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Decompositions of Complete Graph into $(2k-1)$-Circles with One Chord

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Abstract: In this paper, we give a unified method to construct G-designs and solve the existence of $C_{2k-1}^{(r)}$-$GD(v)$ for $v \equiv 1 \ (mod\ 4k)$, where the graph $C_{2k-1}^{(r)}$, $1 \le r \le k-2$, denotes a circle of length $2k-1$ with one chord and r is the number of vertices between the ends of the chord.

Key words: graph design; holey graph design; difference.
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1. Introduction

Let K_v be the *complete graph* with v vertices and G be a finite simple graph. A G-*design* of K_v, denoted by G-$GD(v)$, is a pair $(X, \mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G and any two distinct vertices in K_v are jointed in exactly one block of $\mathcal{B}$.

Let $DK_{n_1,n_2,\cdots,n_h}$ be the *complete partitegraph* with vertex set $X = \bigcup_{i=1}^{h} X_i$, where X_i, $1 \le i \le h$, are disjoint sets with $|X_i| = n_i$ and where two vertices x and y from different sets X_i and X_j are jointed by exactly one edge $\{x, y\}$. A *holey* G-*design*, briefly denoted by G-$HD(T)$, is a triple $(X, \{X_i; 1 \le i \le h\}, \mathcal{A})$ with $X = \bigcup_{i=1}^{h} X_i$, where $T = n_1^1 n_2^1 \cdots n_h^1$ is the type of the holey G-design, $\mathcal{A}$ is a collection of edge-disjoint subgraphs of $DK_{n_1,n_2,\cdots,n_h}$, called *blocks*, such that each block is isomorphic to G and each edge of $DK_{n_1,n_2,\cdots,n_h}$ is jointed in exactly one block of $\mathcal{A}$. Usually, the type is denoted by exponential form, for example, the type $1^i 2^r 3^k \cdots$ denotes that 1 occurs i times, 2 occurs r times, etc..

In this paper, the discussed graphs are $C_{2k-1}^{(r)}$, i.e., one circle of length $2k-1$ with one chord, where r, $1 \le r \le k-2$, is the number of vertices between the ends of the chord. For given graph $C_m^{(r)}$, it is easy to see that the graph $C_m^{(r)}$ is the same graph as $C_m^{(m-2-r)}$. So, if $r > \lfloor \frac{m-2}{2} \rfloor$, we often use $C_m^{(m-2-r)}$ to express the graph. Obviously, there is no subgraph of K_v which is isomorphic to $C_{2k-1}^{(r)}$ when $v < 2k-1$. Therefore, we only consider the complete graphs with at least $2k-1$ vertices. It is easy to see that the following lemmas hold.

Lemma 1.1[1] *The necessary conditions to exist a G-$GD(v)$ are $v(v-1) \equiv 0 \ (mod\ 2e(G))$, $v \ge v(G)$, where $e(G)$ and $v(G)$ are the number of the edges and the vertices of G, respectively.*

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Lemma 1.2 *The necessary conditions to exist a $C_m^{(r)}$-$GD(v)$ are $v(v-1)\equiv 0 \pmod{m+1}$ and $v\geq m$.*

For $r=\lfloor\frac{k-2}{2}\rfloor$, the existence of $C_{2k-1}^{(r)}$-$GD(v)$, which called theta graphs, has been discussed in [2] and [3]. And the existence of $C_m^{(r)}$-$GD(v)$ for $4\leq m\leq 8$ has been discussed in [4–7], which can be summarized as follows:

Lemma 1.3[4–7] *For $4\leq m\leq 8$, the necessary conditions to exist a $C_m^{(r)}$-$GD(v)$ are also sufficient except $(v,m,r)=(5,4,1)$ and $(9,8,3)$.*

2. General Structures and overall arrangement

In this section, we will give some unified methods to construct G-designs. The definition of BIBD and GDD can be found in [1].

