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Strichartz Estimates for Schrödinger Equations with Non-degenerate Coefficients**

Yu MIAO*

Abstract In the present paper, the full range Strichartz estimates for homogeneous Schrödinger equations with non-degenerate and non-smooth coefficients are proved. For inhomogeneous equation, the non-endpoint Strichartz estimates are also obtained.

Keywords Schrödinger equation, Strichartz estimates, Parametrix
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1 Introduction and Main Results

In this paper, we shall provide a proof of Strichartz estimates for Schrödinger equations with non-degenerate and non-smooth coefficients. Such kinds of equations are of their physical background, such as the Zakharov-Schulman system in water wave problems (see [30]) and Ishimori equations in ferromagnetism (see [12]).

We first introduce some assumptions. Consider the initial value problem

$$\begin{cases} i\partial_t u - \displaystyle\sum_{j,k=1}^{n} \partial_j(a_{jk}(t,x)\partial_k u) + \sum_{j=1}^{n} b_j(t,x)\partial_j u + c(t,x)u = 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

with the following hypotheses

(H1) $a_{jk}(t,x) \in C^1(\mathbb{R}, C^3(\mathbb{R}^n))$, $a_{jk} = \varepsilon_j\delta_{jk}$ outside $\mathbb{R}\times B(0,1)$ with $\varepsilon_j \in \{1,-1\}$ and $B(0,1) = \{x : |x| \le 1\}$. And the matrix $(a_{jk})_{n\times n}$ is real, symmetric and non-degenerate, that is, there exists a constant $\nu > 0$, such that

$$\forall\,(t,x,\xi) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n, \quad \frac{1}{\nu}|\xi| \le |(a_{jk})\xi| \le \nu|\xi|.$$

(H2) $b_j(t,x) \in L^\infty(\mathbb{R}, C_0^1(\mathbb{R}^n))$; $c(t,x) \in L^\infty(\mathbb{R}, C_0^1(\mathbb{R}^n))$.

Then there exists a constant c_0 such that

$$|b_j| \le c_0\kappa(|x|), \quad \forall\, j = 1,\cdots,n$$

with $\kappa(|x|) = (1+|x|^2)^{-N/2}$ for some integer $N > n$.

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*School of Mathematical Sciences, Fudan University, Shanghai 200433, China. E-mail: yumiao@fudan.edu.cn
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Let P denote the Schrödinger operator

$$P = i\partial_t + L = i\partial_t - \sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k + \sum_{j=1}^{n} b_j \partial_j + c \tag{1.2}$$

and $a(t,x,\xi)$ be the principal symbol of operator $A = -\sum\limits_{j,k=1}^{n} \partial_j a_{jk} \partial_k$, i.e.,

$$a(t,x,\xi) = \sum_{j,k=1}^{n} a_{jk}(t,x)\xi_j \xi_k.$$

Denote by H_a the Hamiltonian vector field

$$H_a = \sum_{j=1}^{n} (\partial_{\xi_j} a \partial_{x_j} - \partial_{x_j} a \partial_{\xi_j}).$$

Then the Hamilton flow (x_h, ξ_h) is defined as

$$\begin{cases} \dfrac{dx_h}{dh} = a_\xi(t, x_h, \xi_h), & x(0) = x, \\ \dfrac{d\xi_h}{dh} = -a_x(t, x_h, \xi_h), & \xi(0) = \xi, \end{cases} \tag{1.3}$$

where h is the natural parameter along the Hamilton flow. We know that the solution to system (1.3) exists in the interval $(-\delta, \delta)$ with $\delta = \delta(t,x,\xi)$.

The last hypothesis is as follows (see also [13]).

(H3) Non-trapping condition: For each $(t,x,\xi) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$, and for each $\mu > 0$, there exists h_0 with $0 < h_0 < \delta$ such that

$$|x_{h_0}(t,x,\xi)| \geq \mu.$$

Remark 1.1 In view of the flatness of $a_{jk}(t,x)$ outside the region $\mathbb{R} \times B(0,1)$, it suffices to assume $\mu = 1$ for bounded $t \in \mathbb{R}$.

The main results of this paper are as follows.

Theorem 1.1 *Let* (H1)–(H3) *be fulfilled. Then for any $T > 0$, the unique solution of* (1.1) *satisfies*

$$\|u\|_{L^q([0,T],L^r(\mathbb{R}^n))} \leq C(T, a_{jk}, \nu, n) \|u_0\|_{L^2(\mathbb{R}^n)} \tag{1.4}$$

for any admissible pair (q,r), such that

$$\frac{2}{q} = n\Big(\frac{1}{2} - \frac{1}{r}\Big), \quad 2 \leq q \leq +\infty \quad \text{for } n \geq 3,$$
$$\frac{2}{q} = 2\Big(\frac{1}{2} - \frac{1}{r}\Big), \quad 2 < q \leq +\infty \quad \text{for } n = 2.$$

Remark 1.2 (a) The regularity of $a_{jk}(t,x)$ can be relaxed to $C^1(\mathbb{R}, C^{2,\varepsilon}(\mathbb{R}^n))$, $\forall \varepsilon > 0$.
(b) The pair $(2, \frac{n-2}{2n})$ is usually called endpoint, and the other pairs are called non-endpoints.

Now we consider the following inhomogeneous equation with homogeneous initial data

$$\begin{cases} i\partial_t u - \displaystyle\sum_{j,k=1}^{n} \partial_j(a_{jk}(t,x)\partial_k u) + \sum_{j=1}^{n} b_j(t,x)\partial_j u + c(t,x)u = F(t,x), \\ u(0) = 0, \end{cases} \tag{1.5}$$

under the same hypotheses (H1)–(H3). By Duhamel's formula, the solution to (1.5) has the form

$$\Phi_F(t,x) = \int_0^t U(t,s)F(s,x)ds$$

with $U(t)$ being the solution operator to equation (1.1). We have the following theorem.

