ and a matrix programming problem[J]. Linear Algebra Appl., 1990, 131: 107–123.

[5] Tian Yongge. The minimal rank of the matrix expression $A - BX - YC$[J]. Missouri J. Math. Sci., 2002, 14(1): 40–48.

[6] Tian Yongge. Ranks of solutions of the matrix equation $AXB = C$[J]. Linear Multilinear Algebra., 2003, 51(2): 111–125.

[7] Tian Yongge. More on maximal and minimal ranks of Schur complements with applications[J]. Appl. Math. Comput., 2004, 152(3): 675–692.

[8] Lin Chunyan, Wang Qingwen. The minimal and maximal ranks of the general solution to a system of matrix equations over an arbitrary division ring[J]. Math. Sci. Res. J., 2006, 10(3): 57–65.

[9] Bhimasankaram P. Common solutions to the linear matrix equations $AX = B$, $XC = D$, and $EXF = G$[J]. Sankhya Ser. A. 1976, 38: 404–409.

[10] Lin Chunyan, Wang Qingwen. New solvable conditions and a new expression of the general solution to a system of linear matrix equations over an arbitrary division ring[J]. Southeast Asian Bull. Math. 2005, 29(5): 755–762.

[11] Xiao Qingfeng, Hu Xiyan, Zhang Lei. The rank-constrained anti-symmetric solution of the matrix equation $AX = B$ and the optimal approximation[J]. J. Math., 2013, 33(5): 803–810.

[12] Xiao Qingfeng, Hu Xiyan, Zhang Lei. The anti-reflexive extremal rank solutions of the matrix equation $AX = B$[J]. Funct. Anal., Appro. Comput., 2012, 4(2): 15–22.

[13] Xiao Qingfeng. The symmetric generalized centro-symmetric extremal rank solutions of the matrix equation $AX = B$[J]. Ann. Univ. Oradea Fasc. Math., 2013, 2: 41–50.

[14] Xiao Qingfeng. The anti-centro-symmetric extremal rank solutions to the matrix equation $AX = B$[J]. Jordanian J. Math. Stat., 2013, 6(3): 197–210.

[15] Xiao Qingfeng. The (P, Q) generalized anti-reflexive extremal rank solutions to a system of matrix equations[J]. Miskolc Math. Notes. 2013, 14(1): 331–340.

[16] Rao C R, Mitra S K. Generalized inverse of matrices and its application[M]. New York: Wiley, 1971.

矩阵方程$AX = B, XC = D$的定秩解

肖庆丰

(东莞职业技术学院基础课部, 广东 东莞 523808)

摘要: 本文研究了一类矩阵方程组解的秩的范围. 利用矩阵的奇异值分解以及Frobenius范数的特征, 得到了解的极值秩以及解的通式, 并就这些问题的特殊情况进行了讨论, 得到了一些结果.

关键词: 最优控制; 极值秩; 奇异值分解; Frobenius范数

MR(2010)**主题分类号:** 65F05; 65F18 **中图分类号:** O241.6