Lemma 2.1 *For given graph G and positive integers h,m, if there exist both a G-$HD(h^m)$ and a G-$GD(h+w)$, then there exists a G-$GD(hm+w)$, where $w=0$ or 1.*

Proof Let $(X,\mathcal{B})$ be a G-$HD(h^m)$, where $X=\bigcup_{i=1}^{m}X_i$ with $|X_i|=h$. Suppose W be a w-set and $X\bigcap W=\emptyset$. For $1\leq i\leq m$, $(X_i\bigcup W,\mathcal{B}_i)$ is the known G-$GD(h+w)$. Letting $\mathcal{A}=\mathcal{B}\bigcup(\bigcup_{i=1}^{m}\mathcal{B}_i)$, then $(X\bigcup W,\mathcal{A})$ is a G-$GD(hm+w)$. □

Lemma 2.2 *For given graph G and $w=0$ or 1, if there exists a $B[s,1;t]$, a G-$HD(h^s)$ and a G-$GD(h+w)$, then there exists a G-$GD(ht+w)$.*

Proof Let X,H and W be t-set, h-set and w-set respectively, $Y=X\times H$ and $Y\bigcap W=\emptyset$. Denote the known designs by

$$\begin{aligned}
&B[s,1;t]=(X,\mathcal{B});\\
&G\text{-}HD(h^s)=(B\times H,\{\{b\}\times H:\ b\in B\},\ \mathcal{A}_B),\quad \forall B\in\mathcal{B};\\
&G\text{-}GD(h+w)=((\{x\}\times H)\bigcup W,\ \mathcal{C}_x),\quad \forall x\in X.
\end{aligned}$$

Define $\mathcal{A}=\{\mathcal{A}_B:\ B\in\mathcal{B}\}\bigcup\{\mathcal{C}_x:\ x\in X\}$, then $(Y\bigcup W,\mathcal{A})$ is a G-$GD(ht+w)$. □

Lemma 2.3 *For given graph G and $w=0$ or 1, if there exists a $B[s,1;t+1]$, a G-$HD(h^s)$ and a G-$GD((s-1)h+w)$, then there exists a G-$GD(ht+w)$.*

Proof Let X,H and W be $(t+1)$-set, h-set and w-set respectively, $Y=X\times H$ and $Y\bigcap W=\emptyset$. Denote the known designs by

$$\begin{aligned}
&B[s,1;t+1]=(X\bigcup\{\infty\},\mathcal{B}_0\bigcup\mathcal{B}_1);\\
&G\text{-}HD(h^s)=(B\times H,\{\{b\}\times H:\ b\in B\},\ \mathcal{A}_B),\quad \forall B\in\mathcal{B}_1;\\
&G\text{-}GD((s-1)h+w)=((B\setminus\{\infty\}\times H)\bigcup W,\mathcal{C}_B),\quad \forall B\in\mathcal{B}_0,
\end{aligned}$$

where $\mathcal{B}_0$ is the blocks containing ∞ and $\mathcal{B}_1$ is the other blocks. Note that $|W|=0$ or 1. Define

$$\mathcal{D}=\{\mathcal{A}_B:\ B\in\mathcal{B}_1\}\bigcup\{\mathcal{C}_B:\ B\in\mathcal{B}_0\},$$

then $((X\times H)\bigcup W,\mathcal{D})$ is a G-$GD(ht+w)$. □

Lemma 2.4 *For given graph G, positive integer i and $w = 0$ or 1, if there exists a $B_i[s,1;t-i]$, a G-$HD(h^s)$, a G-$HD(h^{s+1})$, a G-$GD(h+w)$ and a G-$GD(ih+w)$, then there exists a G-$GD(ht+w)$.*

Proof Let $(X,\mathcal{B})$ be a $B_i[s,1;t-i]$ with i parallel classes $\mathcal{P}_1,\mathcal{P}_2,\cdots,\mathcal{P}_i$. Suppose $a_1,\cdots,a_i$ be distinct points that not belong to X. Adding the point a_j to each block B in $\mathcal{P}_j$, $1\le j\le i$, we get a $\{s,s+1\}$-$PBD(t)=(X\bigcup\{a_1,\cdots,a_i\},\mathcal{D})$. Assign a weight h to each point $x\in X$ and denote the obtained h-set by Y_x. Similarly, assign a weight h to each point in $\{a_1,\cdots,a_i\}$ and denote the obtained (hi)-subset by $Y^{'}$. Define $Y=Y^{'}\bigcup(\bigcup\limits_{x\in X}Y_x)$, which contains ht elements.