Theorem 1.2 *Let (q,r), (ρ,γ) be any non-endpoint admissible pairs and $F(t,x) \in L^{\rho'}(\mathbb{R}, L^{\gamma'}(\mathbb{R}^n))$ with $\frac{1}{\rho} + \frac{1}{\rho'} = 1$. Then for any $T > 0$,*

$$\|\Phi_F(t,x)\|_{L^q([0,T],L^r(\mathbb{R}^n))} \le C(T, a_{jk}, \nu, n)\|F(t,x)\|_{L^{\rho'}([0,T],L^{\gamma'}(\mathbb{R}^n))}. \tag{1.6}$$

The inequalities (1.4) and (1.6) are known as Strichartz estimates, which are essential tools in studying nonlinear evolution equations, such as, Schrödinger equation, wave equation and KdV equation. For free Schrödinger equation, it was first given by Strichartz [22] when $q = r = \frac{2n+4}{n}$ and then extended to non-endpoint case by Ginibre and Velo [9]. For inhomogeneous case, it was obtained by Yajima [29] and Cazenave-Weissler [3]. The endpoint estimates were recently proved by Keel and Tao [14]. When the operator A is an elliptic one with variable coefficients, Staffilani and Tataru [23] proved (1.4) for compactly supported perturbation of flat Laplacian. Similar results were also got by Burq-Gérard-Tzvetkov [1], Hassel-Tao-Wunsch [10], Robbiano-Zuily [18] and Salort [19]. When A is just non-degenerated, to the author's knowledge, it is still open.

We now would like to have a quick review of the well-posedness and local smoothing for Schrödinger operator with variable smooth coefficients. For the elliptic case, they were proved by Doi [7, 8] and Craig-Kappeler-Strauss [6] (see also the references therein). Very recently, Kenig-Ponce-Rolvung-Vega proved local solutions for non-degenerate case in [13], but no Strichartz inequality is given. In present paper, to be self-contained, we shall provide a short proof for local smoothing effect through the similar method of Doi and Kenig-Ponce-Rolvung-Vega.

Next we will give the outline of our work. In Section 2, local smoothing estimates (see Theorem 2.1) for equation (1.1) is proved. In Section 3, a constant coefficient dispersive estimate is obtained. More precisely, if $f(t,x)$ is spatially compactly supported for each t, then the solution to the equation

$$\begin{cases} i\partial_t w - \displaystyle\sum_{j=1}^{n} \varepsilon_j \partial_{jj} w = f(t,x), \\ w(0) = w_0 \end{cases} \tag{1.7}$$

satisfies the following estimate

$$\|w\|_{L^q([0,T],L^r(\mathbb{R}^n))} \lesssim \|w_0\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^2([0,T],H^{-1/2}(\mathbb{R}^n))}. \tag{1.8}$$

We give a new proof to (1.8) for non-endpoint case, which is based on smoothing effect of homogeneous Schrödinger equation with constant coefficients. In Section 4, the following a priori estimate

$$\|v\|_{L^q([0,T],L^r(\mathbb{R}^n))} \lesssim \|v\|_{L^2([0,T],H^{1/2}(\mathbb{R}^n))} + \|(i\partial_t + A)v\|_{L^2([0,T],H^{-1/2}(\mathbb{R}^n))} \tag{1.9}$$

is proved for any v supported in $[0,T] \times B(0,1)$, which is the main result of our work. The inequality (1.9) was first established for elliptic case in [23]. The main approach there is to construct the approximate parametrix through FBI transform. For non-degenerated case, there is something different. A more careful analysis is necessary. More precisely, one needs to illustrate that the gradient flows, which start from the characteristic surface $K = \{(t,\tau,x,\xi) \mid p(t,\tau,x,\xi) = 0\}$, can uniformly reach the surface $K_d = \{(t,\tau,x,\xi) \mid p(t,\tau,x,\xi) = d\}$ for some constant $d \neq 0$. Once this is proved, the estimates are reduced to the same settings in [23]. However in this paper, as proposed by Tataru, we would like to obtain the dispersive estimates through the parametrix constructed in [15]. The details will be illustrated in this section. At last we conclude the proof of Strichartz inequalities (1.4) and (1.6) with the aid of the estimates (1.8) and (1.9).

2 Well-Posedness and Local Smoothing

We first introduce the notation of pseudo-differential operator with nonregular symbols (see [25]) and a key lemma. Given a dyadic decomposition

$$1 = \sum_{\lambda=2^j\ j\in\mathbb{Z}} s\Big(\frac{\xi}{\lambda}\Big) = \sum_{\lambda} s_\lambda(\xi),$$

we define $S_\lambda u = s_\lambda(D)u$.

One can easily verify the following property

Proposition 2.1 *If $u \in C^r(\mathbb{R}^n)$, then*

$$\sup_\lambda \lambda^r \|S_\lambda u\|_{L^\infty} < +\infty. \tag{2.1}$$

The following definition of pseudo-differential operator with nonregular symbol and the Sobolev boundedness theorem are due to Taylor [25].

We define the Zygmund space $C^r_*(\mathbb{R}^n)$ composed of u such that (2.1) is finite, namely

$$\|u\|_{C^r_*} = \sup_\lambda \lambda^r \|S_\lambda u\|_{L^\infty} < +\infty.$$

Then $C^r_* = C^r$ if $r \in \mathbb{R}^+\backslash\mathbb{Z}^+$, $C^r_* \subsetneq C^r$ if $r \in \mathbb{Z}^+$. Then one can introduce the pseudo-differential operator with nonregular symbol.

Definition 2.1 *Set $r \in (0,+\infty)$, we say $p(x,\xi) \in C^r_* S^m(\mathbb{R}^n)$ provided for each multiindex $\alpha \in \mathbb{Z}^n_+$*

$$|D^\alpha_\xi p(x,\xi)| \le c_\alpha \langle\xi\rangle^{m-|\alpha|}, \tag{2.2}$$

$$\|D^\alpha_\xi p(\,\cdot\,,\xi)\|_{C^r_*(\mathbb{R}^n)} \le c_\alpha \langle\xi\rangle^{m-|\alpha|} \tag{2.3}$$

with $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$.

We define also the pseudo-differential operator Ψ_p with the symbol $p(x,\xi)$ by

$$\Psi_p u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} p(x,\xi)\widehat{u}(\xi)d\xi.$$

Then the following boundedness result in Sobolev spaces holds (see [25, p. 52])

Lemma 2.1 *If* $p(x,\xi) \in C_*^r S^m$, *then*

$$\|\Psi_p\, u\|_{H^s} \le c\|u\|_{H^{s+m}}$$

provided $-r < s < r$.

Another key lemma is concerning with non-trapping condition.