For any block $B\in\mathcal{D}$ with the weight type h^{s+1} (or h^s), there exists an ingredient G-$HD(h^{s+1})$ (or G-$HD(h^s)$) with block set $\mathcal{A}_B$. Suppose W be a w-set and $W\bigcap Y=\emptyset$. For every point $x\in X$, there exists an ingredient G-$GD(h+w)=(Y_x\bigcup W,\mathcal{A}_x)$. Similarly, for the set $\{a_1,\cdots,a_i\}$, there exists an ingredient G-$GD(ih+w)=(Y^{'}\bigcup W,\mathcal{A}^{'})$. Let

$$\mathcal{A}=\mathcal{A}^{'}\bigcup\{\mathcal{A}_B:\ B\in\mathcal{D}\}\bigcup\{\mathcal{A}_x:\ x\in X\}.$$

Then $(Y\bigcup W,\mathcal{A})$ is a G-$GD(ht+w)$. $\square$

Now, we will give some results of the holey designs.

Lemma 2.5[7] *For integers k, t and r, $t\ge 1$, $k\ge 3$ and $1\le r\le k-2$, there exists a $C_{2k-1}^{(r)}$-$HD((2k)^{2t+1})$ and a $C_{2k-1}^{(r)}$-$HD((4k)^{2t+1})$.*

Lemma 2.6 *There exists a $C_{2k-1}^{(r)}$-$HD((4k)^u)$ for integer $u\equiv 0,1 \pmod 3$ and $u\ge 3$, where $k\ge 3$ and $1\le r\le k-2$.*

Proof By [1], there exists a $\{3\}$-$GDD(2^u)$ for $u\equiv 0,1$ $(mod\ 3)$, $u\ge 3$. Suppose $(X,\mathcal{G},\mathcal{B})$ be a $\{3\}$-$GDD(2^u)$, where $X=\bigcup_{i=1}^u X_i$ and $\mathcal{G}=\{X_i:\ 1\le i\le u\}$, $|X_i|=2$. Assign a weight $2k$ to each point $x\in X_i$, $1\le i\le u$ and denote the obtained $4k$-set by Y_i. Let $Y=\bigcup_{i=1}^u Y_i$, which contains $4ku$ elements. For each weighted block $B\in\mathcal{B}$, there exists an ingredient $C_{2k-1}^{(r)}$-$HD((2k)^3)$ with block set $\mathcal{A}_B$ by Lemma 2.5. Define

$$\mathcal{A}=\{\mathcal{A}_B:\ B\in\mathcal{B}\} \text{ and } \mathcal{G}'=\{Y_i:\ 1\le i\le u\}.$$

Then $(Y,\mathcal{G}',\mathcal{A})$ is a $C_{2k-1}^{(r)}$-$HD((4k)^u)$. $\square$

Lemma 2.7[1] *(1) There exists a $B[3,1;v]$ if and only if $v\equiv 1,3 \pmod 6$ and $v\ge 3$.*

(2) For $v\equiv 3 \pmod 6$, there exist $B_i[3,1;v]$ with i parallel classes, where $1\le i\le \frac{v-1}{2}$.

Lemma 2.8 *For given graph G and $w=0$ or 1, if there exists a G-$HD(h^3)$, a G-$HD(h^4)$, a G-$GD(ih+w)$ with $i=1,2,5$, then there exists a G-$GD(ht+w)$ for any $t\ge 1$.*

Proof We consider the existence of G-$GD(ht+w)$ from the following cases.

(1) For $t\equiv 1,3 \pmod 6$, there exists a $B[3,1;t]$ by Lemma 2.7. Thus, there exists a G-$GD(ht+w)$ by the known G-$HD(h^3)$, G-$GD(h+w)$ and Lemma 2.2.

(2) For $t\equiv 0,2 \pmod 6$, there exists a $B[3,1;t+1]$ by Lemma 2.7. Thus, there exits a G-$GD(ht+w)$ by the known G-$HD(h^3)$, G-$GD(2h+w)$ and Lemma 2.3.