Lemma 2.2 *Suppose* (H1)–(H3) *hold and* $\kappa(|x|) = (1+|x|^2)^{-N/2}$ $(N > n)$. *Then there exists a constant* $c > 0$ *and a real classical symbol* $q(t,x,\xi) \in C^1([0,T], S^0)$, *such that*

$$H_a q \ge \kappa(|x|)|\xi| - c, \quad \forall\, (t,x,\xi) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n. \tag{2.4}$$

The elliptic version of above lemma is originally due to Doi [8]. And Kenig-Ponce-Rolvung-Vega proved Lemma 2.2 for non-degenerate case in [13]. In the present paper, one can give a simple proof as in [23].

Now we turn to the proof of local smoothing estimate. Consider the inhomogeneous equation

$$\begin{cases} i\partial_t u - \sum\limits_{j,k=1}^n \partial_j(a_{jk}(t,x)\partial_k u) + \sum\limits_{j=1}^n b_j(t,x)\partial_j u + c(t,x)u = h(t,x), \\ u(0) = u_0. \end{cases} \tag{1.1$'$}$$

Then we have the following a priori estimate.

Lemma 2.3 *Assume that* (H1)–(H3). *Then there exists a constant* c *depending on* n, ν, a_{jk} *and* T, *such that for all* $u \in C([0,T], \mathcal{S}(\mathbb{R}^n))$ *the following estimates holds*

(1)
$$\begin{aligned} &\sup_{0\le t\le T} \|u(t)\|_{L^2}^2 + \int_0^T \int_{\mathbb{R}^n} |J^{1/2}u(x,t)|^2 \kappa(|x|)dxdt \\ &\le c\Big(\|u(0)\|_{L^2}^2 + \int_0^T \|(i\partial_t + L)u\|_{L^2}^2 dt\Big), \end{aligned} \tag{2.5}$$

(2)
$$\sup_{0\le t\le T} \|u(t)\|_{L^2} \le c\Big(\|u(0)\|_{L^2} + \int_0^T \|(i\partial_t + L)u\|_{L^2} dt\Big) \tag{2.6}$$

with $J = (I-\Delta)^{1/2}$.

Proof Setting $h(t,x) = \partial_t u - iLu$, one obtains

$$\partial_t u = iLu + h(x,t). \tag{2.7}$$

By Lemma 2.2, there exists a real symbol $p = (n+1)c_0 q \in C^1([0,T], S^0)$ and $c > 0$, such that

$$H_a p \ge (n+1)c_0\kappa(|x|)|\xi| - c. \tag{2.8}$$

Set $k(t,x,\xi) = e^{p}$, $\widetilde{k}(t,x,\xi) = e^{-p}$ and correspondingly $K(t,x,D) = \Psi_k$, $\widetilde{K}(t,x,D) = \Psi_{\widetilde{k}}$. Observe that $k(t,x,\xi)$, $\widetilde{k}(t,x,\xi) \in C^1([0,T],S^0)$. We also define

$$N(u)^2 = \|Ku\|_{L^2}^2 + \|u\|_{-1}^2. \tag{2.9}$$

In what follows, we shall show that $N(\cdot)$ is an equivalent norm of the standard L^2 norm.

Noticing that

$$\widetilde{K}K = I + \Psi_{r_1}, \quad r_1 \in S^{-1}, \tag{2.10}$$

one has

$$\|u\|_{L^2} = \|\widetilde{K}Ku - \Psi_{r_1}u\|_{L^2} \lesssim \|Ku\|_{L^2} + \|u\|_{-1}. \tag{2.11}$$

The last part of (2.11) comes from the Sobolev Boundedness of classical pseudo-differential operators, which verifies the equivalence.

Next we will set up the inequalities in Lemma 2.3 for $N(u)$. First, we have

$$\frac{d}{dt}|N(u)|^2 = \frac{d}{dt}\|Ku\|_{L^2}^2 + \frac{d}{dt}\|u\|_{-1}^2 = \mathrm{I} + \mathrm{II}. \tag{2.12}$$

For II, one has

$$\begin{aligned}\frac{d}{dt}\|u\|_{-1}^2 &= 2\mathrm{Re}\,(J^{-1}\partial_t u, J^{-1}u) = 2\mathrm{Re}\,(iJ^{-1}Lu + J^{-1}h, J^{-1}u)\\ &\lesssim 2\mathrm{Re}\,(i[J^{-1},A]u, J^{-1}u) + N(u)^2 + (J^{-1}h, J^{-1}u),\end{aligned}$$

where $[\cdot\,,\,\cdot]$ is the commutator. Noticing $[J^{-1},A] \in C_*^2S^0$, we can employ the Lemma 2.1. It follows that

$$\frac{d}{dt}\|u\|_{-1}^2 \lesssim N(u)^2 + N(h)^2. \tag{2.13}$$

For I, one has

$$\begin{aligned}\frac{d}{dt}\|Ku\|_{L^2}^2 &= 2\mathrm{Re}\,(\partial_t(K(t,x,D)u), K(t,x,D)u)\\ &= 2\mathrm{Re}(\Psi_{\partial_t k}\,u, Ku) + 2\mathrm{Re}\,(K\partial_t u, Ku) \le c_1N(u)^2 + 2\mathrm{Re}\,(K(iLu+h), Ku)\\ &\le c_2N(u)^2 + 2\mathrm{Re}\,(i[K,A]u, Ku) + 2\mathrm{Re}\Big(i\sum_{j=1}^n b_j\partial_jKu, Ku\Big) + N(h)N(u)\\ &= c_2N(u)^2 + 2\mathrm{Re}\,(i[K,A]u, Ku) + 2\mathrm{Re}\,(\Psi_bKu, Ku) + N(h)N(u)\end{aligned} \tag{2.14}$$

with $b = \sum_{j=1}^n b_j\xi_j$. Notice that

$$i[K,A]u = \Psi_q + \Psi_{r_2}$$

with $r_2 \in C_*^1S^0$ and $q \in C_*^2S^1$, i.e.,

$$q = \{k,a\} = \sum_{j=1}^n \partial_{\xi_j}k\partial_{x_j}a - \partial_{x_j}k\partial_{\xi_j}a. \tag{2.15}$$

Since

$$\{k, a\} = -\{a, p\}e^p = -\{a, p\}k,$$

we can get

$$i[K, A] = -\Psi_{\{a,p\}}K + \Psi_{r_2} \tag{2.16}$$

with $r_2 \in C^1_* S^0$. By (2.8) and the hypothesis (H2),

$$H_a p + \sum_{j=1}^{n} b_j \xi_j \geq (n+1)c_0\kappa(|x|)|\xi| - c - nc_0\kappa(|x|)|\xi| = \kappa(|x|)|\xi| - c. \tag{2.17}$$