(3) For $t\equiv 3+i \pmod 6$, $i=1,2$, there exists a $B_i[3,1;t-i]$ by Lemma 2.7. So, letting $t-i=6u+3$, the $RB[3,1;t-i]$ is just a $B_{3u+1}[3,1;t-i]$. By Lemma 2.4, there exits a G-$GD(ht+w)$ if $3u+1\ge 1$ (for $i=1$) or $3u+1\ge 2$ (for $i=2$) except for the case $(i,u)=(2,0)$, i.e., $t=3+2=5$. But, G-$GD(5h+w)$ is known. $\square$

Theorem 2.9 *For $w=0$ and 1, $k\geq 5$, if there exists a $C^{(r)}_{2k-1}$-$GD(4k+w)$ and a $C^{(r)}_{2k-1}$-$GD(8k+w)$, then there exists a $C^{(r)}_{2k-1}$-$GD(v)$ for $v\equiv 0,1 \pmod{4k}$.*

Proof By Lemmas 2.5 and 2.6, there exist $C^{(r)}_{2k-1}$-$HD((4k)^u)$ for $u=3,4,5$ and $1\leq r\leq k-2$. So, there exist $C^{(r)}_{2k-1}$-$GD(5\cdot 4k+w)$ for $w=0$ or 1 by Lemma 2.1. Then, there exists a $C^{(r)}_{2k-1}$-$GD(v)$ by Lemma 2.8. □

3. The construction of $C^{(r)}_{2k-1}$-$GD(v)$

In this section, we will give a unified method to construct $C^{(r)}_{2k-1}$-$GD(v)$ for $k\geq 5$. In the construction of $C^{(r)}_{2k-1}$-$GD(v)$ on the set X, we will give the base blocks D. Denote the blocks in $\mathrm{dev}(D)$ as $(a_0,a_1,\cdots,a_{2k-2})$. Then, the chord is denoted by $(r,d)=\{a_i,a_{i+r+1}\}$, where a_i and a_{i+r+1} are the ends of the chord and the difference $d=|a_{i+r+1}-a_i|$. For integer $i\leq j$, $A[i,j]$ denotes $(i,-(i+1),\cdots,(-1)^{j-i}j)$ and $A[i,j]^{-1}$ denotes $(j,-(j-1),\cdots,(-1)^{j-i}i)$.

Lemma 3.1 *There exists a $C^{(r)}_{2k-1}$-$GD(4k+1)$ for $1\leq r\leq k-2$.*

Construction Let $X=Z_{4k+1}$. Considering the number of the block set, we only need to construct one base block.

Case 1 (k even) Let $D=(A([1,2k]\setminus\{d,k\}),k)$. Choose the chord in the blocks as

$$(r,d)=\begin{cases}(4i+1,2i+2)=\{a_0,a_{4i+2}\}, & 0\leq i\leq \frac{k-4}{2}\\ (4i+2,2i+2)=\{a_0,a_{4i+3}\}, & 0\leq i\leq \lfloor\frac{k-6}{4}\rfloor\\ (4i+3,2i+2)=\{a_{2i+2},a_{6i+6}\}, & 0\leq i\leq \lfloor\frac{k-4}{4}\rfloor\end{cases}$$

Case 2 (k odd)

$$\text{Let } D=\begin{cases}(A([1,2k]\setminus\{2i+1,k-1\}),k-1), & 0\leq i\leq \frac{k-3}{2}\\ (A([2,2k-2]\setminus\{2i+2,k\}),2k-1,1,-k,2k), & 0\leq i\leq \frac{k-5}{2}\end{cases}$$

Choose the chord in the blocks as $(r,d)=\begin{cases}(4i+1,2i+1)=\{a_0,a_{4i+2}\}, & 0\leq i\leq \frac{k-3}{2}\\ (4i+3,2i+2)=\{a_0,a_{4i+4}\}, & 0\leq i\leq \frac{k-5}{2}\end{cases}$.

Proof Obviously, each difference in Z_{4k+1} appears exactly once in D or as the chord difference. In order to show that the range of r is filled full indeed, we present the following table.

D	r	range of r
Case 1	$4i+1$ $(0\leq i\leq \frac{k-4}{2})$	$[1,k-3]_4\cup[4,k-4]_4$ $(k\equiv 0 \pmod 4)$ $[1,k-5]_4\cup[4,k-2]_4$ $(k\equiv 2 \pmod 4)$
	$4i+2$ $(0\leq i\leq \lfloor\frac{k-6}{4}\rfloor)$	$[2,k-6]_4$ $(k\equiv 0 \pmod 4)$ $[2,k-4]_4$ $(k\equiv 2 \pmod 4)$
	$4i+3$ $(0\leq i\leq \lfloor\frac{k-4}{4}\rfloor)$	$[3,k-5]_4\cup\{k-2\}$ $(k\equiv 0 \pmod 4)$ $[3,k-3]_4$ $(k\equiv 2 \pmod 4)$
Case 2	$4i+1$ $(0\leq i\leq \frac{k-3}{2})$	$[1,k-4]_4\cup[2,k-3]_4$ $(k\equiv 1 \pmod 4)$ $[1,k-2]_4\cup[2,k-5]_4$ $(k\equiv 3 \pmod 4)$
	$4i+3$ $(0\leq i\leq \frac{k-5}{2})$	$[3,k-2]_4\cup[4,k-5]_4$ $(k\equiv 1 \pmod 4)$ $[3,k-4]_4\cup[4,k-3]_4$ $(k\equiv 3 \pmod 4)$

Table 1

Below, what we need to do is to verify that all vertices in $\widetilde{D}_0$ are distinct, which implies that D is a CDC.

In Case 1, the vertex set of $\widetilde{D}_0$ is $[-k,k]\setminus\{-(i+1),\frac{k}{2}\}$.

In Case 2, the vertex set of $\widetilde{D}_0$ is

$$\begin{cases} [-k,k]\setminus\{-(\frac{k-1}{2}),i+1\} & (\text{for } 0\le i\le \frac{k-3}{2}) \\ [-(k-3),0]\cup([2,k+1]\setminus\{i+2,\frac{k+3}{2}\})\cup\{-k,-(k+1),-2k\} & (\text{for } 0\le i\le \frac{k-5}{2}). \end{cases}$$

Lemma 3.2 *There exists a $C^{(r)}_{2k-1}$-$GD(8k+1)$ for $1\le r\le k-2$.*

Construction Let $X=Z_{8k+1}$. Considering the number of the block set, we only need to construct two base blocks.

Case 1 (k odd)

Subcase 1.1 ($r\equiv 1,2 \pmod 4$)

(1) Let $D_1^1=(k-3,A([3,k-4]\setminus\{d\}),A([k-2,2k-1]\setminus\{k,k+1\}),3k,-2k,-(3k+1),-1,2)$ and choose the chord as $(r,d)=(4i+2,k-4-2i)=\{a_0,a_{4i+3}\},\ 0\le i\le\frac{k-7}{2}$;

$$D_1^2=(k-3,A([4,2k-1]\setminus\{k-3,k,k+1\}),3k,-2k,-(3k+1),-2,3)$$

with the chords $(r,d)=(4i+2,1)=\{a_0,a_{4i+3}\},\ i=\frac{k-5}{2},\ \frac{k-3}{2}$.

(2) Let $D_2=(3k-1,A([2k+1,3k-2]\setminus\{d\}),A[3k+2,4k],-k,-(k+1))$ with the chords $(r,d)=(4i+2,2k+2i+1)=\{a_0,a_{4i+3}\},\ 0\le i\le\frac{k-3}{2}$.

Subcase 1.2 ($r\equiv 0,3 \pmod 4$)

(1) Let $D_1=(k+2,-1,3,-A[5,k-1],-A([k+3,2k-2]\setminus\{d\}),$
$-(2k+3),-(k+1),-(2k+1),2k+4,-k,2k+2)$

with the chords $(r,d)=\begin{cases} (r,d)=(4i+4,k+5+2i)=\{a_0,a_{4i+5}\},\ 0\le i\le\frac{k-7}{2} \\ (r,d)=(4i+4,k+5)=\{a_0,a_{4i+5}\},\ i=\frac{k-5}{2} \end{cases}$.

(2) Let $D_2=(3k,-2,4,A([2k+5,4k]\setminus\{3k-1,d\})^{-1},-2k,-(2k-1))$ with the chords $(r,d)=(4i+4,3k+3+2i)=\{a_0,a_{4i+5}\},\ 0\le i\le\frac{k-5}{2}$.

Case 2 (k even)

Subcase 2.1 ($r\equiv 2,3 \pmod 4$)

(1) Let $D_1=(k+1,2k+2,-(2k+1),A([k+2,2k-2]\setminus\{d\})^{-1},A[2,k-2]^{-1},-k,-(k-1))$ with the chords $(r,d)=(4i+2,k+2+2i)=\{a_0,a_{4i+3}\},\ 0\le i\le\frac{k}{2}-2$.