Then one can apply sharp Gårding inequality to $\Psi_{(\{a,p\}+b)}$, that is,

$$\begin{aligned} -(\Psi_{(\{a,p\}+b)}Ku, Ku) &\leq -(\kappa(|x|)J^1 Ku, Ku) + cN(u)^2 \\ &\leq -c_3\|\kappa(|x|)^{1/2}J^{1/2}Ku\|^2_{L^2} + cN(u)^2 \\ &\leq -c_3\|\kappa(|x|)^{1/2}J^{1/2}u\|^2_{L^2} + cN(u)^2. \end{aligned} \tag{2.18}$$

By (2.14), (2.16) and (2.18), we have

$$\frac{d}{dt}\|Ku\|^2_{L^2} \leq -c_4\|\kappa(|x|)^{1/2}J^{1/2}u\|^2_{L^2} + c_5 N(u)^2.$$

Then

$$\frac{d}{dt}N(u)^2 \leq -c_4\|\kappa(|x|)^{1/2}J^{1/2}u\|^2_{L^2} + c_6 N(u)^2 + N(h)N(u). \tag{2.19}$$

To obtain (2.5), one finds that

$$\frac{d}{dt}N(u)^2 + \|\kappa(|x|)^{1/2}J^{1/2}u\|^2_{L^2} \lesssim N(u)^2 + N(h)N(u). \tag{2.20}$$

Integrating it with respect to t from 0 to T yields (2.5).

Since (2.19) also implies

$$\frac{d}{dt}N(u)^2 \lesssim N(u)^2 + N(h)N(u),$$

one easily gets the estimates (2.6) by Gronwall inequality. This ends the proof.

Remark 2.1 The regularity of a_{jk} can be relaxed to $C^{2,\varepsilon}$. To convince this, it suffices to notice that $r_2 \in C^\varepsilon_* S^0$ is enough in (2.16).

Without difficulties, we can obtain the following theorem through Lemma 2.3 and the argument in [11, Chapter 23, §1].

Theorem 2.1 *Assume that* (H1)–(H3) *hold. Then*

(1) *If* $h(t,x) \in L^1([0,T], L^2(\mathbb{R}^n))$, *then* $(1.1)'$ *has a unique solution* $u \in C([0,T], L^2(\mathbb{R}^n))$ *satisfying*

$$\sup_{0\leq t\leq T}\|u(t)\|_{L^2} \leq c(T, a_{jk}, \nu, n)\Big(\|u_0\|_{L^2} + \int_0^T \|h(t,x)\|_{L^2}dt\Big).$$

(2) *If* $h(t,x)\in L^2([0,T],L^2(\mathbb{R}^n))$, *then* $(1.1)'$ *has a unique solution* $u\in C([0,T],L^2(\mathbb{R}^n))$ *satisfying*

$$\sup_{0\le t\le T}\|u(t)\|_{L^2}^2+\int_0^T\int_{\mathbb{R}^n}|J^{1/2}u(x,t)|^2\kappa(|x|)dxdt\le c(T,a_{jk},\nu,n)\Big(\|u_0\|_{L^2}^2+\int_0^T\|h(t,x)\|_{L^2}^2dt\Big)$$

with $J=(I-\Delta)^{1/2}$.

3 The Estimates of Constant Coefficients Equation

Theorem 3.1 *The unique solution* w *to equation* (1.7) *satisfies the estimate*

$$\|w\|_{L^q([0,T],L^r(\mathbb{R}^n))}\lesssim\|w_0\|_{L^2(\mathbb{R}^n)}+\|f\|_{L^2([0,T],H^{-1/2}(\mathbb{R}^n))}. \tag{3.1}$$

The inequality was originally established in [23]. It is essential that $f(t,x)$ is spatially compactly supported. Here we give a new proof to the non-endpoint case. For endpoint case, the construction is the same as the elliptic one in [23] and we will not repeat it. We first introduce a technical lemma of Christ and Kiselev in [5].

Lemma 3.1 *Let* X *and* Y *be Banach spaces and assume that* $k(t,s)$ *is a continuous function valued in* $B(X,Y)$. *Suppose that* $-\infty\le a<b\le+\infty$, *and set*

$$Tf(t)=\int_a^b k(t,s)f(s)ds,\quad Wf(t)=\int_a^t k(t,s)f(s)ds.$$

Assume that $1\le p<q\le+\infty$ *and*

$$\|Tf\|_{L^q([0,T],Y)}\le C\|f\|_{L^p([0,T],X)}.$$

Then

$$\|Wf\|_{L^q([0,T],Y)}\le\frac{2^{1-2(1/p-1/q)}}{1-2^{-2(1/p-1/q)}}C\|f\|_{L^p([0,T],X)}.$$

We now prove Theorem 3.1. Without loss of generality, we assume $w_0=0$. Set $A_0=-\sum\limits_{j=1}^n\varepsilon_j\partial_{jj}$. Then w can be expressed in the form

$$w(t,x)=\int_0^t e^{i(t-s)A_0}f(s,x)ds.$$

From Lemma 3.1, it suffices to prove

$$\Big\|\int_0^T e^{i(t-s)A_0}f(s,x)ds\Big\|_{L^q([0,1],L^r(\mathbb{R}^n))}\lesssim\|f\|_{L^2([0,T],H^{-1/2}(\mathbb{R}^n))}. \tag{3.2}$$

By Strichartz estimates for flat Schrödinger operator,

$$\begin{aligned}\Big\|\int_0^T e^{i(t-s)A_0}f(s,x)ds\Big\|_{L^q([0,1],L^r(\mathbb{R}^n))}&=\Big\|e^{itA_0}\int_0^T e^{-isA_0}f(s,x)ds\Big\|_{L^q([0,1],L^r(\mathbb{R}^n))}\\&\lesssim\Big\|\int_0^T e^{-isA_0}f(s,x)ds\Big\|_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{3.3}$$

Then the estimate (3.2) is reduced to

$$\Big\|\int_0^T e^{-isA_0}f(s,x)ds\Big\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2([0,T],H^{-1/2}(\mathbb{R}^n))}. \tag{3.4}$$

Since $f(t,\cdot)$ is compactly supported, the inequality (3.4) can be rewritten as

$$\Big\|\int_0^T e^{-isA_0}\chi_2(x)f(s,x)ds\Big\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2([0,T],H^{-1/2}(\mathbb{R}^n))}, \tag{3.4$'$}$$

where $\chi_2(x)$ is a smooth cutoff function.