(2) Let $D_2=(3k+1,A([2k+3,4k]\setminus\{3k+1,d\})^{-1},-2k,-(2k-1))$ with the chords $(r,d)=(4i+2,3k+1+2i)=\{a_0,a_{4i+3}\},\ 0\le i\le\frac{k}{2}-2$.

Subcase 2.2 ($r\equiv 0,1 \pmod 4$)

(1) Let $D_1^1=(k-2,-4k,4k-2,A([3,2k-3]\setminus\{d,k-2,k+1,k+2\}),-2k,-1,2,2-2k,2k-1)$ with the chords $(r,d)=(4i+4,k-5-2i)=\{a_0,a_{4i+5}\},\ 0\le i\le\frac{k}{2}-4$;

$$D_1^2=(k-2,-4k,4k-2,A([4,2k-3]\setminus\{k-2,k+1,k+2\})-2k,-2,3,-(2k-2),2k-1)$$

with the chords $(r,d)=(4i+4,1)=\{a_0,a_{4i+5}\}\ i=\frac{k}{2}-3,\ \frac{k}{2}-2$.

(2) Let $D_2=(A([2k+1,4k-4]\setminus\{d,3k+2\}),-(k+1),-(k+2))$ with the chords $(r,d)=(4i+4,3k-2i-1)=\{a_0,a_{4i+5}\},\ 0\le i\le\frac{k}{2}-2$.

Proof In the Case 1, the construction requests $k \geq 7$. The construction for $k = 5$, i.e., $C_9^{(r)}$-$GD(41)$, $r = 1, 2, 3$, will be given in the following examples. Obviously, each difference in Z_{8k+1} appears exactly once in $D_1 \cup D_2$ or as one of the chord differences. The following table will show that all vertices in each number-tuple are distinct.

D		$\{\widetilde{D}\}$
case 1.1	D_1^1	$[-2, k-3] \cup ([k, \frac{3k-7}{2}] \setminus \{\frac{3k-7}{2} - i\}) \cup [\frac{3k+3}{2}, 2k] \cup \{\frac{3k-3}{2}, 3k, -(3k+1)\}$
	D_1^2	$[-1, k-3] \cup [k+1, \frac{3k-7}{2}] \cup [\frac{3k+3}{2}, 2k] \cup \{\frac{3k-3}{2}, 3k, -(3k+1), -3\}$
	D_2	$([-(3k+1), -\frac{5k+5}{2}] \setminus \{-(3k-i+1)\}) \cup [2k+1, 3k-1] \cup [-\frac{5k-3}{2}, -2k] \cup \{0, k+1\}$
case 1.2	D_1	$([0, \frac{k-3}{2}] \setminus \{\frac{k-5-2i}{2}\}) \cup ([\frac{k+5}{2}, 2k-1] \setminus \{k\}) \cup \{-(3k+6), -(2k+2), -(k+5), -(k+2), -4\}$
	D_2	$[3k, 4k-1] \cup ([-\frac{3k-3}{2}, 1-k] \setminus \{-\frac{3k-2i-5}{2}\}) \cup [3-2k, -\frac{3k+1}{2}] \cup \{0, 2k-1, 3k-2\}$
case 2.1	D_1	$[k+1, \frac{5k-4}{2}] \cup ([\frac{5k+4}{2}, 3k] \setminus \{\frac{5k+2i+4}{2}\}) \cup \{0, k-1, 3k+3\}$
	D_2	$[3k+1, 4k-1] \cup [1-2k, -\frac{3k+2}{2}] \cup ([-\frac{3k-2}{2}, -k] \setminus \{-\frac{3k-2i-2}{2}\}) \cup \{2k-1, 0\}$
case 2.2	D_1^1	$[-3, k-4] \cup [1-2k, -3k-2] \cup ([k-2, 2k-2] \setminus \{\frac{3k-2i-10}{2}, \frac{3k-6}{2}, \frac{3k-2}{2}\})$
	D_1^2	$[-2, k-4] \cup ([k, 2k-2] \setminus \{\frac{3k-6}{2}, \frac{3k-2}{2}\}) \cup \{1-2k, -4, -3k-2, k-2\}$
	D_2	$[2k+3, 3k] \cup ([-3k, -2k-2] \setminus \{-\frac{5k+2i+2}{2}, -\frac{5k-2}{2}, 3-k, k+2\})$

Table 2

In order to show that the range of r is filled full indeed, we present the following table.