By the duality argument, this is equivalent to

$$\|\chi_2(x)e^{-itA_0}g(x)\|_{L^2([0,T],H^{1/2}(\mathbb{R}^n))} \lesssim \|g\|_{L^2(\mathbb{R}^n)},$$

which is nothing else but the local smoothing estimates. This concludes our proof.

4 The Localized Variable Coefficients Estimates

Theorem 4.1 *Assume that* (H1)–(H3) *are fulfilled. Then for any* $v(t,x)$ *supported in* $[0,T]$ $\times B(0,1)$, *we have*

$$\|v\|_{L^q([0,T],L^r(\mathbb{R}^n))} \lesssim \|v\|_{L^2([0,T],H^{1/2}(\mathbb{R}^n))} + \|(i\partial_t + A)u\|_{L^2([0,T],H^{-1/2}(\mathbb{R}^n))}, \tag{4.1}$$

whenever the right hand side is finite.

This section is organized as follows. First we reduce the estimates (1.9) to a dyadic one as in [23]. Then we decompose time interval into small pieces and obtain the pointwise estimates in a unit tube through the exact parametrix (see Lemma 4.1). Reuniting the piecewise estimates, one can obtain the inequality (1.9). The key point of the proof is Lemma 4.2, which is firstly obtained in [15]. But for the sake of completeness, we would like to repeat the proof. The last part is devoted to the proof of Theorem 1.1 and Theorem 1.2. It is worthwhile to mention that in this section we would always assume $a_{jk} \in C^1([0,T],C^2(\mathbb{R}^n))$ and $T = 1$.

Before going into the details of the proof, we would like to introduce a parametrix constructed in [15].

Lemma 4.1 *Assume that the symbol* $a(t,y,\eta)$ *satisfies is measurable in* t *and satisfies the bounds*

$$|\partial_y^\alpha \partial_\eta^\beta a(t,y,\eta)| \le c_{\alpha\beta}, \quad |\alpha| + |\beta| \ge 2, \tag{4.2}$$

and (x^t,ξ^t) *is the Hamilton flow for* $D_t + a$, *that is,*

$$\begin{cases} \dfrac{dx^t}{dt} = a_\xi(t,x^t,\xi^t), & x(0) = x, \\ \dfrac{d\xi^t}{dt} = -a_x(t,x^t,\xi^t), & \xi(0) = \xi. \end{cases}$$

The kernel K of the fundamental solution operator $D_t + a(t, y, D)$ can be represented in the form

$$K(t, y_1, y) = \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\xi^t (y_1 - x^t)} e^{i\Psi(t,x,\xi)} G(t, x, y_1, \xi) e^{i\xi(x-y)} e^{-(x-y)^2/2} dx d\xi, \tag{4.3}$$

where the function G satisfies

$$|(x^t - y_1)^\gamma \partial_x^\alpha \partial_{y_1}^\beta \partial_\xi^\nu G(t, x, y_1, \xi)| \le C_{\alpha\beta\gamma\nu}, \tag{4.4}$$

and the real phase function Ψ is defined by

$$\frac{d\Psi}{dt} = -a + \xi^t a_\xi(t, y^t, \eta^t), \quad \Psi(0) = 0.$$

Given the dyadic decomposition as in Section 2, we define the partial summation operator $U_{\sqrt{\lambda}} = \sum_{\mu \le \sqrt{\lambda}} S_\mu$ and the new coefficients $a_{jk}^\lambda = U_{\sqrt{\lambda}} a_{jk}$. Since a_{jk}^λ is a small perturbation of a_{jk}, the non-degenerate condition still holds for these new coefficients. And the derivatives of a_{jk}^λ satisfy

$$\begin{aligned} &|\partial_y^\alpha a_{jk}^\lambda(t, y)| \le C(\alpha) \lambda^{(|\alpha|-2)/2}, && |\alpha| \ge 2, \\ &|\partial_y^\alpha a_{jk}^\lambda(t, y)| \le C(\alpha), && 0 \le |\alpha| \le 2. \end{aligned}$$

Choose another smooth cutoff function $\widetilde{s}(\eta)$ which is equal to 1 on the support of $s(\eta)$. Then (4.1) is reduced to the dyadic one

$$\|S_\lambda u\|_{L^q([0,1], L^r(\mathbb{R}^n))} \lesssim \lambda^{1/2} \|S_\lambda u\|_{L^2([0,1]\times\mathbb{R}^n)} + \lambda^{-1/2} \|P_\lambda S_\lambda u\|_{L^2([0,1]\times\mathbb{R}^n)} \tag{4.5}$$

with $P_\lambda = D_t + \sum_{j,k=1}^n a_{jk}^\lambda \partial_j \partial_k \widetilde{S}_\lambda(D) = D_t + a^\lambda(t, y, D)$. A useful observation is that it suffices to prove (4.5) for $\lambda \gg 1$.

For fixed frequency λ, we decompose time interval into small pieces, that is,

$$[0,1] = \bigcup_{k=1}^{\lambda} \Big[\frac{k-1}{\lambda}, \frac{k}{\lambda}\Big] = \bigcup_{k=1}^{\lambda} I_k.$$

We will concentrate our attention in one small time interval I_k. Without loss of generality, one can take I_1 for example. Suppose $u(t, y)$ solves the equation

$$\begin{cases} D_t u + a^\lambda(t, y, D) u = 0, & 0 \le t \le \dfrac{1}{\lambda}, \\ u(0) = u_0, \end{cases} \tag{4.6}$$

and define the solution operator $U(t, s) u(s, y) = u(t, y)$. By L^2 conservation, it is easy to know that $U(t, s)$ is L^2 isometrics. Set

$$\widetilde{t} = \lambda t, \quad \widetilde{y} = \sqrt{\lambda} y, \quad \widetilde{u}(\widetilde{t}, \widetilde{y}) = u\Big(\frac{\widetilde{t}}{\lambda}, \frac{\widetilde{y}}{\sqrt{\lambda}}\Big),$$

and the symbol $\widetilde{a}(\widetilde{t},\widetilde{y},\eta)=\lambda^{-1}a^{\lambda}(\frac{\widetilde{t}}{\lambda},\frac{\widetilde{y}}{\sqrt{\lambda}},\sqrt{\lambda}\eta)$. Then $\widetilde{u}$ solves the equation

$$\begin{cases} D_{\widetilde{t}}\widetilde{u}+\widetilde{a}(\widetilde{t},\widetilde{y},D)\widetilde{u}=0, \quad \widetilde{t}\in[0,1], \\ \widetilde{u}(0)=u_0\Big(\dfrac{\widetilde{y}}{\sqrt{\lambda}}\Big). \end{cases} \tag{4.7}$$

Define also the solution operator $\widetilde{U}(\,\cdot\,,\,\cdot\,)$ as before. Then the following pointwise estimates hold.

Lemma 4.2 *There exists a constant $0\le M<1$ which is independent of λ, such that for any $|t-s|\le M$,*

$$\|\widetilde{U}(t,s)\psi_{\sqrt{\lambda}}(D)u(s,y)\|_{L^\infty(\mathbb{R}^n_y)}\lesssim\frac{1}{|t-s|^{n/2}}\|u(s,y)\|_{L^1(\mathbb{R}^n_y)}, \tag{4.8}$$

where $\psi(\eta)$ is chosen to be a smooth cutoff function

$$\psi(\eta)=\begin{cases} 1, \quad \dfrac{1}{2}\le|\eta|\le 2, \\ 0, \quad |\eta|\le\dfrac{1}{4}\ \text{or}\ |\eta|\ge 4, \end{cases}$$

and $\widetilde{s}(\eta)\equiv 1$ on the support of $\psi(\eta)$.

Proof It suffices to prove it at $s=0$. Let $u(t,y)=\widetilde{U}(t)\psi_{\sqrt{\lambda}}(D)u_0(y)$. We would prove

$$\|u(t_0,y)\|_{L^\infty(\mathbb{R}^n_y)}\lesssim\frac{1}{t_0^{n/2}}\|u_0(y)\|_{L^1(\mathbb{R}^n_y)} \tag{4.9}$$

at any time $t_0\in[0,M]$ (M to be determined). For short range time $0\le t_0\le\frac{1}{\lambda}$,

$$\begin{aligned}\|u(t_0,y)\|_{L^\infty(\mathbb{R}^n_y)}&=\Big\|\psi_{\sqrt{\lambda}}(D)u_0(y)+\int_0^{t_0}\widetilde{a}(\tau,y,D)u(\tau)d\tau\Big\|_{L^\infty(\mathbb{R}^n_y)}\\ &\lesssim\lambda^{n/2}\|u_0(y)\|_{L^1(\mathbb{R}^n_y)}+\lambda\int_0^{t_0}\|u(\tau,y)\|_{L^\infty(\mathbb{R}^n_y)}d\tau.\end{aligned}$$

By Gronwall's inequality, one has

$$\|\widetilde{U}(t_0)\psi_{\sqrt{\lambda}}(D)u_0(y)\|_{L^\infty(\mathbb{R}^n_y)}\lesssim\lambda^{n/2}\|u_0(y)\|_{L^1(\mathbb{R}^n_y)}\lesssim\frac{1}{t_0^{n/2}}\|u_0(y)\|_{L^1(\mathbb{R}^n_y)}.$$

Then we need to prove (4.9) for long range $\frac{1}{\lambda}\le t_0\le M$. Set

$$\begin{aligned}&t_1=\frac{t}{t_0},\quad y_1=\frac{y}{\sqrt{t_0}},\quad \eta_1=\sqrt{t_0}\eta,\\ &u_1(t_1,y_1)=u(t_0t_1,y_1\sqrt{t_0}),\quad a_1(t_1,y_1,\eta_1)=t_0\widetilde{a}\Big(t_0t_1,y_1\sqrt{t_0},\frac{\eta_1}{\sqrt{t_0}}\Big).\end{aligned}$$

Then u_1 solves the equation

$$\begin{cases} D_{t_1}u_1+a_1(t_1,y_1,D)u_1=0, \\ u_1(0)=\psi_{\sqrt{\lambda t_0}}(D)u_0(y_1\sqrt{t_0})\triangleq v(y_1). \end{cases}$$

The estimate (4.9) is reduced to

$$\|u_1(1,y_1)\|_{L^\infty} \lesssim \|v(y_1)\|_{L^1}. \tag{4.10}$$

It is easy to verify

$$|\partial_{y_1}^\alpha \partial_{\eta_1}^\beta a_1(t_1,y_1,\eta_1)| \le C(\alpha,\beta), \quad \forall\, |\alpha|+|\beta| \ge 2.$$

Then one has

$$\begin{aligned}
u_1(t,y_1) &= \int e^{i\xi^t(y_1-x^t)} e^{i\Psi(t,x,\xi)} G(t,x,y_1,\xi) e^{i\xi(x-y)} e^{-(x-y)^2/2} \psi_{\sqrt{\lambda t_0}}(D) v(y)\,dy\,dx\,d\xi \\
&= \int e^{i\xi^t(y_1-x^t)} e^{i\Psi(t,x,\xi)} G(t,x,y_1,\xi) \int e^{-(\eta-\xi)^2/2} e^{-ix\eta} \psi\Big(\frac{\eta}{\sqrt{\lambda t_0}}\Big) \widehat{v}(\eta)\,d\eta\,dx\,d\xi \\
&= \int_{\mathbb{R}^n_x} \int_W e^{i\xi^t(y_1-x^t)} e^{i\Psi(t,x,\xi)} G(t,x,y_1,\xi) \int e^{-(\eta-\xi)^2/2} e^{-ix\eta} \psi\Big(\frac{\eta}{\sqrt{\lambda t_0}}\Big) \widehat{v}(\eta)\,d\eta\,d\xi\,dx \\
&\quad + \int_{\mathbb{R}^n_x} \int_{\mathbb{R}^n_\xi \setminus W} e^{i\xi^t(y_1-x^t)} e^{i\Psi(t,x,\xi)} G(t,x,y_1,\xi) \int e^{-(\eta-\xi)^2/2} e^{-ix\eta} \psi\Big(\frac{\eta}{\sqrt{\lambda t_0}}\Big) \widehat{v}(\eta)\,d\eta\,d\xi\,dx \\
&= \mathrm{I}(t) + \mathrm{II}(t)
\end{aligned}$$

with $W = \{\xi \in \mathbb{R}^n \mid \frac{\sqrt{\lambda t_0}}{8} \le |\xi| \le 8\sqrt{\lambda t_0}\}$.