D		r	range of r
case 1.1	D_1^1	$4i+2\ (0 \leq i \leq \frac{k-7}{2})$	$[2, 2k-4]_4 =$
	D_1^2	$4i+2\ (i = \frac{k-5}{2}, \frac{k-3}{2})$	$[1, k-4]_4 \cup [2, k-3]_4 \quad (k \equiv 1 \pmod 4)$
	D_2	$4i+2\ (0 \leq i \leq \frac{k-3}{2})$	$[1, k-2]_4 \cup [2, k-5]_4 \quad (k \equiv 3 \pmod 4)$
case 1.2	D_1	$4i+4\ (0 \leq i \leq \frac{k-5}{2})$	$[4, 2k-6]_4 =$
			$[3, k-2]_4 \cup [4, k-5]_4 \quad (k \equiv 1 \pmod 4)$
	D_2	$4i+3\ (0 \leq i \leq t-1)$	$[3, k-4]_4 \cup [4, k-3]_4 \quad (k \equiv 3 \pmod 4)$
case 2.1	D_1	$4i+2\ (0 \leq i \leq \frac{k-4}{2})$	$[2, 2k-6]_4 =$
			$[2, k-2]_4 \cup [3, k-5]_4 \quad (k \equiv 0 \pmod 4)$
	D_2	$4i+3\ (0 \leq i \leq t-1)$	$[2, k-4]_4 \cup [3, k-3]_4 \quad (k \equiv 2 \pmod 4)$
case 2.2	D_1^1	$4i+4\ (0 \leq i \leq \frac{k-8}{2})$	$[2, 2k-4]_4 =$
	D_1^2	$4i+4\ (i = \frac{k-6}{2}, \frac{k-4}{2})$	$[1, k-3]_4 \cup [4, k-4]_4 \quad (k \equiv 0 \pmod 4)$
	D_2	$4i+4\ (0 \leq i \leq \frac{k-4}{2})$	$[1, k-5]_4 \cup [4, k-2]_4 \quad (k \equiv 2 \pmod 4)$

Table 3

Example $C_9^{(r)}$-$GD(41)$ with $r = 1, 2, 3$.

Construction Let $X = Z_{41}$. We should construct two base blocks D_1 and D_2.

$$D_1^1 = (17, -18, 2, 3, -7, 8, -10, -4, 9);$$

$$D_1^2 = (-17, 18, 1, 2, 3, 7, 8, 9, 10);$$

$$D_2 = (14, A([11, 13] \setminus \{d\}), 15, -16, 19, -20, -5, -6).$$

Choose the chords $(r, d) = (1, 1) = \{a_0, a_2\}$ in the blocks of $dev(D_1^1)$ and $(r, d) = (1, 11) = \{a_0, a_7\}$ in the blocks of $dev(D_2)$.

Choose the chords $(r, d) = (2, 1) = \{a_0, a_3\}$ in the blocks of $dev(D_1^1)$ and $(r, d) = (2, 13) = \{a_0, a_3\}$ in the blocks of $dev(D_2)$.

Choose the chords $(r,d)=(3,4)=\{a_0,a_4\}$ in the blocks of $dev(D_1^2)$ and $(r,d)=(3,13)=\{a_0,a_4\}$ in the blocks of $dev(D_2)$.

Theorem 3.3 *For $v\equiv 1 \pmod{4k}$ and $1\le r\le k-2$, the necessary conditions to exist a $C_{2k-1}^{(r)}$-$GD(v)$ are also sufficient.*

Proof By Lemmas 3.1 and 3.2, there exists a $C_{2k-1}^{(r)}$-$GD(4k+1)$, a $C_{2k-1}^{(r)}$-$GD(8k+1)$, respectively. Then, we obtain the conclusion by Theorem 2.9. □

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完备图分拆为带一条弦的 $(2k-1)$- 长圈

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摘要: 本文给出了构造 G- 设计的一个统一方法及当 $v\equiv 1(\mathrm{mod}\ 4k)$ 时的 $C_{2k-1}^{(r)}$-$GD(v)$ 的存在性，其中 $C_{10}^{(r)}$，$1\le r\le k-2$ 表示带一条弦的 $2k-1$ 长圈， r 表示弦两个端点间的顶点个数.

关键词: 图设计；带洞图设计；差.