Since $\psi(\eta)$ is supported on $\{\frac14 \le |\eta| \le 8\}$,

$$\begin{aligned}
&\|\mathrm{II}(1)\|_{L^\infty(\mathbb{R}^n_{y_1})} \\
&\le \Big\| \int_{\mathbb{R}^n_x} \int_{\mathbb{R}^n_\xi \setminus W} |G(1,x,y_1,\xi)| \int e^{-(\eta-\xi)^2/2} \psi\Big(\frac{\eta}{\sqrt{\lambda t_0}}\Big) |\widehat{v}(\eta)|\,d\eta\,d\xi\,dx \Big\|_{L^\infty(\mathbb{R}^n_{y_1})} \\
&\le e^{-c\lambda t_0} \Big\| \int_{\mathbb{R}^n_x} \int_{\mathbb{R}^n_\xi \setminus W} (1+|x^1-y_1|)^{-N} \int e^{-(\eta-\xi)^2/4} \psi\Big(\frac{\eta}{\sqrt{\lambda t_0}}\Big) |\widehat{v}(\eta)|\,d\eta\,d\xi\,dx \Big\|_{L^\infty(\mathbb{R}^n_{y_1})}.
\end{aligned}$$

Claim 4.1 There exists a finite $\theta = \theta(n) > 0$ such that

$$\int (1+|x^1-y_1|)^{-N} dx \lesssim (1+\lambda t_0)^\theta.$$

Proof of Claim 4.1 Since

$$x^1 = x + \int_0^1 \dot{x}^\tau d\tau = x + \int_0^1 \partial_\xi a_1(x^\tau,\xi^\tau) d\tau,$$

one has

$$\begin{aligned}
(1+|x^t-y|)^{-N} &= \Big(1+\Big|x+\int_0^1 \partial_\xi a_1(x^\tau,\xi^\tau)d\tau - y_1\Big|\Big)^{-N} \\
&\le (1+|x-y_1|)^{-N} \Big(1+\Big|\int_0^1 \partial_\xi a_1(x^\tau,\xi^\tau)d\tau\Big|\Big)^N \\
&\le (1+|x-y_1|)^{-N}(1+\lambda t_0)^{N/2}.
\end{aligned} \tag{4.11}$$

To obtain the last inequality of (4.11), it suffices to notice that $a_1(x,\xi)$ is compactly supported in ξ. This ends the proof of this claim.

Combining Claim 4.1, one has

$$\begin{aligned}\|\mathrm{II}(1)\|_{L^\infty(\mathbb{R}^n_{y_1})} &\lesssim e^{-c\lambda^2}\lambda^\theta \int_{\mathbb{R}^n} \psi\Big(\frac{\eta}{\sqrt{\lambda t_0}}\Big)|\widehat{v}(\eta)|d\eta \lesssim e^{-c\lambda^2}\lambda^{\theta+n/2}\|\widehat{v}(\eta)\|_{L^\infty}\\ &\lesssim e^{-c\lambda^2}\lambda^{\theta+n/2}\|v(y_1)\|_{L^1} \le c\|v(y_1)\|_{L^1},\end{aligned}$$

where c is independent of λ.

We rewrite $\mathrm{I}(t)$ as

$$\begin{aligned}\mathrm{I}(t) &= \int_W e^{i\xi^t(y_1-x^t)}e^{i\Psi(t,x,\xi)}G(t,x,y_1,\xi)e^{i\xi(x-y)}e^{-(x-y)^2/2}\psi_{\sqrt{\lambda t_0}}(D)v(y)dydxd\xi\\ &= \int_{\mathbb{R}^n_y} K(t,y_1,y)\psi_{\sqrt{\lambda t_0}}(D)v(y)dy.\end{aligned}$$

The inequality (4.10) would follow from

$$\|K(1,y_1,y)\|_{L^\infty(\mathbb{R}^n_{y_1})} \lesssim 1. \tag{4.12}$$

Claim 4.2

$$\int_W (1+|x^1-y_1|)^{-N}d\xi \lesssim 1.$$

Proof of Claim 4.2 We need to study the dependence of x^t on ξ. Set $X(t) = \frac{\partial x^t}{\partial \xi}$ and $\Xi(t) = \frac{\partial \xi^t}{\partial \xi}$. Then $X(t)$ and $\Xi(t)$ satisfy the ODE system

$$\begin{cases}\dfrac{\partial X(t)}{\partial t} = \partial_{\xi x}a_1 X + \partial_{\xi\xi}a_1\Xi, & X(0)=0,\\ \dfrac{\partial \Xi(t)}{\partial t} = -\partial_{xx}a_1 X - \partial_{\xi x}a_1\Xi, & \Xi(0)=I.\end{cases} \tag{4.13}$$

Noticing $\partial_{\xi x}a_1 = \frac{t_0}{\lambda}a^\lambda_{x,\xi}\big(\frac{t_0t_1}{\lambda}, \frac{x\sqrt{t_0}}{\sqrt{\lambda}}, \frac{\xi\sqrt{\lambda}}{\sqrt{t_0}}\big)$, $\partial_{xx}a_1 = \frac{t_0^2}{\lambda^2}a^\lambda_{x,x}\big(\frac{t_0t_1}{\lambda}, \frac{y\sqrt{t_0}}{\sqrt{\lambda}}, \frac{\xi\sqrt{\lambda}}{\sqrt{t_0}}\big)$ and $(\partial_{\xi\xi}a_1)_{n\times n} = \big(a^\lambda_{\xi,\xi}\big(\frac{t_0t_1}{\lambda}, \frac{y\sqrt{t_0}}{\sqrt{\lambda}}, \frac{\xi\sqrt{\lambda}}{\sqrt{t_0}}\big)\big) = (a^\lambda_{jk})$ in the region W, one gets

$$\frac{\partial X(t)}{\partial t} = \partial_{\xi\xi}a_1(t) + O(t_0) = \partial_{\xi\xi}a_1(0) + \int_0^t \frac{\partial}{\partial\tau}\partial_{\xi\xi}a_1(\tau)d\tau + O(t_0) = \partial_{\xi\xi}a_1(0) + O(t_0).$$

Then we have

$$X(1) = \partial_{\xi\xi}a_1(0) + O(t_0).$$

Noticing non-degenerate condition, we can choose a positive constant M such that for each $|t_0| < M$

$$\mathrm{Det}(X(1)) \ge C > 0, \quad \forall\, \xi \in B.$$

It is important to notice that the constants M and C are independent of λ. Hence we obtain

$$\int_W (1+|x^1-y_1|)^{-N}d\xi = \int_W (1+|x^1-y_1|)^{-N}\frac{1}{\mathrm{Det}(X(1))}dx^1 \lesssim 1.$$

Now we return to the proof of (4.12),

$$\begin{aligned}\|K(1,y_1,y)\|_{L^\infty(\mathbb{R}^n_{y_1})} &= \Big\|\int_W\int_{\mathbb{R}^n_x} e^{i\xi^1(y_1-x^1)}e^{i\Psi(t,x,\xi)}G(1,x,y_1,\xi)e^{i\xi(x-y)}e^{-(x-y)^2/2}dxd\xi\Big\|_{L^\infty(\mathbb{R}^n_{y_1})}\\ &\lesssim \int_{\mathbb{R}^n_x}\int_W (1+|x^1-y_1|)^{-N}e^{-(x-y)^2/2}d\xi dx\\ &\lesssim 1.\end{aligned}$$

This completes the proof of Lemma 4.2.

Rescaling in time and space, one obtains the following corollary.

Corollary 4.1 *There exits a constant $M>0$ which is independent of λ, such that for each $|t-s|\le\frac{M}{\lambda}$*

$$\|U(t,s)\psi_\lambda(D)u(s,y)\|_{L^\infty(\mathbb{R}^n_y)}\lesssim\frac{1}{|t-s|^{n/2}}\|u(s,y)\|_{L^1(\mathbb{R}^n_y)}.$$

Based on the corollary above, one can get piecewise dispersive estimates in I_1 as follows. We first have the trivial formulae

$$D_tu+A^\lambda u=P_\lambda u,\quad 0\le t\le\frac{1}{\lambda}\tag{4.14}$$

and

$$D_t\psi_\lambda(D)u+A^\lambda\psi_\lambda(D)u=\psi_\lambda(D)P_\lambda u+[A^\lambda,\psi_\lambda(D)]u.$$

Then

$$\psi_\lambda(D)u(t)=\int_0^t U(t,s)\{\psi_\lambda(D)P_\lambda u+[A^\lambda,\psi_\lambda(D)]u\}ds,\quad 0\le t\le\frac{1}{\lambda}.$$

More precisely, the time interval should be $[0,\frac{M}{\lambda}]$, but this makes no difference. Replacing u by $S_\lambda u$ and noticing that $\psi_\lambda(D)S_\lambda u=S_\lambda u$, we have

$$\begin{aligned}S_\lambda u(t)&=\int_0^t U(t,s)\{\psi_\lambda(D)P_\lambda S_\lambda u+[A^\lambda,\psi_\lambda(D)]S_\lambda u\}ds\\ &=\int_0^t\psi_\lambda(D)U(t,s)\psi_\lambda(D)P_\lambda S_\lambda u ds+\int_0^t\psi_\lambda(D)U(t,s)[A^\lambda,\psi_\lambda(D)]S_\lambda u ds.\end{aligned}$$

By Corollary 4.1 and Keel-Tao's argument in [14], one has the following estimates

$$\|S_\lambda u\|_{L^q([0,1/\lambda],L^r)}\lesssim\|P_\lambda S_\lambda u\|_{L^1([0,1/\lambda],L^2)}+\|[A^\lambda,\psi_\lambda(D)]S_\lambda u\|_{L^1([0,1/\lambda],L^2)}.\tag{4.15}$$

Then the following piecewise estimates holds:

$$\|S_\lambda u\|_{L^2([0,-1/\lambda],L^{2n/(n-2)})}\lesssim\lambda^{1/2}\|P_\lambda S_\lambda u\|_{L^2([0,1/\lambda],L^2)}+\lambda^{1/2}\|S_\lambda u\|_{L^2([0,1/\lambda],L^2)}.$$

Similarly, one has

$$\|S_\lambda u\|_{L^2(I_k,L^{2n/(n-2)})}\lesssim\lambda^{1/2}\|P_\lambda S_\lambda u\|_{L^2(I_k,L^2)}+\lambda^{-1/2}\|S_\lambda u\|_{L^2(I_k,L^2)}.\tag{4.16}$$

To end the proof of Theorem 4.1, it suffices to square the both sides of (4.16) and sum on k.

Next we will conclude the proof of Strichartz inequalities (1.4) and (1.6) based on the local smoothing estimates, Theorem 3.1 and Theorem 4.1.

Proof of Theorem 1.1 Choose a cutoff function $\chi(x) \in C_c^\infty(\mathbb{R}^n)$, i.e.,

$$\chi(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| > 2. \end{cases}$$

We split u into two parts

$$u = \chi(x)u + (1-\chi(x))u = u_1 + u_2. \tag{4.17}$$

Then u_2 solves the equation

$$\begin{cases} i\partial_t u_2 + A_0 u_2 = [\chi, A_0]u - (1-\chi(x))\sum\limits_{j=1}^n b_j\partial_j u - (1-\chi(x))cu = f_1, \\ u_2(0) = (1-\chi(x))u_0. \end{cases} \tag{4.18}$$

Correspondingly, u_1 satisfies the equation

$$\begin{cases} i\partial_t u_1 + Au_1 = -[\chi, A_0]u + \chi(x)\sum\limits_{j=1}^n b_j\partial_j u + \chi(x)cu = f_2, \\ u_1(0) = \chi(x)u_0. \end{cases} \tag{4.19}$$

By local smoothing estimate (see Theorem 2.1), one has

$$\|[\chi, A_0]u\|_{L^2([0,T],H^{-1/2}(\mathbb{R}^n))} \lesssim \|u_0\|_{L^2(\mathbb{R}^n)}, \tag{4.20}$$

$$\|b_j\partial_j u\|_{L^2([0,T],H^{-1/2}(\mathbb{R}^n))} \lesssim \|\kappa(|x|)J^{1/2}u\|_{L^2(\mathbb{R}^{n+1})} \lesssim \|u_0\|_{L^2(\mathbb{R}^n)}. \tag{4.21}$$

Combining Theorem 3.1, Theorem 4.1 and inequalities (4.20), (4.21), we have

$$\begin{aligned} \|u\|_{L^q([0,T],L^r(\mathbb{R}^n))} &\le \|u_1\|_{L^q([0,T],L^r(\mathbb{R}^n))} + \|u_2\|_{L^q([0,T],L^r(\mathbb{R}^n))} \\ &\lesssim \|\chi u\|_{L^2(H^{1/2})} + \|[\chi, A_0]u\|_{L^2(H^{-1/2})} + \sum_{j=1}^n \|b_j\partial_j u\|_{L^2(H^{-1/2})} \\ &\lesssim \|u_0\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

This concludes the proof of Theorem 1.1.

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